Disentangling the Measurement of Risk and Uncertainty*

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Abstract

In this paper, we study whether it is possible to measure Knightian uncertainty and disentangle it from the conventional risk measure in the location-scale family of normal distributions. First, we establish and prove the impossibility theorems that no quantitative uncertainty measure would be mutually agreeable among investors with general risk-and-uncertainty-averse preferences. Then, we show it is possible to partially rank the uncertainty of the location-scale family of normal distributions by the first, second, fourth, and sixth moments. Lastly, we propose two ways to assess the stock uncertainty, the ranges of location and scale of the family of normal distributions computed from the 1st, 2nd, 4th and 6th moments, and a composite uncertainty measure proposed by Izhakian (2020), but with necessary and critical modification. We assess the uncertainty of stock market indices, S&P 500 in the U.S and CSI 300 in China, and show that the uncertainty and risk measures are indeed distinct.

Keywords: Knightian Uncertainty, Risk, Location-Scale Family, Stochastic Dominance

1 Introduction

Investors and economic agents have to constantly face risk and uncertainty everywhere in the economy and financial markets. Albeit Knight (1921) has emphasized that risk (unknown consequences with a known probability distribution) and uncertainty (unknown consequences and unknown probability distributions) are "radically distinct"[1] more than one hundred years ago, the conventional analysis frameworks in economics and finance assume that economic agents face only risk and no uncertainty. The paradox proposed by Ellesberg (1951) show that the decisions made under uncertainty can not be rationalized under the risk-only context, but are actually rational in the context that they know neither the consequences nor the probability distribution. De Groot and Thurik (2018) find that many studies in psychology and related fields claim to measure decision-making under risk actually measure decision making under uncertainty instead of risk. Over the past century, researchers in finance and economics have made important theoretical progress in the decision making and asset pricing under Knightian uncertainty. They show that uncertainty and risk are indeed priced differently in the financial market (Hansen, 2007) and affect the decisions of economic agents in fundamentally different ways[2]. However, direct empirical evidence on the different impacts of risk and uncertainty on investment decisions and asset prices remains rare.

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[1] "But Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk, from which it has never been properly separated."

[2] See Guidolin and Rinaldi (2013) for a comprehensive review of theoretical research asset pricing and portfolio choice under uncertainty or ambiguity.

In practice, investors struggle with how to deal with "the unknown and unknowable" risk (Zeckhauser, 2010). The challenge is to measuring uncertainty when the probability distribution is unknown.

In this paper, we first establish and prove that it is impossible to find a quantitative measure of uncertainty that is independent of preferences within the location-scale family of normal distribution. This result is analogy to the findings of Ma and Wong (2010) that it is impossible to find a quantitative measure of risk that is independent of preferences for risk in the risk-only context.

Albeit it is impossible to get unanimous rank of uncertainty or risk between two arbitrary events, it still makes sense for empirical asset pricing studies where a set of statistics exist to characterize the uncertainty in the Knightian-uncertainty context, such as using the standard deviation or variance in the risk-only context to characterize the risk.

The core question of asset pricing is what risk is priced in the market and how. Hence the measure of risk is the fundamental question in empirical asset pricing studies. In the risk-only context, where the returns are assumed to follow a unique distribution, in particular normal distribution, the variance or standard deviation (second order moment) measures the risk and the asset pricing studies focus on the relationship between risk and expected return (first order moment). However, the distribution of realized stock returns usually exhibit fatter tail than normal distribution. In the risk-only context, this problem is usually addressed by assuming normal distribution, such as Levy distribution, or combination of normal distribution and jump process. However, the standard deviation of Levy distribution is infinite, and it is difficult to separate the latent jump process from the normal distribution when we only observation the composition of the two distributions. Some empirical asset pricing studies use higher-moments, such as skewness or kurtosis to measure tail risk and/or uncertainty, but lack of statistical or economic foundation.

In this paper, we show that under the assumption stock returns fall into the class of location-scale family of Gaussian distributions, and the investors' beliefs about the probabilities of Gaussian distributions are uniform, expected variance[3] of a family of probability density functions proposed by Izhakian (2020) helps to measure uncertainty. This measure is equivalent to the Shannon entropy[4] of a family of Gaussian distributions, as the relative entropy of a Gaussian distribution is just a function of its variance ($\frac{1}{2}\log(2\pi\sigma^2)+\frac{1}{2}$).

Furthermore, we show that sufficient statistics of a family of Gaussian distributions are the mean of the first, second, fourth, and sixth moments, as these four moments can fully recover the location and scale of Gaussian distributions[5] along with the characteristic function and all other higher order moments. We also show that the measure proposed by Izhakian (2020) is inconsistent and null when the possible means and standard deviations are randomly drawn from a range with infinite or continuum support[6]. Our new measure corrects the problem of Izhakian's measure and converges to the expected variances of probability distributions when the support goes to infinity. Brenner and Izhakian (2018) propose an empirical implementation of Izhakian's measure by discretizing the normal distributions and computing the sample analogy of the expected variance of a family normal distribution. We show that Brenner and Izhakian's approach overweights average variances of the p.d.fs when range of standard deviations is large, introducing a spurious positive correlation between stock uncertainty and conventional measure of risk when the risk is high. We propose an algorithm to compute the uncertainty directly from means and standard deviations of the daily stock returns estimated from high-frequency data.

Lastly, we compute the stock uncertainty of market indices, S&P 500 in the U.S. and CSI 300 in

[3] The expectation is taken with respect to the average of this family of probability distributions.

[4] Shannon(1948) proposes to use entropy to measure "how uncertain we are of the outcome", and the relative entropy of a continuous random variable with a probability density function $f(x)$ is defined as $\int -f(x)\log f(x)dx$. Among all real random variables with expectation μ and variance σ^2 the the Gaussian distribution attains the maximum entropy.

[5] If the means and variances of the family of Gaussian distributions fall into rectangle sets.

[6] This result is consistent with the findings of Fu et al. (2023)

China, using high frequency trading data. We find that the relationship between the stock uncertainty and risk measured by sample variance are nonlinear and negative, contrary to the common belief that the higher the risk, the larger the uncertainty. More importantly, we show that dispersion in the means is a more reasonable measure of risk, while the conventional measure of risk significantly negatively correlates with the expected return (estimated by the sample mean). In addition, we show that the nonlinear transformation of the 2nd moment (conventional measure of risk), 4th moment (conventional measure of tail risk) and 6th moment combined with the stock uncertainty measure can potentially explain the variation in the expected returns.

Our findings imply that the asset pricing model in the context of Knightian uncertainty should be fundamentally different from the linear factor models in the risk-only context. As the properties of a family of normal distributions are fully characterized by the 1st, 2nd, 4th and 6th moments, the analogy of the one-factor CAPM in the risk-only context should be a three-factor model.

2 Impossibility Theorems

When encountering decision making in a context of risk and uncertainty, economists tend to take a subjective view on risk and uncertainty assessment. This view can be traced back to Savage (1954), Gilboa(1987), Gilboa and Schmeidler (1989), and most recently Kopylov (2010) among others for their axiomatic foundation on subjective probability, non-additive and set-valued probabilities, along with utility representation of preferences in presence of risk and uncertainty.

The risk and uncertainty assessment is a subjective matter that largely depend on how investors rank the asset returns towards their investment decisions. Presumably, when asset returns r and r' with uncertainty respectively modeled through location-scale sets Δ and Δ', the rank between the two returns should reflect decision makers' view on the degree of uncertainty involved in the two sets.

The analysis in this section is restricted to asset returns falling into the location-scale family formulated in Wong and Ma (2008) among others. We mainly present and prove two versions of impossibility theorems on the existence of uncertainty measurement, respectively, among investors falling into the class of Savage's subjective expected utilities (SEU) and the expected utility under uncertain probabilities (EUUP) recently developed by Izhakian (2017). We also provide robustness checks to several other popular classes of utility functions either with SEU as subclass.or are subclasses of SEU.

2.1 Model setup

Let ε_t be an i.i.d. random process follows standard normal distribution[7] in an filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the following location-scale specifications on logarithm of asset returns ($r_t = \ln R_t$):

$$r_t = \mu + \sigma\varepsilon_t \tag{R\&U}$$

where (μ, σ) is a pair of mean and standard deviation of return r_t. At each point of time, the pair (μ, σ) takes some value in a set Δ, and the actual selection within the set may or may not be governed by a probability distribution. ε on $(\Omega, \mathcal{F}, \mathbb{P})$ is referred to as the random seed that captures the risk aspect of asset returns. This is the typical framework of Knightian uncertainty on Δ in the presence of risk ε.

The followings are three typical examples of Δ:

Example 1 $\Delta = \left\{R \equiv \sqrt{\mu^2+\sigma^2} \in \left[\underline{R}, \overline{R}\right]; \theta \equiv \tan^{-1}\left(\frac{\mu}{\sigma}\right) \in \left[\underline{\theta}, \overline{\theta}\right]\right\}$

[7] The Gaussian (normal) distribution assumption is not crucial for the main theory developed in this paper. It is assumed mainly for the sake of empirical implementation.

Example 2 $\Delta = \left\{(\mu,\sigma) \in \left[\underline{\mu},\overline{\mu}\right] \times \left[\underline{\sigma},\overline{\sigma}\right]\right\}$

Example 3 $\Delta = \left\{(\mu_i,\sigma_i), i = 1,2,\cdots,n\right\}$

as illustrated in Figure 1

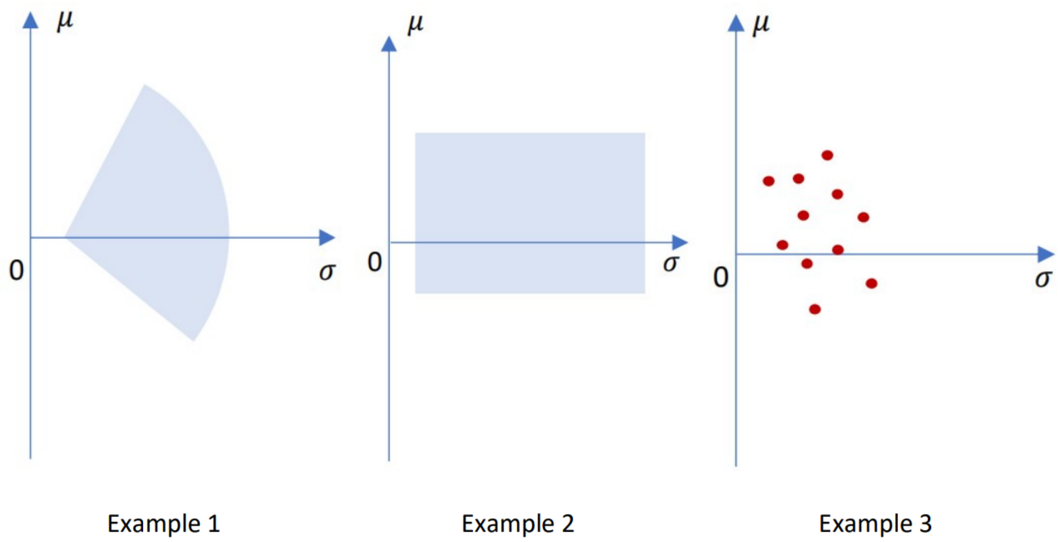

Figure 1. Different Sets of Means and Standard Deviations

The mathematical formulation of the model uncertainty is not unique depending on the interpretation of the uncertainty, which is reflected through the definition of the universal state space. Broadly speaking, the literature contains the following two ways in defining the universal state space:

First, the return process r_t is simply taken as an *i.i.d.* process on

$$\mathcal{E} \triangleq \left(\Omega \times \Delta, \mathcal{F} \times \mathcal{B}(\Delta), \mathbb{P} \times \mathcal{Q}\right)$$

under some set-valued beliefs $\mathcal{Q} \subseteq \mathcal{M}(\Delta)$, $\mathcal{B}(\Delta)$ is a Borel σ-algebra of Δ, and $\mathcal{M}(\Delta)$ is the set of all $\mathcal{B}(\Delta)$-measurable probability measures on Δ.

Second, the universal state space is taken to be

$$\mathcal{E}' \triangleq \left(\Omega \times \mathcal{M}, \mathcal{F} \times \mathcal{B}(\mathcal{M}), \mathbb{P} \times \xi\right)$$

in which $\mathcal{M} = \mathcal{M}(\Delta)$, $\mathcal{B}(\mathcal{M})$ is an σ-algebra of $\mathcal{M}$ under some well-defined topology on $\mathcal{M}$, and ξ is a second-layer belief on $\mathcal{M}$ that constitutes a $\mathcal{B}(\mathcal{M})$-measurable probability measure ξ on $\mathcal{M}$. More precisely, $\xi \in \mathcal{B}(\mathcal{M})$ is a measure on the set of all probability measures on Δ. It can be interpreted as a compound lottery on Δ in the sense of Segal (1987).

The return process r_t are *i.i.d.* in the sense that, for each $\mathbb{Q} \in \mathcal{M}$ or $\xi \in \mathcal{B}(\mathcal{M})$, the random process r_t under $\mathbb{P} \times \mathbb{Q}$ or $\mathbb{P} \times \xi$ are identically distributed and independent over time.

In either approach, the uncertainty is on the set of possible (if any) probability measures $\mathbb{Q} \in \mathcal{M}(\Delta)$ that govern the distribution of (μ,σ) in Δ. When Δ is finite and contains n pairs of (μ,σ) as in Example 3, then $\mathcal{M}(\Delta)$ is a simplex defined as following,

$$\mathcal{M}(\Delta) = \left\{\mathbf{q} \in \mathbb{R}^n : \sum_{i=1}^{n} q_i = 1, q_i \geq 0\right\}$$

where q_i is the probability of selecting the pair (μ_i,σ_i) in Δ. When n is 3, that is, Δ contains three pairs of (μ,σ), $\mathcal{M}(\Delta)$ is a 2-D simplex in a 3-D space, as illustrated in Figure 2.

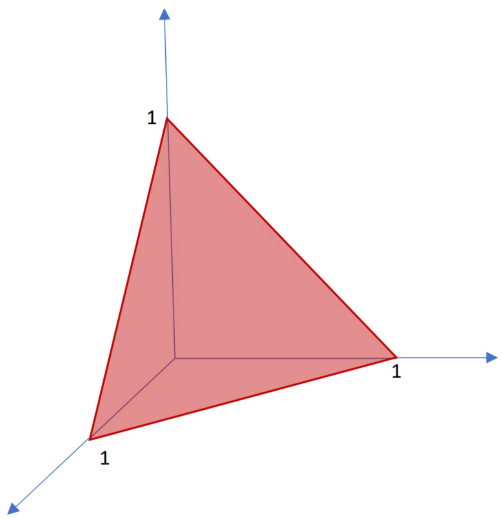

Figure 2. Space of Probability Distributions over a Set of Three Pairs of Means and Standard Deviations of Asset Returns

Note that the conventional risk framework is a special case of model (R&U) where the selection of the mean-standard-deviation pair (μ, σ) within the set Δ is governed by a known or unique probability distribution, that is, $\mathcal{M}(\Delta)$ is a singleton. In particular, if Δ is a singleton ($n = 1$), $\mathcal{M}(\Delta) = \{1\}$ is also a singleton, this is a special case where there is no uncertainty. If Δ contains more than one pair of (μ, σ), as long as the the selection of the pair (μ, σ) within the set Δ is governed by a known or unique probability distribution $\mathbb{Q}$, that is, $\mathcal{M}(\Delta) = \{\mathbb{Q}\}$ is a singleton, then there is no uncertainty and the model is still within the conventional risk framework.

To carry out uncertainty assessment, we restrict preferences of investors to a broad class of expected and non-expected utility functions defined over the location-scale family. Let $\mathcal{U}$ denotes utility functions that satisfy the usual conditions, namely, monotonicity and risk/uncertainty aversion. We may wish to further distinguish between the two classes of utility functions by the above mentioned state-space formulation; namely, the class of expected and/or non-expected utility under Savage's subjective beliefs on $\mathcal{E}$, denoted $\mathcal{U}_S$; and the class of expected and/or non-expected utility functions under second-layer beliefs on $\mathcal{E}'$, denoted $\mathcal{U}_{NE}$.

Typical examples of utility functions belonging to $\mathcal{U}_S$ include

1. Savage (1954) subjective expected utility (SEU):

$$V(r) = \iint\limits_{\Delta} u(\mu, \sigma)\, d\mathbb{Q}(\mu, \sigma)$$

for some $\mathbb{Q} \in \mathcal{M}(\Delta)$ and $\mathcal{M}(\Delta)$ is the set of all probability measures on Δ;

2. Gilboa and Schmeidler (1989) maximin expected utility:

$$V(r) = \min_{\mathbb{Q} \in \mathcal{Q}} \iint\limits_{\Delta} u(\mu, \sigma)\, d\mathbb{Q}(\mu, \sigma)$$

where $\mathcal{Q} \subseteq \mathcal{M}(\Delta)$ is a non-empty convex subset of $\mathcal{M}(\Delta)$.

3. Chew's (1983) betweenness utility:[8] $V(r)$ is a solution to

$$\iint\limits_{\Delta} \mathbb{E}_P \left[H \left(r, V \left(r \right) \right) \right] d\mathbb{Q} \left(\mu, \sigma \right) = 0$$

for some $\mathbb{Q} \in \mathcal{M}(\Delta)$, where $\mathbb{E}_P$ is the expectation with respect to ε conditional on (μ, σ) and $H(x, y)$ is a betweenness function that increases in x, (strictly) decreases in y with $H(x, x) \equiv 0$. For weighted utility $H(x, y) = w(x)(v(x) - v(y))$, it yields

$$V(r) = v^{-1} \left(\frac{\iint\limits_{\Delta} w \circ v(\mu, \sigma) \, d\mathbb{Q}(\mu, \sigma)}{\iint\limits_{\Delta} w(\mu, \sigma) \, d\mathbb{Q}(\mu, \sigma)} \right)$$

in which $w(\mu, \sigma) = \mathbb{E}_P[w(r)]$ and $w \circ v(\mu, \sigma) = \mathbb{E}_P[w(r) v(r)]$.

The well known utility functions belonging to $\mathcal{U}_{NE}$ include

1. Smooth Utility of Klibanoff, Marinacci and Mukerjii (2005):

$$V(r) = \int_{\mathcal{M}} \psi \left(\iint\limits_{\Delta} u(\mu, \sigma) \, d\mathbb{Q}(\mu, \sigma) \right) d\xi(\mathbb{Q})$$

for some $\xi \in \mathcal{B}(\mathcal{M})$, where $\psi(x)$ is a monotonic increasing concave or convex function in x, corresponding to uncertainty averse or uncertainty loving preference, respectively.

2. Expected Utility under Uncertain Probabilities (EUUP) of Izhakian (2017):[9]

$$V(r) = \int_{\mathcal{M}} \iint\limits_{\Delta} u(\mu, \sigma) \, d\mathbb{Q}(\mu, \sigma) \, d\xi(\mathbb{Q}) \tag{2.1}$$

where ξ is uniform on $\mathcal{M}(\cdot)$, as a special $\mathcal{B}(\mathcal{M})$-measurable probability measure.

In summary, utility functions in either class are formulated through the following two-step procedure:

Step one, it involves utility representation in a pure context of risk; that is, taking risk source ε as given, for any fixed bundle of (μ, σ) in Δ, let $u(r; \mu, \sigma)$ represents the expected or non-expected utility of $r_t = \mu + \sigma\varepsilon_t$. These include, for instance, Markowitz's (1954) mean-variance utility functions, Von Neumann and Morgenstern (1947) expected utility, betweenness utility of Chew (1983,1989) and Dekel (1986), anticipated utility of Quiggin (1982) and Yaari (1987), and the prospect theory of Kahneman and Tversky(1979). Presumably, the location-scale utility function $u(r; \mu, \sigma)$ on Δ summarizes an investor's risk attitudes.

Step two, it involves defining utility function in the presence of uncertainty to the location-scale coefficients (μ, σ) on Δ. The utility $V(r)$ defined over $r = \mu + \sigma\varepsilon$ with set-valued coefficients $(\mu, \sigma) \in \Delta$

[8] Betweenness utility function under risk and uncertainty is formulated by Ma (2002), who discusses on its usefulness in resolving both Allais' (1951) paradox under risk and Ellsberg's (1961) paradox under risk and uncertainty. It reduces to SEU with the specification of $H(x, y) = u(x) - u(y)$. See also Wong and Ma (2008) for betweenness utility functions defined over the location-scale family.

[9] Precisely, this is a variation to the EUUP of Izhakian (2017), as we do not allow for distortion on subjective probability $\mathbb{Q} \in \mathcal{Q}$. Instead, one may take the location-scale utility $u(\mu, \sigma)$ as expected utility under some distorted probability with respect to the Gaussian normal distribution under which ε is defined. Without loss of generality, it is always possible to introduce distortions on subjective beliefs $\mathbb{Q}$ as well as that on the objective probability $\mathbb{P}$, and one may allow for different distortions when utility payoffs are set below or above some reference point.

is jointly determined by the subjective beliefs over the set Δ and subjective expected or non-expected utilities $u(r;\mu,\sigma)$.

All utility functions listed above accommodate Savage's subjective expected utility as a subclass, except for the EUUP which assumes the second-layer distribution is uniform over $\mathcal{M}(\cdot)$. Hence, when unanimous ranking between r and r' are reached among all SEU investors with some $\mathbb{Q} \in \mathcal{M}(\Delta)$, the same ranking will carry forward to EUUP investors as well; however, the converse is, in general, not true.

As a final remark, it is natural to require the assessment of uncertainty not to be differentiated by investors with long positions or short positions. In other words, an uncertainty measure, if exists, would rank the uncertainty of r and $-r$ in the same fashion. In other words, if r is more uncertain than r', then $-r$ should be more uncertain than $-r'$ as well[10].

Definition 1 *Given some risk sources ε and ε', let r and r' be asset returns with randomness driven by ε and ε' and uncertainty governed by Δ and Δ', respectively. Δ' is said to be more uncertain than Δ if*

$$(a)\ r \succcurlyeq_i r';\ and\ (b) - r \succcurlyeq_i -r' \tag{2.2}$$

for all $i \in \mathcal{U}$, which hold with strict preference for some i. Moreover, a function $\rho : \Delta \to \mathbb{R}$ is an uncertainty measure *(independent of preference) if*

$$\rho(\Delta') > \rho(\Delta) \quad \Leftrightarrow \Delta' \text{ is more uncertain than } \Delta \tag{2.3}$$

in the sense of the unanimous agreements (a) and (b) between Δ and Δ' are reached among all $i \in \mathcal{U}$.

Given the above definition of *uncertainty* that is distinct from *risk*, we can formally examine the existence of quantifiable uncertainty. Two critical questions have to be answered: First, is it possible to reach an unanimous ranking of uncertainty among all decision makers in the class? Second, does it exist a quantifiable measure that is consistent with such a unanimous ranking among all decision makers? Note that a sensible uncertainty measure, if exists, would at least apply to Examples 1-3 and yield a finite positive measure for each case.

Both questions received wide attention and have been thoroughly studied in literature. However, existing literature mainly focus in the risk-only context, thus only address the issue of risk measure. In the world with risk only, the randomness of location and scale variables are governed by some objective probabilities. Ma and Wong (2010) follow the footsteps of Rothschild and Stiglitz (1970) and show that not all r and r' can be ranked unanimously by decision makers within SEU class. Furthermore, when unanimous ranking can be reached, the conventional measure of risk, that is, the standard deviation or variance, in most circumstance, is found to generate assessment not contradicting to the ranking reached by the decision makers.

In this paper, we focus on two questions in the context with both risk and uncertainty. First, whether an unanimous ranking can be reached. Second, in case when an unanimous rank can be achieved, whether a quantifiable measure of uncertainty consistent with the ranking can be constructed. For the first question, we answer with two impossibility theorems which extend the findings of Ma and Wong (2010) to the general context with both risk and uncertainty. For the second question, we find that it is impossible to rank uncertainty based on one composite measure such as the one proposed by Izhakian (2020), but a profile criteria of the first six moments. In particular, we show that if the set Δ of possible means and standard deviations is rectangular, we can rank the uncertainty based on the first, second,

[10]Here, we take $-r$ as an approximation for the return obtained by the short sellers. When short seller's margin deposit ratio (m) and rebate rate r_b are not ignorable, the precise rate of return for the short seller is $\ln\left(1 + m\left(1 - e^{r_t}\right)e^{-r_b}\right) \approx -m\left(1 - r_b\right)r_t$.

fourth and sixth moments. Furthermore, concerning the empirical implementation of the uncertainty measure of Izhakian (2020), we suggest some necessary modifications keeping in mind of its pitfalls.

2.2 Impossibility Theorem for SEU as a Subclass

We start with Savage's SEU functions that belong to $\mathcal{U}_S$. We show that it is impossible to find a quantifiable uncertainty measure that support unanimous ranking of uncertainty agreeable by SEU investors. Moreover, when no agreement can be reached among SEU investors, then no agreement can be reached among utility functions in $\mathcal{U}_S$ which include SEU as a subclass. Hence, it is impossible to find a quantifiable uncertainty measure that support unanimous ranking of uncertainty agreeable by $\mathcal{U}_S$ investors.

The following impossibility theorem is obtained by restricting to SEU investors who prefer more to less ($\mathcal{U}_S^1$), and to SEU investors who prefer more to less and are risk averse ($\mathcal{U}_S^2$), respectively.

Theorem 1 *1. An unanimous rank of uncertainty between two arbitrary sets $\Delta \neq \Delta'$ among SEU investors cannot be always reached; particularly,*

(a) unanimous rank of uncertainty between Δ and Δ' among all $\mathcal{U}_S^1$ investors is reached if and only if $\Delta = \Delta'$;

(b) unanimous rank of uncertainty between Δ and Δ' among all $\mathcal{U}_S^2$ investors is reached if and only if Δ and Δ' share a singleton support on the location coefficient μ, and the intersection of two volatility intervals contains no interior point, if not empty.

2. So long as $\mathcal{U}_S$ contains $\mathcal{U}_S^1$ and $\mathcal{U}_S^2$ as a subclass, a quantitative measure $\rho : \Delta \to \mathbb{R}$ satisfying condition (2.3) does not exist even when an unanimous rank of uncertainty between Δ and Δ' is reached.

Proof. Statement 1 is a direct consequence of statements (a) and (b). We prove (a) and (b) by contradiction. Suppose an unanimous rank between r and r' can be reached under Savage subjective expected utility class so that

$$
\begin{aligned}
\mathbb{E}_{Q\times P}\left[u\left(r\right)\right] &\geq \mathbb{E}_{Q'\times P}\left[u\left(r'\right)\right] && (2.4)\\
\mathbb{E}_{Q\times P}\left[u\left(-r\right)\right] &\geq \mathbb{E}_{Q'\times P}\left[u\left(-r'\right)\right] && (2.5)\\
\forall u' &> 0 \text{ (and } u''<0 \text{ if risk aversion is imposed)}\\
\forall Q &\in \mathcal{M}\left(\Delta\right), \forall Q' \in \mathcal{M}\left(\Delta'\right)
\end{aligned}
$$

Case 1: Utility functions are restricted to $\mathcal{U}_S^1$ (i.e., $u'>0$). Conditions (2.2) are equivalent to r FSD r' and $(-r)$ FSD $(-r')$; that is, $\forall x \in (-\infty,\infty), \forall Q' \in \mathcal{M}\left(\Delta'\right), \forall Q \in \mathcal{M}\left(\Delta\right),$

$$
\begin{aligned}
\iint\limits_{\Delta'} \Phi\left(\frac{x-\mu}{\sigma}\right) dQ'\left(\mu,\sigma\right) &\geq \iint\limits_{\Delta} \Phi\left(\frac{x-\mu}{\sigma}\right) dQ\left(\mu,\sigma\right)\\
\iint\limits_{\Delta'} \Phi\left(\frac{x+\mu}{\sigma}\right) dQ'\left(\mu,\sigma\right) &\geq \iint\limits_{\Delta} \Phi\left(\frac{x+\mu}{\sigma}\right) dQ\left(\mu,\sigma\right)
\end{aligned}
$$

These are equivalent to, for all $x \in (-\infty, \infty)$,

$$\begin{aligned}
\sup_{(\mu,\sigma)\in\Delta} \Phi\left(\frac{x-\mu}{\sigma}\right) &\leq \inf_{(\mu,\sigma)\in\Delta'} \Phi\left(\frac{x-\mu}{\sigma}\right) \\
\sup_{(\mu,\sigma)\in\Delta} \Phi\left(\frac{x+\mu}{\sigma}\right) &\leq \inf_{(\mu,\sigma)\in\Delta'} \Phi\left(\frac{x+\mu}{\sigma}\right)
\end{aligned} \tag{2.6}$$

that is, $\forall x \in (-\infty, \infty)$, $\forall (\mu, \sigma) \in \Delta, \forall (\mu', \sigma') \in \Delta'$,

$$\Phi\left(\frac{x \pm \mu}{\sigma}\right) \text{ FSD } \Phi\left(\frac{x \pm \mu'}{\sigma'}\right).$$

This holds when and *only when* both of the following conditions are satisfied:

(i) Δ' and Δ share a common singleton support on location coefficients μ;[11]

(ii) Δ' and Δ share a common singleton support on scale coefficients σ.

Conditions (i) and (ii) combined imply that Δ and Δ' must be identical with a singleton support (μ, σ). This violates the non-degeneracy condition $\Delta \neq \Delta'$.

Case 2: Utility functions are restricted to $\mathcal{U}_S^2$ (i.e., $u' > 0$ and $u'' < 0$). Conditions 2.2 are equivalent to r SSD r' and $-r$ SSD $-r'$; that is, $\forall x \in (-\infty, \infty)$, $\forall Q' \in \mathcal{M}(\Delta')$, $\forall Q \in \mathcal{M}(\Delta)$,

$$\int_{-\infty}^{x} \iint_{\Delta'} \Phi\left(\frac{x \mp \mu}{\sigma}\right) dQ'(\mu, \sigma)\, dx \geq \int_{-\infty}^{x} \iint_{\Delta} \Phi\left(\frac{x \mp \mu}{\sigma}\right) dQ(\mu, \sigma)\, dx$$

or

$$\iint_{\Delta'} \int_{-\infty}^{x} \Phi\left(\frac{x \mp \mu}{\sigma}\right) dx dQ'(\mu, \sigma) \geq \iint_{\Delta} \int_{-\infty}^{x} \Phi\left(\frac{x \mp \mu}{\sigma}\right) dx dQ(\mu, \sigma)$$

This is equivalent to, $\forall x \in (-\infty, \infty)$, $\forall (\mu, \sigma) \in \Delta, (\mu', \sigma') \in \Delta'$,

$$0 \leq \int_{-\infty}^{x} \left(\Phi\left(\frac{x \mp \mu'}{\sigma'}\right) - \Phi\left(\frac{x \mp \mu}{\sigma}\right)\right) dx \tag{2.7}$$

that is,

$$\Phi\left(\frac{x \mp \mu}{\sigma}\right) \text{ SSD } \Phi\left(\frac{x \mp \mu'}{\sigma'}\right)$$

This holds when and only when both of the following conditions are satisfied:

(i) Δ' and Δ share a common singleton support on location coefficients μ;

(ii)' the maximum volatility (scale coefficient) in Δ does not exceed the minimum volatility in Δ'; that is,

$$\sup\{\sigma : (\mu, \sigma) \in \Delta\} \leq \inf\{\sigma : (\mu, \sigma) \in \Delta'\}.$$

Conditions (i) and (ii)' imply that Δ and Δ' share a singleton support on the location coefficient μ, and the intersection of two volatility intervals contains no interior point, if not empty.

It remains to prove statement 2. We only need to consider Case 2 for $\mathcal{U}_S^2$ investors: Suppose a quantifiable uncertainty measure ρ exists so that, to all Δ and Δ' satisfying conditions (i) and (ii)', it holds $\rho(\Delta) \leq \rho(\Delta')$. For arbitrary $\Delta_1 \subset \Delta$ and $\Delta_1' \subset \Delta'$, conditions (i) and (ii)' are satisfied for Δ_1 and Δ_1'. Accordingly, we must have $\rho(\Delta_1) \leq \rho(\Delta_1')$. We can thus rank the uncertainty between Δ_1

[11] We have

$$\begin{aligned}
\inf\{\mu : (\mu, \sigma) \in \Delta\} &\geq \sup\{\mu : (\mu, \sigma) \in \Delta'\} \\
\sup\{\mu : (\mu, \sigma) \in \Delta\} &\leq \inf\{\mu : (\mu, \sigma) \in \Delta'\}
\end{aligned}$$

These together imply condition (i) holds.

and Δ, and between Δ_1' and Δ' as well. These, however, contradict to statement 1-(b) as condition (ii)' does not hold for Δ_1 and Δ, nor for Δ_1' and Δ'. These enable us to conclude that there exists no single quantitative uncertainty measure satisfying (2.3). ■

The ideas for the proofs of Theorem 1 follow the standard arguments on stochastic dominance. To reach an unanimous agreement among all investors with preference in a class, it has to be the case that, under arbitrary given subjective beliefs over the set, the investors must prefer one set to the other. That is, r and $-r$ must stochastically dominate r' and $-r'$ respectively under all subjective beliefs over the set Δ and Δ'. We first show that these are not possible for first order stochastic dominance (FSD) among SEU investors who prefer more to less. We then show that these are not possible for second order stochastic dominance (SSD) among all risk averse investors who prefer more to less, and derive some necessary and sufficient conditions for unanimous ranking between Δ and Δ'. We show that unanimous ranking among risk averse investors can be reached if and only if conditions (i) and (ii)' hold, but the quantitative measure of uncertainty must be set-valued criteria, and a single-value measure satisfying (2.3) does not exist.

The following additional remarks are in order:

Remark 1 *Theorem 1 holds for all utility functions in $\mathcal{U}_S$ that encompass SEU as a subclass. The impossibility theorem also applies to smooth utility of Klibanoff, Marinacci and Mukerji (2005) because it contains SEU as a subclass where the distribution ξ on $\mathcal{B}(\mathcal{M})$ is of singleton support (i.e., ξ assigns probability one to a probability measure $Q \in \mathcal{M}(\Delta)$).*

Remark 2 *Theorem 1 holds for constant relative risk aversion (CRRA) utilities as a subclass of SEU; that is,*

$$V(r) = \frac{1}{\alpha}\mathbb{E}_{Q\times P}\left[R_t^{\alpha}\right] = \frac{1}{\alpha}\iint\limits_{\Delta} e^{\alpha\mu + \frac{\alpha^2\sigma^2}{2}} dQ(\mu,\sigma)$$

for all $0 \neq \alpha \leq 1$. See Appendix 1 for proof.

Remark 3 *The set-valued Sharpe-ratios $\left[\underline{\rho}(\Delta), \overline{\rho}(\Delta)\right]$ with*

$$\begin{aligned} \underline{\rho}(\Delta) &= \inf\left\{\frac{\mu}{\sigma} : (\mu,\sigma) \in \Delta\right\} \\ \overline{\rho}(\Delta) &= \sup\left\{\frac{\mu}{\sigma} : (\mu,\sigma) \in \Delta\right\} \end{aligned}$$

as uncertainty measures are equivalent to conditions (i) and (ii)' combined. Δ' is more uncertain than Δ implies that $\underline{\rho}(\Delta) > \overline{\rho}(\Delta')$. However, this does not constitute a quantitative measure as it involves set-valued criteria summarized by $\left[\underline{\rho}(\Delta), \overline{\rho}(\Delta)\right]$.

Remark 4 *If the minimum of expected returns of all assets are negative and maximum of expected returns are positive, then conditions (i) are not met, and we can not obtain an unanimous agreement among all investors. Even if there is no uncertainty in location (i.e., condition (i) holds), when all investors are risk averse, the average volatilities in the set Δ as a quantifiable measure, which do not contradict to condition (ii)', is however not equivalent to (ii)'.*

In summary, the impossibility theorem on uncertainty measure among Savage's SEU investors follows the standard arguments on FSD and SSD. Unlike Rothschild and Stiglitz (1970) who built up the equivalence between SSD and mean-preserving-spread (for the case of constant location) in a pure risk context, and established standard deviation as a quantitative measure of risk that never contradict to SSD among risk averse investors, it is not applicable to use the average volatilities as a quantitative measure of uncertainty in a context of risk and uncertainty. One must look into different criteria on the magnitude and dispersions of both location and scale coefficients.

2.3 Impossibility Theorem for EUUP Class

Next, we establish impossibility theorem for the EUUP class. When unanimous ranking is reached among SEU investors, the same ranking will be reached among EUUP investors. However, the converse does not always hold. An unanimous agreement among EUUP class does not necessarily imply unanimous agreement among SEU. Some weaker conditions for unanimous agreement among EUUP are thus possible.

EUUP assume uniform second-layer subjective beliefs ξ over $\mathcal{M}(\Delta)$, that is, the belief of investors with EUUP on Δ are homogeneous. As investors' second layer belief under EUUP is homogeneous, one may speculate that unanimous ranking of uncertainty among EUUP investors are relatively easier to be reached. Indeed, the stochastic dominance argument as for Theorem 1 are less obvious to hold for EUUP investors. Below we aim to establish the arguments for cases with finite and continuum supports of Δ.

2.3.1 Finite Supports

Izhakian (2020) proposes an uncertainty measure assuming each Δ is with finite n number of supports. We show that it is impossible to find an unanimous rank of uncertainty between two arbitrary finite sets $\Delta = \{(\mu_i, \sigma_i)\}_{i=1}^{n}$ and $\Delta' = \left\{\left(\mu'_j, \sigma'_j\right)\right\}_{j=1}^{m}$.

Theorem 2 *An unanimous rank of uncertainty between two arbitrary finite sets $\Delta \neq \Delta'$ cannot be always reached among EUUP investors. Even when such a rank is reached, a quantitative measure $\rho : \Delta \to \mathbb{R}$ satisfying condition (2.3) does not exist.*

The proofs proceed in three steps. First, we establish an "if and only if" connection on an unanimous ranking between Δ and Δ' among EUUP investors and stochastic dominance between the two mean-average normal distributions $\overline{\Phi}_r(x) \triangleq \frac{1}{n}\sum_{i=1}^{n} \Phi\left(\frac{x-\mu_i}{\sigma_i}\right)$ and $\overline{\Phi}_{r'}(x) \triangleq \frac{1}{m}\sum_{j=1}^{m} \Phi\left(\frac{x-\mu'_j}{\sigma'_j}\right)$. Second, following the standard argument for first order stochastic dominance (FSD), we show that no unanimous ranking between Δ and Δ' are possible among EUUP investors who prefer more to less. In the third step, we apply the argument for SSD to derive some necessary conditions for unanimous ranking between Δ and Δ' among risk averse EUUP investors who prefer more to less. These are summarized by the following three propositions respectively, and the proofs of these propositions are provided in the Appendix.

Proposition 1 *For all Δ and Δ' with finite supports, $V(r) \geq V(r')$ for $u \in EUUP$ if and only if*

$$\int_{-\infty}^{\infty} u(x)\, d\overline{\Phi}_r(x) \geq \int_{-\infty}^{\infty} u(x)\, d\overline{\Phi}_{r'}(x) \tag{2.8}$$

Particularly, unanimous ranking among monotone (monotone and risk averse) EUUP investors are reached if and only if $\overline{\Phi}_r$ first-order (second-order) stochastically dominate $\overline{\Phi}_{r'}$.

Proposition 2 *The following statements are equivalent:*

1. *Δ' is more uncertain than Δ for all EUUP investors who prefer more to less (i.e., $u'(x) \geq 0$);*
2. *$\overline{\Phi}_r$ FSD $\overline{\Phi}_{r'}$ and $\overline{\Phi}_{-r}$ FSD $\overline{\Phi}_{-r'}$, that is,*

$$\overline{\Phi}_r(x) \leq \overline{\Phi}_{r'}(x), \overline{\Phi}_{-r}(x) \leq \overline{\Phi}_{-r'}(x) \tag{2.9}$$

   for all $x \in \mathbb{R}$.

3. *$\Delta = \Delta'$.*

Proposition 3 *The statements 1 and 2 below are equivalent, and both imply statement 3:*

1. *Δ' is more uncertain than Δ for all EUUP investors who prefer more to less and risk averse (i.e., $u'(x) \geq 0, u''(x) \leq 0$) ;*

2. *$\overline{\Phi}_r$ SSD $\overline{\Phi}_{r'}$ and $\overline{\Phi}_{-r}$ SSD $\overline{\Phi}_{-r'}$; that is,*

$$\int_{-\infty}^{x} \left[\overline{\Phi}_r(x) - \overline{\Phi}_{r'}(x)\right] dx \leq 0 \text{ and } \int_{-\infty}^{x} \left[\overline{\Phi}_{-r}(x) - \overline{\Phi}_{-r'}(x)\right] dx \leq 0 \tag{2.10}$$

   for all $x \in \mathbb{R}$.

3. *Let $\overline{\Theta}_r$ be the characteristic function for $\overline{\Phi}_r$.[12] For all $x \in \mathbb{R}$ it holds true that*

$$\overline{\Theta}_r(x) \leq \overline{\Theta}_{r'}(x); \tag{2.11}$$

   particularly, the following conditions hold

   (a) $\frac{1}{n}\sum_{i=1}^{n}\mu_i = \frac{1}{m}\sum_{j=1}^{m}\mu'_j$;

   (b) $\max\left\{\sigma^2 : (\mu,\sigma) \in \Delta\right\} \leq \max\left\{\sigma^2 : (\mu,\sigma) \in \Delta'\right\}$;

   (c) $\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i^2 + \sigma_i^2\right) \leq \frac{1}{m}\sum_{j=1}^{m}\left(\mu_j'^2 + \sigma_j'^2\right)$;

   (d) for arbitrary positive integer N, if the equality

$$\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k(\mu_i,\sigma_i) = \frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_k\left(\mu'_j,\sigma'_j\right) \tag{2.12}$$

   hold for $k = 0, 1, \cdots, 2N$, then the equality (2.12) must hold for $k = 2N+1$ as well, in addition to the inequality

$$\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+2}(\mu_i,\sigma_i) \leq \frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_{2N+2}\left(\mu'_j,\sigma'_j\right) \tag{2.13}$$

   for $k = 2N+2$.

Condition (3) in Proposition 2 for FSD violates the non-degeneracy condition for uncertainty assessment. So, no mutual agreeable ranking are possible between two arbitrary finite sets $\Delta \neq \Delta'$ among investors with monotone EUUP utility functions.

According to Proposition3, it is necessary to draw two average characteristic functions $\overline{\Theta}_r(x)$ and $\overline{\Theta}_{r'}(x)$ and check if one is always below the other, before reaching a consensus for an unanimous ranking if ever applies. Condition 3 is necessary for SSD criterion (2.10).[13] Moreover, conditions (a), (b) and (c) listed in statement 3 of Proposition3 are necessary and readily verifiable for any arbitrary two location-scale Δ and Δ' sets of finite supports. Violation to any of these conditions would make the assessment of uncertainty between Δ and Δ' unattainable.

Condition (a) is on the average locations; condition (b) is on the maximum volatility in the support; and (c) constitutes an restriction on the radius R^2 of Δ, which governs the magnitude or depth of uncertainty. Further to (a), (b) and (c), the necessary moment conditions extend to arbitrary higher order moments as prescribed in (d).

Thus, no single criterion or uncertainty measure can be constructed to assess if one set Δ' is more uncertain than the other Δ. A set of criteria as illustrated by (a), (b), (c) and (d) must be verified.

[12] $\overline{\Theta}_r(x) = \frac{1}{n}\sum_{i=1}^{n}\Theta(\mu_i,\sigma_i;x)$ with $\Theta(\mu,\sigma;x) = \sum_{k=0}^{\infty}\frac{\mathfrak{m}_k(\mu,\sigma)}{k!}(-x)^k$ and $\mathfrak{m}_k(\mu,\sigma)$ the k-th moment for $N(\mu,\sigma)$.

[13] Condition 3, as it turns out, is *necessary and sufficient* for stochastic dominance among risk averse investors with constant RRA utility functions, i.e. $u(R_t) = \frac{R_t^\gamma}{\gamma}, \gamma \leq 1$, as a subclass of risk averse family.

Violation to any of these conditions, would make the unanimous rank between the two finite sets Δ and Δ' non-attainable. Combining Propositions 2 and 3, we obtain Theorem 2.

The followings are a couple of examples to illustrate the viability of our Impossibility Theorem 2 to the extreme cases of no uncertainty in either location or scale coefficient.

Example 4 *We assume location coefficient μ be constant (i.e., there is no uncertainty on the location coefficient). With $\Theta(x) = e^{-\mu x + \frac{\sigma^2 x^2}{2}}$ the rank between the two mean-average characteristic functions $\overline{\Theta}_r$ and $\overline{\Theta}_{r'}$ (with common μ) reduces to that between $\frac{1}{n}\sum_{i=0}^{n} e^{\frac{\sigma_i^2 x^2}{2}}$ and $\frac{1}{m}\sum_{j=0}^{m} e^{\frac{\sigma_j'^2 x^2}{2}}$ for all $x \in \mathbb{R}$. Conditions (c) and (d) in Proposition 3 as necessary conditions for inequality (2.11) are simplified into*

(c) $\frac{1}{n}\sum_{i=0}^{n} \sigma_i^2 \leq \frac{1}{m}\sum_{j=0}^{m} \sigma_j'^2$;

(d) if $\frac{1}{n}\sum_{i=0}^{n} \sigma_i^{2k} = \frac{1}{m}\sum_{j=0}^{m} \sigma_j'^{2k}$ for all $k = 1, \cdots, N$, then[14]

$$\frac{1}{n}\sum_{i=0}^{n} \sigma_i^{2N+2} \leq \frac{1}{m}\sum_{j=0}^{m} \sigma_j'^{2N+2}.$$

Example 5 *We assume σ^2 be constant (i.e., there is no uncertainty on the scale coefficient). The rank between the two mean-average characteristic functions $\overline{\Theta}_r$ and $\overline{\Theta}_{r'}$ reduces to that between $\frac{1}{n}\sum_{i=0}^{n} e^{-\mu_i x}$ and $\frac{1}{m}\sum_{j=0}^{m} e^{-\mu_j' x}$ for all $x \in \mathbb{R}$. The necessary conditions (a)-(d) as in Proposition 3 respectively become*

(a) $\frac{1}{n}\sum_{i=0}^{n} \mu_i = \frac{1}{m}\sum_{j=0}^{m} \mu_j'$;

(b) $\mu_{\max} \leq \mu'_{\max}$ if the maximum supports are positive; and $\mu_{\min} \geq \mu'_{\min}$ if the minimum supports are negative;[15]

(c) $\frac{1}{n}\sum_{i=1}^{n} \mu_i^2 \leq \frac{1}{m}\sum_{j=1}^{m} \mu_j'^2$;

(d) If $\frac{1}{n}\sum_{i=0}^{n} \mu_i^k = \frac{1}{m}\sum_{j=0}^{m} \mu_j'^k$ for $k = 1, \cdots, 2N$, then[16]

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{n} \mu_i^{2N+1} &= \frac{1}{m}\sum_{j=0}^{m} \mu_j'^{2N+1} \\ \frac{1}{n}\sum_{i=0}^{n} \mu_i^{2N+2} &\leq \frac{1}{m}\sum_{j=0}^{m} \mu_j'^{2N+2}. \end{aligned}$$

[14] When $\frac{1}{n}\sum_{i=0}^{n} \sigma_i^{2k} = \frac{1}{m}\sum_{j=0}^{m} \sigma_j'^{2k}$ for all positive integers $k = 0, 1, \cdots, \infty$, it has to be the case that the two characteristic functions coincide with $\Delta = \Delta'$. In fact, since it involves only finite number of volatility coefficients, we speculate that one may only need to check finite number of moment conditions, say up to $\max\{2n, 2m\}$.

[15] It is noted that, with $\mu_{\max} \geq 0 \geq \mu_{\min}$ and constant scale coefficient σ, the speeds of convergence for $\frac{1}{n}\sum_{i=0}^{n} e^{-\mu_i x}$ at $x \to \pm\infty$ is totally governed by the maximum $\mu_{\max}$ and the minimun $\mu_{\min}$ of the location coefficients respectively.

[16] At the extreme case when $\frac{1}{n}\sum_{i=0}^{n} \mu_i^k = \frac{1}{m}\sum_{j=0}^{m} \mu_j'^k$ for all positive integers, it has to be the case that the two characteristic functions coincide with $\Delta = \Delta'$. In fact, since it involves only finite number of parameters, one may only need to check a finite number of moment conditions, say up to $2N+1$ and $N = \max\left\{\left[\frac{n}{2}\right], \left[\frac{m}{2}\right]\right\}$.

2.3.2 Continuum support Δ

Assume Δ is a compact set (i.e., bounded and closed) in the Euclidean space $\mathbb{R} \times \mathbb{R}_+$ so that $|\Delta| = \iint_{\Delta} d\mu d\sigma$ takes a finite value.[17] Define

$$\overline{\Phi}_r(x) = \frac{1}{|\Delta|} \iint_{\Delta} \Phi\left(\frac{x-\mu}{\sigma}\right) d\mu d\sigma \tag{2.14}$$

$$\overline{\Theta}_r(x) = \frac{1}{|\Delta|} \iint_{\Delta} \exp(-\mu x + \frac{\sigma^2 x^2}{2}) d\mu d\sigma \tag{2.15}$$

as the *mean-average cumulative distribution function (c.d.f.)* and the *mean-average characteristic function* for r with location-scale range Δ.[18] Let

$$\overline{\mathfrak{m}}_k = (-1)^k \, \overline{\Theta}_r^{(k)}(0) = \frac{1}{|\Delta|} \iint_{\Delta} \mathfrak{m}_k(\mu, \sigma) \, d\mu d\sigma$$

be the mean-average k-th order moment for the mean-average cdf $\overline{\Phi}_r$.

Following the same argument as for the finite support case we can readily establish the equivalence between the unanimous ranking (in uncertainty) between Δ and Δ' and the stochastic dominance between the two mean-average c.d.f.s $\overline{\Phi}_r$ and $\overline{\Phi}_{r'}$; that is, investors in the EUUP class all prefer Δ to Δ' if and only if $\overline{\Phi}_r$ stochastically dominates $\overline{\Phi}_{r'}$ in the sense of Eq.(2.8). Accordingly, all statements made in Propositions 2 and 3 and Theorem 2 remain valid except that we replace "average" with "mean-average" for $\overline{\Phi}_r(x)$ and $\overline{\Theta}_r(x)$ respectively. So, in accordance with Theorem 2, we obtain the following theorem that applies to arbitrary compact set Δ and Δ':

Theorem 3 *An unanimous rank of uncertainty between two arbitrary compact sets $\Delta \neq \Delta'$ cannot be always reached among EUUP investors. Even when such a rank is reached, a quantitative measure $\rho : \Delta \to \mathbb{R}$ satisfying condition (2.3) does not exist. Nevertheless, when Δ' is ranked to be more uncertain than Δ by all monotone and risk averse investors in EUUP, it has to be the case that*

$$\overline{\Theta}_r(x) \leq \overline{\Theta}_{r'}(x), \forall x \in \mathbb{R} \tag{2.16}$$

which implies the followings:[19]

1. $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}_1'$ *and* $\overline{\mathfrak{m}}_2 \leq \overline{\mathfrak{m}}_2'$*;*

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[17] Here, we define $|\Delta|$ for Example 2 when the location-scale is defined on the (μ, σ) plane. It works with other axis system adopted for the local-scale family as well. For instance, one may set $|\Delta| = \iint_{\Delta} dR d\theta$ on (R, θ) plane as in Example 1, and set $|\Delta| = \iint_{\Delta} d\mu d\nu$, $\nu = \sigma^2$, on the (μ, ν) plane. One system can be relatively less complex than the other depending on the geometric structure and/or the shape of Δ.

[18] In accordence with Riemman-Stejjiel integration, the mean-average c.d.f. is obtained by taking the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \Phi\left(\frac{x-\mu_i}{\sigma_i}\right) = \frac{1}{|\Delta|} \iint_{\Delta} \Phi\left(\frac{x-\mu}{\sigma}\right) d\mu d\sigma$$

for arbitrary division of Δ into finite (n) disjoint subset $\cup_{i=1}^{n} \Delta_i$ of equal area, and $(\mu_i, \sigma_i) \in \Delta_i$ are arbitrary. The same applies to mean-average characteristic functions.

[19]

$$\overline{\mathfrak{m}}_1 = \frac{1}{|\Delta|} \iint_{\Delta} \mu d\mu d\sigma, \overline{\mathfrak{m}}_2 = \frac{1}{|\Delta|} \iint_{\Delta} \left(\mu^2 + \sigma^2\right) d\mu d\sigma, \cdots$$

2. for arbitrary positive integer N if $\overline{\mathfrak{m}}_k = \overline{\mathfrak{m}}'_k$ for all $k = 1, \cdots, 2N$, then it must hold true that

$$\overline{\mathfrak{m}}_{2N+1} = \overline{\mathfrak{m}}'_{2N+1} \text{ and } \overline{\mathfrak{m}}_{2N+2} \leq \overline{\mathfrak{m}}'_{2N+2}. \tag{2.17}$$

Moreover, for rectangles Δ and Δ', if $\overline{\mathfrak{m}}_k = \overline{\mathfrak{m}}'_k$ for $k = 1, 2, 4, 6$, then the two mean-average characteristic function must be identical (i.e., $\overline{\Theta}_r = \overline{\Theta}_{r'}$) with $\Delta = \Delta'$.

Before prove Theorem 3, we first provide an intuitive explanation. Note that necessary conditions for inequality 2.16, Conditions 1 and 2, are the counterpart to the case of finite support as in Theorem 2. It is straightforward to show that $\overline{\Theta}_r(x)$ is positive and convex with initial conditions $\overline{\Theta}_r(0) = 1$, $\overline{\Theta}'_r(0) = -\overline{\mathfrak{m}}_1$ and $\overline{\Theta}''_r(0) = \overline{\mathfrak{m}}_2$. As illustrated in Figure 3a and 3b, the two smooth convex functions $\overline{\Theta}_r(x)$ and $\overline{\Theta}_{r'}(x)$ meet at $x = 0$. Inequality 2.16 implies the two curves must tangent at $x = 0$. This yields $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}'_1$. The local convexity $\overline{\mathfrak{m}}_2 \leq \overline{\mathfrak{m}}'_2$ determines if one curve is locally above the other. When $\overline{\mathfrak{m}}_2 = \overline{\mathfrak{m}}'_2$, we must look into higher order moment conditions as illustrated in Condition 2.

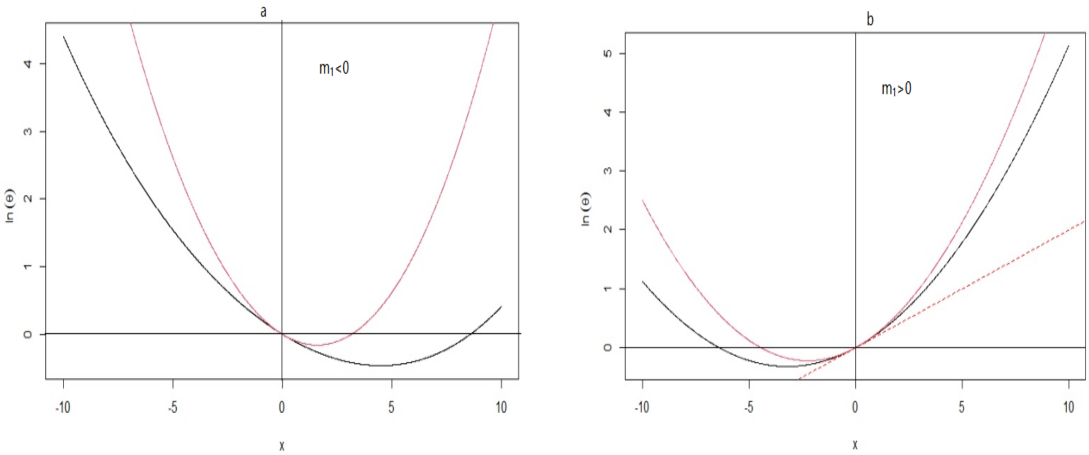

Figure 3. Two mean-average characteristic functions with same first-order moment ($\overline{\mathfrak{m}}_1$) but different second-order moment ($\overline{\mathfrak{m}}_2$), with $\overline{\mathfrak{m}}_2$ of the red line larger than that of the black line.

Proof. We just need to prove the validity of the last statement of Theorem 3. Set $\Delta = [\underline{\mu}, \overline{\mu}] \times [\underline{\nu}, \overline{\nu}]$ with $\nu = \sigma^2$, then the mean-average characteristic function can be explicitly expressed as a function of location-scale coefficients $\{\underline{\mu}, \overline{\mu}, \underline{\nu}, \overline{\nu}\}$ as follow:

$$\begin{aligned} \overline{\Theta}_r(x) &= \frac{1}{|\Delta|} \int_{\underline{\mu}}^{\overline{\mu}} \int_{\underline{\nu}}^{\overline{\nu}} \exp(-x\mu + \frac{x^2 \nu}{2}) d\mu d\nu \\ &= \frac{2}{x^3} \frac{\exp(-x\underline{\mu} + \frac{x^2}{2}\overline{\nu}) + \exp(-x\overline{\mu} + \frac{x^2}{2}\underline{\nu}) - \exp(-x\underline{\mu} + \frac{x^2}{2}\underline{\nu}) - \exp(-x\overline{\mu} + \frac{x^2}{2}\overline{\nu})}{(\overline{\mu} - \underline{\mu})(\overline{\nu} - \underline{\nu})}. \end{aligned} \tag{2.18}$$

The mean-average characteristic function also admits the following Taylor expansion

$$\overline{\Theta}_r(x) = 1 - \overline{\mathfrak{m}}_1 x + \frac{\overline{\mathfrak{m}}_2}{2!} x^2 - \frac{\overline{\mathfrak{m}}_3}{3!} x^3 + \frac{\overline{\mathfrak{m}}_4}{4!} x^4 + \cdots$$

with which we obtain its mean-average moments $\{\overline{\mathfrak{m}}_k\}_{k=1}^{\infty}$ in terms of location-scale coefficients $\{\underline{\mu}, \overline{\mu}, \underline{\nu}, \overline{\nu}\}$. Thus, if we can find four mean-average moments that uniquely determine $\{\underline{\mu}, \overline{\mu}, \underline{\nu}, \overline{\nu}\}$, then these four moments also uniquely determine the mean-average characteristic function $\overline{\Theta}_r(x)$. ■

The following Lemma on mean-average characteristic function for random return r under rectangle uncertainty $\Delta = [\underline{\mu}, \overline{\mu}] \times [\underline{\nu}, \overline{\nu}]$ is sufficient for our main conclusion.

Lemma 1 *The following conditions are equivalent:*

1. r *and* r' *are identically distributed under* $\mathcal{E}' = (\Omega \times \mathcal{M}, \mathcal{F} \times \mathcal{B}(\mathcal{M}), \mathbb{P} \times \xi)$*;*
2. $\Delta = \Delta'$*;*
3. $\overline{\Theta}_r(x) = \overline{\Theta}_{r'}(x), \forall x \in \mathbb{R}$*;*
4. $\overline{\mathfrak{m}}_n = \overline{\mathfrak{m}}'_n$ *for all* $n = 0, 1, \cdots$*;*
5. $\overline{\mathfrak{m}}_n = \overline{\mathfrak{m}}'_n, n = 1, 2, 4, 6.$[20]

Proof. We just need to prove Condition 5 as a sufficient condition for conditions 1-4. It is suffice to show that, the 1st, 2nd, 4th and 6th mean-average moments uniquely determine $\{\underline{\mu}, \overline{\mu}, \underline{\nu}, \overline{\nu}\}$.

For Gaussian normal seed variable with rectangle location-scale uncertainty set Δ, we can readily verify that $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}'_1, \overline{\mathfrak{m}}_2 = \overline{\mathfrak{m}}'_2$ implies $\overline{\mathfrak{m}}_3 = \overline{\mathfrak{m}}'_3$, and if further that $\overline{\mathfrak{m}}_4 = \overline{\mathfrak{m}}'_4$, then $\overline{\mathfrak{m}}_5 = \overline{\mathfrak{m}}'_5$; that is, the 3rd and 5th moments are redundant given the 1st, 2nd and 4th moments.

From equation (2.18), we can compute the first six moments. Denoted by $x = \frac{\underline{\nu}+\overline{\nu}}{2}$ and $y = \underline{\nu}\overline{\nu}$. We have:[21]

$$\begin{aligned}
\overline{\mathfrak{m}}_1 &= \frac{\underline{\mu}+\overline{\mu}}{2} \\
\overline{\mathfrak{m}}_2 &= \frac{4\overline{\mathfrak{m}}_1^2}{3} - \frac{\underline{\mu}\overline{\mu}}{3} + x \\
\overline{\mathfrak{m}}_3 &= 3\overline{\mathfrak{m}}_1\overline{\mathfrak{m}}_2 - 2\overline{\mathfrak{m}}_1^3 \\
\overline{\mathfrak{m}}_4 &= C_4 + B_4 x - \frac{1}{5}x^2 - y \\
\overline{\mathfrak{m}}_5 &= 5\overline{\mathfrak{m}}_1\overline{\mathfrak{m}}_4 - 20\overline{\mathfrak{m}}_1^3\overline{\mathfrak{m}}_2 + 16\overline{\mathfrak{m}}_1^5 \\
\overline{\mathfrak{m}}_6 &= C_6 + B_6 x + A_6 x^2 - \frac{48}{7}x^3 - 15\overline{\mathfrak{m}}_2 y
\end{aligned}$$

where

$$\begin{aligned}
C_4 &= \frac{1}{5}\left(3\overline{\mathfrak{m}}_2 + 2\overline{\mathfrak{m}}_1^2\right)^2 - 4\overline{\mathfrak{m}}_1^4 \\
B_4 &= \frac{12}{5}\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right) \\
C_6 &= \frac{1}{7}\left(3\overline{\mathfrak{m}}_2 + 4\overline{\mathfrak{m}}_1^2\right)^3 - 48\overline{\mathfrak{m}}_1^4\overline{\mathfrak{m}}_2 \\
B_6 &= \frac{36}{7}\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right)\left(3\overline{\mathfrak{m}}_2 + 4\overline{\mathfrak{m}}_1^2\right) \\
A_6 &= \frac{144}{7}\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right) - 3\overline{\mathfrak{m}}_2
\end{aligned}$$

Hence the third and fifth moments ($\overline{\mathfrak{m}}_3$ and $\overline{\mathfrak{m}}_5$) are redundant.

We can recover all model coefficients $\{\underline{\mu}, \overline{\mu}, \underline{\nu}, \overline{\nu}\}$ from $\{\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4, \overline{\mathfrak{m}}_6\}$ by solving the above non-linear equations system. We follow a three-step procedure:

1. Eliminate y from $\overline{\mathfrak{m}}_4$ and $\overline{\mathfrak{m}}_6$ to obtain a cubic equation for x:

$$\left(x - \overline{\mathfrak{m}}_2 + \overline{\mathfrak{m}}_1^2\right)^3 = \frac{7}{48}\left(15\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_4 - \overline{\mathfrak{m}}_6 - 30\overline{\mathfrak{m}}_2^3 + 16\overline{\mathfrak{m}}_1^6\right) \tag{2.19}$$

[20] As it turns out, for Gaussian normal seed variable with rectangle location-scale uncertainty set Δ, $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}'_1, \overline{\mathfrak{m}}_2 = \overline{\mathfrak{m}}'_2$ implies $\overline{\mathfrak{m}}_3 = \overline{\mathfrak{m}}'_3$, and if further $\overline{\mathfrak{m}}_4 = \overline{\mathfrak{m}}'_4$, implies $\overline{\mathfrak{m}}_5 = \overline{\mathfrak{m}}'_5$; that is, the 3rd and 5th moments in the presence of the first four moments are redundant.

[21] Mathematical derivations of these are available upon request.

which has a unique solution given by

$$x = \frac{\underline{\nu} + \overline{\nu}}{2} = \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - C \tag{2.20}$$

in which $C \triangleq \sqrt[3]{\frac{7}{48}\left(\overline{\mathfrak{m}}_6 - 15\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_4 + 30\overline{\mathfrak{m}}_2^3 - 16\overline{\mathfrak{m}}_1^6\right)}$.

2. Substitute $x = \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - C$ into the equations for $\overline{\mathfrak{m}}_2$ and $\overline{\mathfrak{m}}_4$ respectively to solve for $\underline{\mu}\overline{\mu}$ and $y = \underline{\nu}\overline{\nu}$. These yield

$$\begin{aligned} \underline{\mu}\overline{\mu} &= \overline{\mathfrak{m}}_1^2 - 3C && (2.21)\\ \underline{\nu}\overline{\nu} &= 4\overline{\mathfrak{m}}_2^2 - 2\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_1^2 - \overline{\mathfrak{m}}_1^4 - \overline{\mathfrak{m}}_4 - 2\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right)C - \frac{1}{5}C^2 && (2.22) \end{aligned}$$

3. Combining these expressions for $\underline{\mu}\overline{\mu}$ and $\underline{\nu}\overline{\nu}$, together with $\underline{\mu}+\overline{\mu} = 2\overline{\mathfrak{m}}_1$ and $\underline{\nu}+\overline{\nu} = 2\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - C\right)$, enable us to solve for $\left\{\underline{\mu}, \overline{\mu}, \underline{\nu}, \overline{\nu}\right\}$:

$$(\frac{\overline{\mu} - \underline{\mu}}{2})^2 = \frac{1}{4}[(\underline{\mu} + \overline{\mu})^2 - 4\underline{\mu}\overline{\mu}] = \overline{\mathfrak{m}}_1^2 - (\overline{\mathfrak{m}}_1^2 - 3C) = 3C$$

hence $C \geq 0$,and

$$\begin{aligned} \underline{\mu} &= \overline{\mathfrak{m}}_1 - \sqrt{3C}, \overline{\mu} = \overline{\mathfrak{m}}_1 + \sqrt{3C} && (2.23)\\ \underline{\nu} &= \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - C - \sqrt{\frac{6}{5}C^2 + \overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4} && (2.24)\\ \overline{\nu} &= \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - C + \sqrt{\frac{6}{5}C^2 + \overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4} && (2.25) \end{aligned}$$

The solution is well-posed, namely $\underline{\mu} \leq \overline{\mu}$ and $0 \leq \underline{\nu} \leq \overline{\nu}$, so long as $C \in \left[0, \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right]$ and

$$\frac{1}{5}C^2 + 2\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right)C \leq 4\overline{\mathfrak{m}}_2^2 - 2\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_1^2 - \overline{\mathfrak{m}}_1^4 - \overline{\mathfrak{m}}_4. \tag{2.26}$$

It is important to observe that C captures the dispersion in location coefficients ($\overline{\mu} - \underline{\mu} = 2\sqrt{3C}$). When

$$\overline{\mathfrak{m}}_6 = 15\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_4 - 30\overline{\mathfrak{m}}_2^3 + 16\overline{\mathfrak{m}}_1^6 \tag{2.27}$$

$C = 0$, that is, there exists no uncertainty in location coefficient, and $\overline{\mu} = \underline{\mu} = \overline{\mathfrak{m}}_1$. In this case, the revealed volatility boundaries

$$\underline{\nu} = \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - \sqrt{\overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4}, \overline{\nu} = \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 + \sqrt{\overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4} \tag{2.28}$$

are non-negative as long as $\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 \geq 0$ and $\overline{\mathfrak{m}}_4 \in \left[3\overline{\mathfrak{m}}_2^2 - 2\overline{\mathfrak{m}}_1^4, 4\overline{\mathfrak{m}}_2^2 - 2\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_1^2 - \overline{\mathfrak{m}}_1^4\right]$. Particularly, when $\overline{\mathfrak{m}}_4 = 3\overline{\mathfrak{m}}_2^2 - 2\overline{\mathfrak{m}}_1^4$, the volatility uncertainty disappears with $\underline{\nu} = \overline{\nu} = \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2$.

In summary, with $\{\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4, \overline{\mathfrak{m}}_6\}$, we can recover the range $\left[\underline{\mu}, \overline{\mu}\right] \times \left[\underline{\nu}, \overline{\nu}\right]$ along with the characteristic function $\overline{\Theta}(x)$ and all other order moments $\{\overline{\mathfrak{m}}_n\}_{n=1}^{\infty}$. ■

Remark 5 *Condition 2.16 constitutes the necessary and sufficient condition for stochastic dominance among all risk averse EUUP investors with constant RRA.*

Remark 6 *For the case of rectangle support* $\Delta = \left[\underline{\mu}, \overline{\mu}\right] \times \left[\underline{\sigma}, \overline{\sigma}\right]$*, as asserted in the last statement of Theorem 3, some weaker conditions than Condition 2 are sufficient for the validity of inequality 2.16. Precisely, we show that, it is sufficient to check the 1st, 2nd, 4th, and 6th mean-average moments* *before reaching a consensus on the validity of inequality 2.16. As illustrated in Lemma 1, with equalities $\overline{\mathfrak{m}}_k = \overline{\mathfrak{m}}'_k$ for $k = 1, 2, 4, 6$, we can fully recover the rectangle set $\Delta = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$, and conclude that the two characteristic functions must coincide to each other with $\Delta = \Delta'$.*[22] *This yields an violation of non-degeneracy $\Delta \neq \Delta'$. In other words, in order to rank two different rectangle sets, $\Delta \neq \Delta'$, it is necessary and sufficient to check the validity of conditions 1 and 2 up to $N = 3$.*

Remark 7 *Interestingly, to be fully consistent with unanimous uncertainty assessment among EUUP investors, conditions 1 and 2 in Theorem 3 constitute an partial order that displays mean-preserving-spread risk aversion (MPS-RA) in the sense of Ma (2011) and Boyle and Ma (2013), a notion that is weaker than that of Rothchild and Stiglitz (1970).*

Remark 8 *The impossibility theorems are proved under the assumption that there are no distortions on the subjective probabilities on Δ as in Izhakian (2017) and (2020). Accordingly, our impossibility theorem applies for the general class of EUUP with distortions. Presumably, if investors in a smaller group can not reach an agreement on the assessment of uncertainty, and cannot find a quantitative uncertainty measure even when unanimous agreement were reached, then there is no hope to reach such agreement with a quantitative measure among a bigger community with richer class of utility functions.*

Although we show that it is impossible to identify a single-value quantitative uncertainty measure, we find the mean-average characteristic function $\overline{\Theta}_r(x)$ an useful measure never contradict to the order of rank between two arbitrary sets whenever applies. More specifically, inequality 2.16 must hold whenever Δ' is ranked more uncertain than Δ. When the location-scale family are restricted to rectangle Δ, the assessment of uncertainty reduces to four different criteria among the first six (except for the third and the fifth) mean-average moments of the distribution $\overline{\Phi}_r(x)$ or characteristic function $\overline{\Theta}_r(x)$. This is true at least when the risk seed ε is Gaussian normal.

In summary, we conclude that Theorem 2 can be extended to set Δ with a continuum range of support.

2.4 Uncertainty Assessment by Mean-Variance Investors

Mean-variance (MV) utility functions of Markowitz (1954) do not belong to SEU nor fall into EUUP class. In the context of risk and uncertainty, we consider MV utility functions by assuming Savage's subjective belief and the second-layer uniform belief as in EUUP, respectively. Intuitively, when utility functions are mean-variance, one may expect "variance" or "standard deviation" as quantitative uncertainty measure. However, under either model specification on subjective belief, it is questionable whether unanimous agreement on uncertainty assessment can be achieved under MV utility. Furthermore, it is questionable whether variance or standard deviation is sensible measure of uncertainty even for the extreme circumstance when unanimous agreement on uncertainty assessment can be reached.

2.4.1 MV class under Savage beliefs

The subjective mean-variance (MV) utility functions under some subjective belief $Q \in \mathcal{M}(\Delta)$ is defined as

$$V(r) = \mathbb{E}_{Q \times P}\left[r - \gamma\left(r - \mathbb{E}_{Q \times P}[r]\right)^2\right] \tag{2.29}$$

where $\gamma \geq 0$ is the risk aversion coefficient. Investors may have different risk aversion coefficient γ and different subjective belief Q.

[22] We prove in Lemma 1 that, $\overline{\mathfrak{m}}_3$ is a redundant statistics as it can be fully determined by the first and second mean-average moments; and that $\overline{\mathfrak{m}}_5$ is also redundant statistics as it can be fully determined by the first, the second and the fourth mean-average moments.

For r and r' with location-scale coefficient set Δ and Δ', respectively, the condition for an unanimous rank between r and r' under MV utility functions is $\forall Q \in \mathcal{M}(\Delta), Q' \in \mathcal{M}(\Delta')$ and $\forall \gamma \geq 0$;

$$\mathbb{E}_{Q\times P}\left[r-\gamma\left(r-\mathbb{E}_{Q\times P}\left[r\right]\right)^2\right] \geq \mathbb{E}_{Q'\times P}\left[r'-\gamma\left(r'-\mathbb{E}_{Q'\times P}\left[r'\right]\right)^2\right]$$

or, equivalently,

$$\mathbb{E}_Q\left[\mu\right]-\mathbb{E}_{Q'}\left[\mu\right] \geq \gamma\left(var_Q\left(\mu\right)+\mathbb{E}_Q\left[\sigma^2\right]-var_{Q'}\left(\mu\right)-\mathbb{E}_{Q'}\left[\sigma^2\right]\right) \tag{2.30}$$

When $\gamma \to 0_+$ we have $\mathbb{E}_Q\left[\mu\right] \geq \mathbb{E}_{Q'}\left[\mu\right]$ for all Q and Q', which implies,

$$\inf\left\{\mu : (\mu,\sigma) \in \Delta\right\} \geq \sup\left\{\mu : (\mu,\sigma) \in \Delta'\right\}. \tag{2.31}$$

Condition (2.31) is a partial order on location uncertainty.

The opposite inequality is obtained from $-r \succeq -r'$; that is,

$$\sup\left\{\mu : (\mu,\sigma) \in \Delta\right\} \leq \inf\left\{\mu : (\mu,\sigma) \in \Delta'\right\} \tag{2.32}$$

Combining conditions (2.31) and (2.32) yields that Δ and Δ' must share a common singleton support on μ.

Moreover, condition (2.30) for any $\gamma \geq 0$ and $\mathbb{E}_Q\left[\mu\right] = \mathbb{E}_{Q'}\left[\mu\right]$ for all Q and Q', implies

$$var_Q\left(\mu\right)+\mathbb{E}_Q\left[\sigma^2\right] \leq var_{Q'}\left(\mu\right)+\mathbb{E}_{Q'}\left[\sigma^2\right] \tag{2.33}$$

Suppose Q and Q' assign unit mass at some $(\mu,\sigma) \in \Delta$ and $(\mu',\sigma') \in \Delta$, respectively, we obtain

$$\sup\left\{\sigma^2 : (\mu,\sigma) \in \Delta\right\} \leq \inf\left\{\sigma^2 : (\mu,\sigma) \in \Delta'\right\}. \tag{2.34}$$

Condition (2.34) is a partial order on uncertainty regarding volatility or scale of Δ.

In analogy to discussion for SEU class in the previous subsection, unanimous rank between two arbitrary sets Δ and Δ' among MV investors can not be always reached. Even when unanimous choices between Δ and Δ' are reached among MV investors, as is the case when Δ and Δ' share a common singleton support on location coefficient and when (2.34) is satisfied, there is no quantifiable measure that would yield a single-value measure of uncertainty consistent with conditions (2.31), (2.32) and (2.34). In other words, a quantitative measure $\rho : \Delta \to \mathbb{R}$ on the degree of uncertainty does not exist among MV investors with Savage's beliefs.

2.4.2 MV class under uniform second-layer belief

We consider MV utility function under second-layer belief of the form

$$V\left(r\right) = \mathbb{E}_{\xi\times\mathbb{P}}\left[r-\gamma\left(r-\mathbb{E}_{\xi\times\mathbb{P}}\left[r\right]\right)^2\right], \gamma \geq 0$$

in which ξ is uniform on $\mathcal{M}(\Delta)$. With

$$\begin{aligned}
\mathbb{E}_{\xi\times\mathbb{P}}\left[r\right] &= \frac{1}{|\Delta|}\iint\limits_{\Delta} \mu d\mu d\sigma = \overline{\mathfrak{m}}_1 \\
\mathbb{E}_{\xi\times\mathbb{P}}\left[r^2\right] &= \frac{1}{|\Delta|}\iint\limits_{\Delta} \left(\mu^2+\sigma^2\right) d\mu d\sigma = \overline{\mathfrak{m}}_2
\end{aligned}$$

we obtain

$$V\left(r\right)=\overline{\mathfrak{m}}_{1}+\gamma\overline{\mathfrak{m}}_{1}^{2}-\gamma\overline{\mathfrak{m}}_{2}. \tag{2.35}$$

For r and r' with location-scale coefficient set Δ and Δ', the condition for an unanimous rank between r and r' becomes

$$\begin{aligned}\overline{\mathfrak{m}}_{1}+\gamma\overline{\mathfrak{m}}_{1}^{2}-\gamma\overline{\mathfrak{m}}_{2} &\geq \overline{\mathfrak{m}}_{1}'+\gamma\overline{\mathfrak{m}}_{1}'^{2}-\gamma\overline{\mathfrak{m}}_{2}' &(2.36)\\ -\overline{\mathfrak{m}}_{1}+\gamma\overline{\mathfrak{m}}_{1}^{2}-\gamma\overline{\mathfrak{m}}_{2} &\geq -\overline{\mathfrak{m}}_{1}'+\gamma\overline{\mathfrak{m}}_{1}'^{2}-\gamma\overline{\mathfrak{m}}_{2}' &(2.37)\end{aligned}$$

for all $\gamma\geq 0$.

When $\gamma\rightarrow 0_{+}$ we have $\overline{\mathfrak{m}}_{1}=\overline{\mathfrak{m}}_{1}'$. Substitute this back into inequality Eq (2.36), we have $\overline{\mathfrak{m}}_{2}\leq\overline{\mathfrak{m}}_{2}'$. So, in this case, unanimous agreement on uncertainty is reached if and only if

$$\overline{\mathfrak{m}}_{1}=\overline{\mathfrak{m}}_{1}' \text{ and } \overline{\mathfrak{m}}_{2}\leq\overline{\mathfrak{m}}_{2}'$$

In this case, even though the mean-average second moment $\overline{\mathfrak{m}}_{2}$ as an quantitative uncertainty measure does not contradict to unanimous rank made by MV investors, we must keep in mind that this works only under the extreme circumstance that the two mean-average first moments coincide and

$$\overline{\mathfrak{m}}_{2}=\frac{1}{|\Delta|}\iint\limits_{\Delta}\left(\mu^{2}+\sigma^{2}\right)d\mu d\sigma \tag{2.38}$$

as uncertainty measure does not fully capture the magnitude and dispersion in volatility, taking the mean-average first moment as given.

If we further restrict short selling, from inequality Eq (2.36) we obtain $\overline{\mathfrak{m}}_{1}\geq\overline{\mathfrak{m}}_{1}'$ and

$$\overline{\mathfrak{m}}_{1}-\overline{\mathfrak{m}}_{1}'\geq\gamma(\overline{\mathfrak{m}}_{2}-\overline{\mathfrak{m}}_{1}^{2}-\overline{\mathfrak{m}}_{2}'+\overline{\mathfrak{m}}_{1}'^{2}) \tag{2.39}$$

for all $\gamma\geq 0$, which implies $\overline{\mathfrak{m}}_{2}-\overline{\mathfrak{m}}_{1}^{2}\leq\overline{\mathfrak{m}}_{2}'-\overline{\mathfrak{m}}_{1}'^{2}$. In summary, the assessment of the uncertainty between Δ and Δ' must jointly involve two criteria on the first and the second mean-average moments■

1. $\frac{1}{|\Delta|}\iint\limits_{\Delta}\mu d\mu d\sigma=\overline{\mathfrak{m}}_{1}\geq\overline{\mathfrak{m}}_{1}'=\frac{1}{|\Delta'|}\iint\limits_{\Delta'}\mu d\mu d\sigma$;

2. $\frac{1}{|\Delta|}\iint\limits_{\Delta}\left(\mu-\overline{\mathfrak{m}}_{1}\right)^{2}d\mu d\sigma+\frac{1}{|\Delta|}\iint\limits_{\Delta}\sigma^{2}d\mu d\sigma\leq\frac{1}{|\Delta'|}\iint\limits_{\Delta'}\left(\mu-\overline{\mathfrak{m}}_{1}'\right)^{2}d\mu d\sigma+\frac{1}{|\Delta'|}\iint\limits_{\Delta'}\sigma^{2}d\mu d\sigma$

Condition 1 and the first term in condition 2 are on the magnitude and dispersion in location coefficient (μ), respectively, while the second term in condition 2 measures the magnitude in scale coefficient (σ^{2}). Hence, a single-value uncertainty measure can not be obtained even when investors only express preferences over mean and variance of asset returns in the context with risk and uncertainty.

2.5 Uncertainty assessment with short-sale constraints

We obtain the impossibility theorems without short-sale constraints through symmetric treatment on short sales. More specifically, we set $r'=-r$ and treat $-r$ as an instantaneous return for an different investment vehicle. Such treatment is particularly convenient when margins on short sales and rebate rates are taken into consideration, and investors face different short selling constraints. We define a weaker notion of uncertainty measure by dropping Condition (b) in Definition 1 and by allowing the perception of uncertainty on r from investors with long positions only:

Definition 2 *Taking as given risk sources ε and ε', let r and r' be asset returns that are respectively driven by ε and ε' with uncertainty respectively governed by Δ and Δ'. Δ' is said to be more uncertain than Δ if $r \succcurlyeq_i r'$ for all $i \in \mathcal{U}$, which hold with strict preference for some i. Moreover, a function $\rho : \Delta \to \mathbb{R}$ is an* uncertainty measure *if*

$$\rho(\Delta') > \rho(\Delta) \text{ whenever } \Delta' \text{ is more uncertain than } \Delta. \tag{2.40}$$

It is straightforward to show that, without symmetric consideration of short sales constraints, the impossibility theorems remain valid. However, for both FSD and SSD in Theorem 1, condition (i) in obtaining unanimous ranking is weakened as condition (i)':

$$\inf\{\mu : (\mu, \sigma) \in \Delta\} \geq \sup\{\mu : (\mu, \sigma) \in \Delta'\} \tag{2.41}$$

with condition (ii) and (ii)' remain unchanged.

Similarly, we can obtain impossibility theorem for EUUP class, Propositions 2 and 3 with short-sale constraints remain by dropping the set of restrictions imposed on $\overline{\Phi}_{-r}$ and $\overline{\Theta}_{-r}$. Even though the nondegeneracy argument in Proposition 2 no longer imply a contradiction, we do obtain some concrete conditions for uncertainty assessment. Accordingly, statements in Theorems 2 and 3 remain valid. The details of the mathematical proofs are thus omitted.

3 Empirical Measure of Uncertainty

In this section, we first propose a way to assess uncertainty of location-scale family of normal distribution using the first, second, forth and sixth moments. Then we discuss problems with the composite uncertainty measure proposed by Izhakian (2020) and empirically implemented by Brenner and Izhakian (2018), and propose a modified composite uncertainty measure that is consistent and unbiased.

3.1 Assessment of Uncertainty using Moments

Theorem 3 provide a natural venue towards uncertainty assessment through moments, particularly when the model uncertainty be characterized by an rectangle set Δ. The precise connections between uncertainty assessment and moments are through the mean-average characteristic function $\overline{\Theta}$. In order to rank two rectangle sets Δ and Δ', say whether one is more uncertain than the other, it is necessary to check if the corresponding mean-average characteristic functions $\overline{\Theta}_r(x)$ and $\overline{\Theta}_{r'}(x)$ cross somewhere along the x-axis, which includes $x = 0$ at which the two curves always meet as shown in Figure 2. No ranking can be made between Δ and Δ' concerning their degree of uncertainty when the two curves cross.

In addition, Lemma 1 shows that the rank between two arbitrary rectangle sets Δ and Δ' can be simplified into the assessment of moments up to the sixth order, which follows a Lexicographic procedure of ordering:

1. $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}'_1$? If no, accept H_1 : no assessment can be made;[23] if yes, goes to step 2:
2. $\overline{\mathfrak{m}}_2 < \overline{\mathfrak{m}}'_2$ (or $\overline{\mathfrak{m}}'_2 < \overline{\mathfrak{m}}_2$)? If yes, do not reject (i.e., accept) H_0 : Δ' (Δ) is more uncertain than Δ (Δ'); if no, goes to step 3:
3. $\overline{\mathfrak{m}}_3 = \overline{\mathfrak{m}}'_3$? If no, accept H_1 : no assessment can be made; if yes, goes to step 4:
4. $\overline{\mathfrak{m}}_4 < \overline{\mathfrak{m}}'_4$ (or $\overline{\mathfrak{m}}'_4 < \overline{\mathfrak{m}}_4$)? If yes, do not reject (i.e., accept) H_0 : Δ' (Δ) is more uncertain than Δ (Δ'); if no, goes to step 5:

[23] More precisely, we shall rank $r \succ r'$ & $-r' \succ -r$ if $\overline{\mathfrak{m}}_1 > \overline{\mathfrak{m}}'_1$; and $r' \succ r$ & $-r \succ -r'$.

5. $\overline{\mathfrak{m}}_5 = \overline{\mathfrak{m}}_5'$? If no, accept H_1 : no assessment can be made; if yes, goes to step 6:

6. $\overline{\mathfrak{m}}_6 < \overline{\mathfrak{m}}_6'$ (or $\overline{\mathfrak{m}}_6' < \overline{\mathfrak{m}}_6$)? If yes, do not reject (i.e., accept) $H_0 : \Delta'$ (Δ) is more uncertain than Δ (Δ'); or else, if $\overline{\mathfrak{m}}_6 = \overline{\mathfrak{m}}_6'$, accept $H_0 : \Delta$ and Δ' are equally uncertain.

So, even though it is impossible to obtain a single composite measure of uncertainty, the assessments summarized by $\{\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4, \overline{\mathfrak{m}}_6\}$ are sufficient to rank the uncertainty of Δ whenever it applies.

Lemma 1 also provides a useful channel to connect information summarized by $\{\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4, \overline{\mathfrak{m}}_6\}$ to compare uncertainty on the location and scale coefficients. With

$$C = \sqrt[3]{\frac{7}{48}\left(\overline{\mathfrak{m}}_6 - 15\overline{\mathfrak{m}}_2\overline{\mathfrak{m}}_4 + 30\overline{\mathfrak{m}}_2^3 - 16\overline{\mathfrak{m}}_1^6\right)} \tag{3.1}$$

from equations (2.23) to (2.25) we obtain

$$\frac{\overline{\mu} + \underline{\mu}}{2} = \overline{\mathfrak{m}}_1 \tag{3.2a}$$
$$\frac{\overline{\mu} - \underline{\mu}}{2} = \sqrt{3C} \tag{3.2b}$$
$$\frac{\overline{\nu} + \underline{\nu}}{2} = \overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2 - C \tag{3.2c}$$
$$\frac{\overline{\nu} - \underline{\nu}}{2} = \sqrt{\frac{6}{5}C^2 + \overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4} \tag{3.2d}$$

Thus, the mean returns ($\overline{\mathfrak{m}}_1$) measures the average of location ($\frac{\overline{\mu}+\underline{\mu}}{2}$). Taking the mean returns ($\overline{\mathfrak{m}}_1$) and the variance of the returns ($\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2$) as given, C measures the deviation of the variance ($\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2$) from the average scale ($\frac{\overline{\nu}+\underline{\nu}}{2}$). $\sqrt{3C}$ gives the dispersion in location ($\frac{\overline{\mu}-\underline{\mu}}{2}$), and the dispersion in the scale ($\frac{\overline{\nu}-\underline{\nu}}{2}$) is given by $\sqrt{\frac{6}{5}C^2 + \overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4}$. The term ($\overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4$) in the square root interpreted as the dispersion in scale in the absence of location uncertainty (i.e., $C = 0$). With

$$\overline{\mathfrak{m}}_4 - 3\overline{\mathfrak{m}}_2^2 + 2\overline{\mathfrak{m}}_1^4 = \left(\overline{\mathfrak{m}}_4 - \overline{\mathfrak{m}}_2^2\right) - 2\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right)^2 - 4\overline{\mathfrak{m}}_1^2\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right)$$

The dispersion in the scale coefficient in the absence of location uncertainty increases in $\left(\overline{\mathfrak{m}}_4 - \overline{\mathfrak{m}}_2^2\right)$, and decreases (quadratically) in $\left(\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1^2\right)$, the relationship between these moments and the dispersion in location coefficient ($\sqrt{3C}$) are just the opposite, Hence, in general, there is no monotone relationship between these moments (including the conventional measure of risk, the variance) and dispersions in the location and the scale coefficients. Thus, it is not necessarily true that the higher the risk the more is the uncertainty.

We use the daily returns of S&P 500 stock market index in the U.S. market, and CSI 300 stock market index to estimate the first, second, forth and sixth moments using expanding window estimation, then using equations (3.2a) to compute the ranges of location and scale coefficients of the family of normal distribution.

Figure 4 shows the 1st, 2nd, 4th and 6th moments of daily returns of the U.S. stock market index S&P 500. The 2nd moment (risk) increases over time, but the 4th and 6th moments increase one time in 1987 due to the "Black Monday" crash. Figure 5 shows the 1987 "Black Monday" crash significantly increased the ranges of the location and scale, the uncertainty, and the impact remains till 2023.

Figure 6 shows the 1st, 2nd, 4th and 6th moments of daily returns of China stock market index CSI 300. All of the four moments increase in 2008 when the four trillion stimulus plan was implemented, then decrease steadily till 2014 when the stock market experienced another boom-and-bust cycle. Figure 7 shows the 2008 stimulus plan has a big impact on the ranges of both location and scale, but the

2014-2015 boom-and-bust does not impact the range of location but mildly increases the range of scale, or uncertainty. Once the uncertainty in the stock market increases, it remains high for long time.

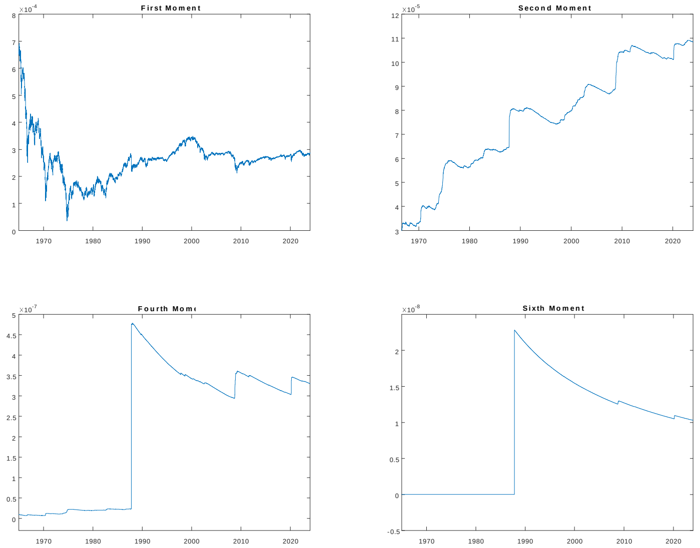

Figure 4. Moments of Daily Returns of the U.S. Stock Market Index S&P 500 from 1965/01/01 to 2023/12/29.

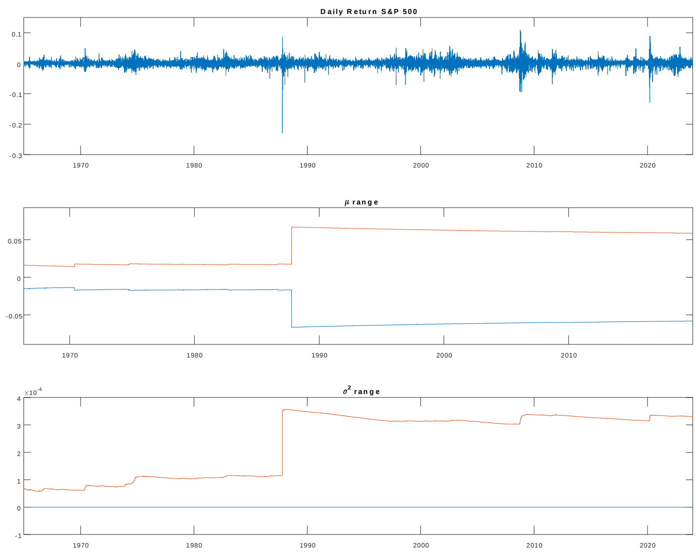

Figure 5. Time Series of Daily Return and Ranges of Location and Scale of the U.S. Stock Market Index S&P 500 from 1965/01/01 to 2023/12/29.

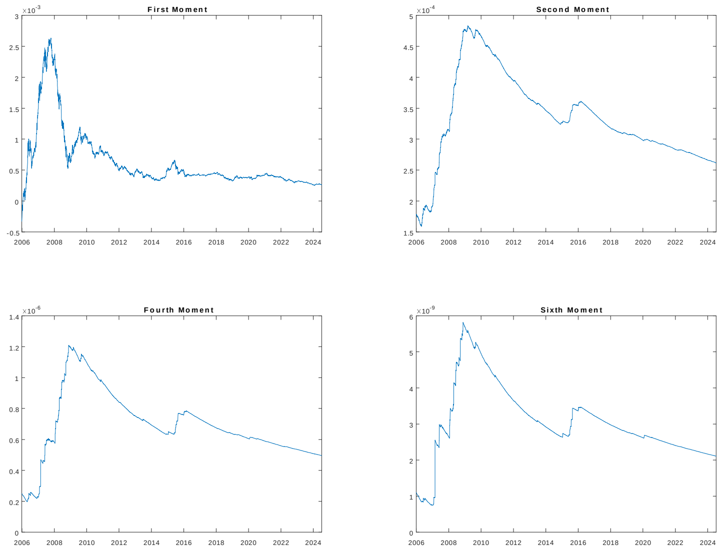

Figure 6. Moments of Daily Returns of China Stock Market Index CSI 300

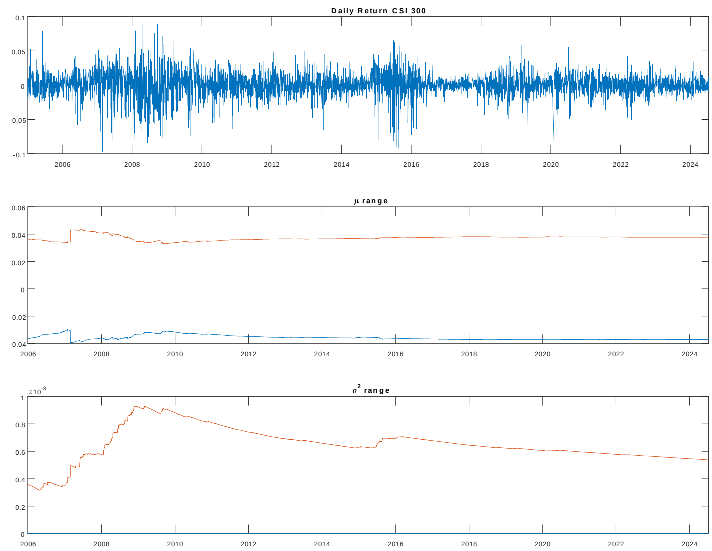

Figure 7. Time Series of Daily Return and Ranges of Location and Scale of China Stock Market Index CSI 300

3.2 A Composite Uncertainty Measure

Lemma 1 shows that the rank between two arbitrary rectangle sets Δ and Δ' can be simplified into the assessment of four moments $\{\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4, \overline{\mathfrak{m}}_6\}$. As the first moment corresponds to the expected return, the second moments corresponds to the conventional measure of risk, the fourth moments corresponds to the conventional measure of tail risk, we lack insight of the sixth moments. Izhakian (2020) proposes to use "the expected volatility of probabilities" as a composite measure of uncertainty. Intuitively, this measure gauges the average distance of probabilities or the "average dispersion" of $(\mu\,\sigma)$ in Δ measures one aspect of uncertainty. In this subsection, we first define a consistent composite measure of uncertainty on Δ with continuum and finite support, and show that measure proposed by Izhakian (2020) is null when the number of supports of Δ goes to infinity. Then, we propose a simple and consistent method to estimate the composite uncertainty measure from data on asset returns, and show that the empirical implementation proposed by Izhakian (2020) overestimate the uncertainty when the range of Δ is large.

Suppose asset returns are governed by a family of normal distributions:

$$\varphi_r(\mu,\sigma;x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) \text{ for } (\mu,\sigma)\in\Delta$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ is the p.d.f. of standard Gaussian distribution.

Let $\mathcal{P} = \mathcal{M}(\Delta)$ be the set of all probability measures on Δ. For each $q\in\mathcal{P}$ we denote

$$\varphi_r(q,x) = \int_\Delta \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)dq(\mu,\sigma)$$

as the "expected p.d.f." with respect to q for asset return r. Moreover, we define the mean and variance of the "expected p.d.f" over the state-space of $\mathcal{P}$ with respect to the uniform probability measure ξ on $\mathcal{P}$.

$$\begin{aligned}
\mathbb{E}[\varphi_r(x)] &= \int_{\mathcal{P}} \varphi_r(q,x)\,d\xi(q) \\
\mathbb{V}ar[\varphi_r(x)] &= \int_{\mathcal{P}} (\varphi_r(q,x) - \mathbb{E}[\varphi_r(x)])^2\,d\xi(q)
\end{aligned}$$

Then the mean of the variance of "expected p.d.f" with respect to the mean of the "expected p.d.f" measures the dispersion of the "expected p.d.f.", one of the important aspects of the uncertainty of return r, that is,

$$\Psi^2[r] = \int_{-\infty}^{\infty} \mathbb{E}[\varphi_r(x)]\,\mathbb{V}ar[\varphi_r(x)]\,dx$$

3.2.1 Composite Uncertainty Measure for Δ with Finite Support

When Δ is with finite support, that is, $\Delta = \{(\mu_i,\sigma_i)\}_{i=1}^n$,denote

$$\varphi_r^i \equiv \varphi_r(\mu_i,\sigma_i;x) = \frac{1}{\sigma_i}\phi\left(\frac{x-\mu_i}{\sigma_i}\right), i=1,\cdots,n$$

we have all the probability distributions on Δ form a (n-1)-dimension simplex in a n-dimension linear space,

$$\mathcal{P} = \left\{\mathbf{q}\in\mathbb{R}_+^n : \sum_{i=1}^n q_i = 1\right\}$$

and

$$\varphi_r(q,x) = \sum_i q_i\varphi_r^i(x) = \mathbf{q}\cdot\boldsymbol{\varphi}(x)$$

Furthermore, we show that the mean and variance of "expected p.d.f" are

$$\begin{aligned}
\mathbb{E}_n[\varphi_r(x)] &= \frac{\int \underset{\mathcal{P}}{\dots} \int \mathbf{q}\cdot\boldsymbol{\varphi}(x)\,d\mathbf{q}}{Vol(\mathcal{P})} = \frac{1}{n}\sum_{i=1}^n \varphi_i(x) \equiv \overline{\varphi}_n(x) \\
\mathbb{V}ar_n[\varphi_r(x)] &= \frac{\boldsymbol{\varphi}(x)^T\left(\int \underset{\mathcal{P}}{\dots}\int(\mathbf{q}-\overline{\mathbf{q}})(\mathbf{q}-\overline{\mathbf{q}})^T d\mathbf{q}\right)\boldsymbol{\varphi}(x)}{Vol(\mathcal{P})} \\
&= \frac{1}{n(n+1)}\sum_{i=1}^n(\varphi_i(x) - \overline{\varphi}_n(x))^2 \equiv \frac{1}{n+1}S_n^2(x)
\end{aligned}$$

where the second equations follow from the integration by parts. It is important to note that, the mean of the "expected p.d.f" ($\mathbb{E}_n [\varphi_r (x)]$) is same as $\overline{\varphi}_n (x)$, the mean of n normal p.d.f.s drawn from Δ with equal probability, but the variance of the "expected p.d.f." is $S_n^2 (x) / (n+1)$, where $S_n^2 (x)$ is defined as the variance of n normal p.d.f.s drawn from Δ with equal probability.

In the following Proposition 4 we show that the uncertainty measure with finite support $\Psi_n^2 [r]$ should be defined as

$$\Psi_n^2 [r] = \int_{-\infty}^{\infty} \overline{\varphi}_n (x) S_n^2 (x) dx = (n+1) \int_{-\infty}^{\infty} \mathbb{E}_n [\varphi_r (x)] \mathbb{V}ar_n [\varphi_r (x)] dx \tag{UNC1}$$

which is $n+1$ times the mean of the variance of "expected p.d.f". However, Izhakian (2020) defines the uncertainty measure with finite support as,

$$\tilde{\Psi}_n^2 [r] = \int_{-\infty}^{\infty} \mathbb{E}_n [\varphi_r (x)] \mathbb{V}ar_n [\varphi_r (x)] dx = \frac{\Psi_n^2 [r]}{n+1} \tag{UNC0}$$

Proposition 4 *When Δ is with finite support, the uncertainty measure $\Psi_n^2 [r]$ defined in (UNC1) has following properties:*

1. *When adding an extra support $(\mu_{n+1}, \sigma_{n+1})$ to the set Δ, we have*

$$\begin{aligned}
\overline{\varphi}_{n+1} &= \overline{\varphi}_n + \frac{1}{n+1} (\varphi_{n+1} - \overline{\varphi}_n) \\
S_{n+1}^2 &= \frac{n}{n+1} S_n^2 + \frac{n}{(n+1)^2} (\varphi_{n+1} - \overline{\varphi}_n)^2 \\
\Psi_{n+1}(r) &= \frac{n}{n+1} \Psi_n + \frac{n}{(n+1)^2} \int_{-\infty}^{\infty} \varphi_{n+1} (x) S_n^2 (x) dx \\
&\quad + \frac{n}{(n+1)^2} \int_{-\infty}^{\infty} \overline{\varphi}_n (x) (\varphi_{n+1} (x) - \overline{\varphi}_n (x))^2 dx \\
&\quad + \frac{n}{(n+1)^3} \int_{-\infty}^{\infty} (\varphi_{n+1} (x) - \overline{\varphi}_n (x))^3 dx.
\end{aligned}$$

2. *$\Psi_n^2 [r] = 0$ if and only if $n = 1$, which corresponds to case with no uncertainty and only risk (**??**), that is, (μ, σ) takes unique value of (μ_0, σ_0) in Δ.*

3. *For Δ with continuum range, we may divide Δ into n region to get n discrete samples and get an uncertainty measure of Δ by taking the limit (if exists) as $n \to +\infty$. If random sample of (μ, σ) on Δ is governed by a specific probability distribution q, then by Law of Large Number, as $n \to +\infty$, we get*

$$\Psi_{q,n}^2 [r] = \int_{-\infty}^{\infty} \overline{\varphi}_{q,n} (x) S_{q,n}^2 (x) dx \underset{n \to +\infty}{\overset{P}{\to}} \int_{-\infty}^{\infty} \mathbb{E}_q [\varphi_r (x)] \mathbb{V}ar_q [\varphi_r (x)] dx$$

   which is positive *unless q is with a unit mass on some $(\mu, \sigma) \in \Delta$.*

   For Example 2, when q is uniform on Δ, the limit becomes

$$\mathbb{E}_q [\varphi_r (x)] = \frac{\int_{\underline{\sigma}}^{\overline{\sigma}} \Phi \left(\frac{x - \overline{\mu}}{\sigma}, \frac{x - \underline{\mu}}{\sigma} \right) d\sigma}{(\overline{\sigma} - \underline{\sigma}) (\overline{\mu} - \underline{\mu})} \tag{3.3}$$

$$\mathbb{V}ar_q [\varphi_r (x)] = \frac{\int_{\frac{\underline{\sigma}}{\sqrt{2}}}^{\frac{\overline{\sigma}}{\sqrt{2}}} \Phi \left(\frac{x - \overline{\mu}}{\sigma}, \frac{x - \underline{\mu}}{\sigma} \right) \frac{d\sigma}{\sigma}}{2\sqrt{\pi} (\overline{\sigma} - \underline{\sigma}) (\overline{\mu} - \underline{\mu})} - (\mathbb{E}_q [\varphi_r (x)])^2 \tag{3.4}$$

   where $\Phi (x, x') = \int_x^{x'} \phi(x) dx$.

4. When ξ on $\mathcal{P}$ governs the random sample of q, probability distribution of (μ, σ) on Δ, by Law of Large Number, as $n \to +\infty$.

$$\Psi_n^2[r] \underset{n\to+\infty}{\overset{P}{\to}} \int_{-\infty}^{\infty} \mathbb{E}[\varphi_r(x)] \mathbb{V}ar[\varphi_r(x)]\, dx = \Psi^2[r]$$

5. Izhakian's (2020) measure defined in (UNC0) [24] *converges to zero as $n \to +\infty$, that is, $\lim_{n\to\infty} \tilde{\Psi}_n^2[r] = 0$. Hence, Izhakian (2020) measure is not a consistent estimate of $\Psi^2[r]$ and is null when Δ is with infinite/continuum support.*

Proof. See Appendix. ■

3.2.2 Empirical Implementation of Uncertainty Measure for Δ with Finite Support

Given the estimates of $(\mu_i, \sigma_i)_{i=1}^n$, we can directly compute the uncertainty measure $\Psi_n^2[r]$ defined in (UNC1) as an explicit function of estimates of $(\mu_i, \sigma_i)_{i=1}^n$.

Proposition 5 *Assume asset returns r are governed by a family of normal distributions with $\{(\mu_i, \sigma_i)\}_{i=1}^n$, which is the finite support of Δ, then the uncertainty measure $\Psi_n^2[r]$ defined in (UNC1) can be computed as*

$$\Psi_n^2(r) = \frac{1}{n^3}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \frac{\phi_i(0)\phi_j(0)\phi_k(0)}{\phi_{i,j,k}(0)}\left[\frac{\phi_j(0)\phi_{i,j,k}(0)}{\phi_k(0)\phi_{i,j,j}(0)} - 1\right] \tag{EUNC1}$$

where

$$\begin{aligned}\phi_i(0) &= \frac{1}{\sigma_i\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\mu_i}{\sigma_i}\right)^2\right\}\\ \phi_{i,j,k}(0) &= \frac{1}{\sigma_{i,j,k}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\mu_{ijk}}{\sigma_{ijk}}\right)^2\right\}\end{aligned}$$

with σ_{ijk}^2 as the harmonic average of σ_i^2, σ_j^2 and σ_k^2, μ_{ijk} as the weighted average of μ_i, μ_j, μ_k,

$$\begin{aligned}\frac{1}{\sigma_{ijk}^2} &= \frac{1}{\sigma_i^2} + \frac{1}{\sigma_j^2} + \frac{1}{\sigma_k^2}\\ \mu_{ijk} &= \frac{\frac{\mu_i}{\sigma_i^2} + \frac{\mu_j}{\sigma_j^2} + \frac{\mu_k}{\sigma_k^2}}{\frac{1}{\sigma_i^2} + \frac{1}{\sigma_j^2} + \frac{1}{\sigma_k^2}}\end{aligned}$$

Proof. See Appendix A. ■

Izhakian (2020) proposes to empirically implement the uncertainty measure (UNC0) by estimating $\overline{\varphi}_n(x)$ by the sample mean and $S_n^2(x)$ by the variance of discretized normal PDFs with μ and σ estimated from the 5-min stock returns in a trading day, under the assumption that the stock return in each trading day follows a normal probability distribution with different μ and σ; that is,

$$\mathbb{E}_n\left[\widehat{\varphi_r(x_m)}\right] = \frac{1}{n}\sum_{i=1}^n \frac{1}{\sigma_i}\Phi\left(\frac{x_{m-1}-\mu_i}{\sigma_i}, \frac{x_m-\mu_i}{\sigma_i}\right) \tag{3.5}$$

$$\widehat{\mathbb{V}ar_n\left[\varphi(x_m)\right]} = \frac{1}{n}\sum_{i=1}^n \left[\frac{1}{\sigma_i}\Phi\left(\frac{x_{m-1}-\mu_i}{\sigma_i}, \frac{x_m-\mu_i}{\sigma_i}\right) - \mathbb{E}_n\left[\widehat{\varphi_r(x_m)}\right]\right]^2 \tag{3.6}$$

[24] This is the special case of Equation (8) in Izhakian (2020) when the support of Δ is finite, Equation (9) in Izhakian (2020) should be an integral instead of finite sum, as the domain of x is still continuous when the support of Δ is finite.

for $m = 0, ...M$ with $x_0 < x_1 < .... < x_M$ divide the range of daily stock returns within a month into M intervals with equal width w. And, the empirical uncertainty measure is computed as[25]

$$\widehat{\tilde{\Psi}_n^2 [r]} = \frac{1}{\sqrt{w(1-w)}} \sum_{m=1}^{M} \mathbb{E}_n \left[\widehat{\varphi (x_m)}\right] \cdot \widehat{\mathbb{V}ar_n \varphi (x_m)} \tag{EUNC0}$$

If we compare the (3.5) and (3.6) with its continuum counterpart (3.3) and (3.4), we find that (EUNC0) of Brenner and Izhakian (2018) was not appropriately adjusted by the product of range of μ and σ,that is, $(\overline{\sigma} - \underline{\sigma}) (\overline{\mu} - \underline{\mu})$, but by $\sqrt{w(1-w)}$. Hence this measure overestimates the uncertainty of stock returns with larger $(\overline{\sigma} - \underline{\sigma}) (\overline{\mu} - \underline{\mu})$, that is when the conventional measure of risk is higher, as $(\overline{\sigma} - \underline{\sigma}) (\overline{\mu} - \underline{\mu})$ is positively related to the sample standard deviation, the conventional estimate of risk.

which appropriately adjust the range of μ and σ, and is a consistent estimate of $\Psi_n^2 [r]$ that avoids the bias in Brenner and Izhakian (2018) empirical measure (EUNC0),

4 Uncertainty of Stock Market Indices

In this section, we first compare our uncertainty measure (UNC1) with that of Izhakian (2020) (UNC0) by simulation and estimate using data on the stock market indices in the U.S. and China. Then we examine the property of the stock uncertainty measure and moments estimated using data on the stock market indices.

We first estimate the means and standard deviations of 5-minute returns in each trading day of the U.S. stock market index S&P 500 and China stock market index CSI 300[26]. Figure 4 shows the histograms and scatter plots of pairs (μ_i, σ_i), means and standard deviations of daily returns of the S&P 500 and CSI 300. It is obvious that (μ_i, σ_i) are not uniformly distributed, but fan-shaped distributed within a circular sector defined by the maximum of σ and ranges of μ/σ. Comparing the top panels with bottom panels of Figure 8, it is clear that the range of (μ_i, σ_i) of the U.S. stock market index is larger but the dispersion is smaller than that of the China stock market index. The range of the fan-shaped area and dispersion within the range govern the uncertainty of stock return. The range can be measured by the maximum of standard deviations and the minimum and maximum of the ratio of means and standard deviations (μ/σ),it is interesting to examine when the stock uncertainty measure helps to measure the dispersion of (μ_i, σ_i) within the range.

[25] Izhakian (2020) equation (12)

[26] The sample period is from Jan. 1, 2008 to December 31, 2022. We obtain 3777 pairs of (μ_i, σ_i) for S&P 500 and 3587 pairs of (μ_i, σ_i) for CSI 300, as there is not sufficient 5-min returns on Janurary 8, 2016, and data in 2018 November and December is missing.

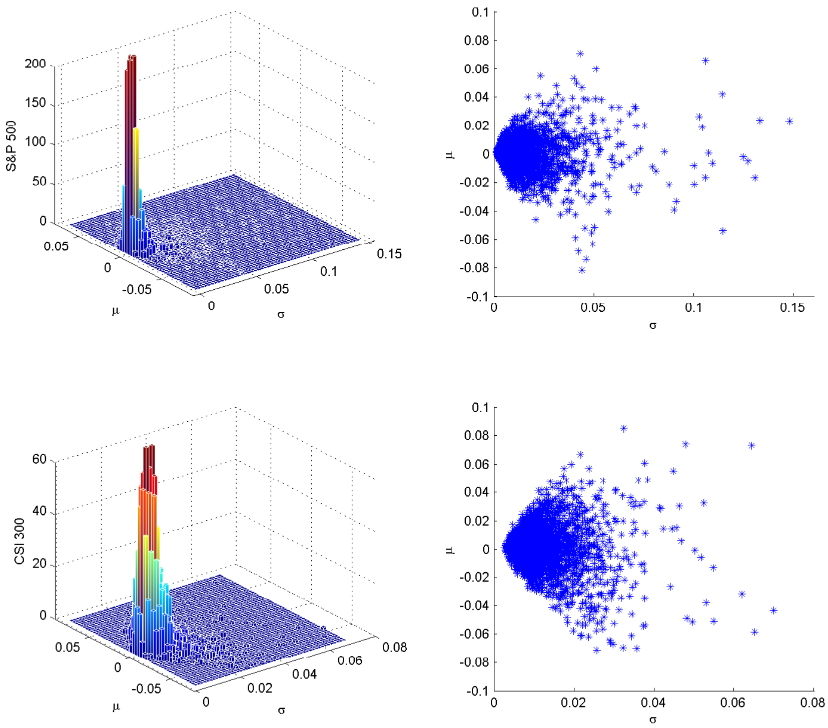

Figure 8. Histograms and Scatter Plots of Means and Standard Deviations of Daily Returns of Stock Market Indices in the U.S. and China.

4.1 Comparison of Stock Uncertainty Measures

We compare our uncertainty measure (UNC1) with that of Izhakian (2020) (UNC0) using the U.S. stock market index S&P 500.

4.1.1 Simulation Results

We use bootstrap to draw n pair of (μ_i, σ_i) from the estimated mean and standard deviation of daily returns of S&P 500, and compute the uncertainty measure (unc1) and Izhakian (2020) measure (unc0) for $n = 2, 5, 10, 20, 40, 60, 80, 200$.Figure 9 shows the scattered plot of $\{(\mu_i, \sigma_i)\}_{i=1}^n$ from one simulation with (UNC0) and (UNC1) reported in the title. It clearly shows that our uncertainty measure (unc1) converges to around 20, but Izhakian's measure (unc0) goes to zero as n gets large.

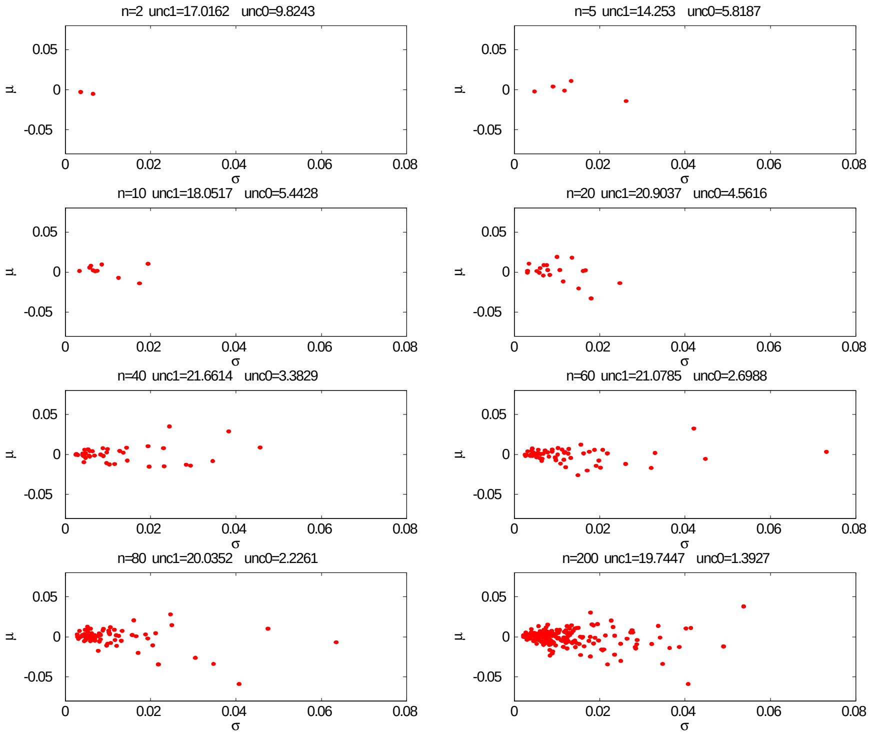

Figure 9. Simulated Mean and Standard Deviation of Daily Stock Returns and Uncertainty Measures for S&P 500.

4.1.2 Estimation results

First, we compute the monthly, quarterly, half-year and annual risk measure, our uncertainty measure (EUNC1) and Izhakian's measure (UNC0) using our algorithm. Under the assumption that the daily stocks in each measuring period (a month, a quarter, half year or a year) follows different normal distributions with different means and standard deviations, the longer is the measuring period, the more is the number of finite support (n) of Δ. More specifically, we calculate the monthly uncertainty measures and risk measure based on the sample means and sample standard deviations of daily returns within a month. There are about 20 trading days in each month, so the number of finite support of Δ is 20 for monthly measures. Similarly, n equals about 60, 120, and 240, for the quarterly, half-year and annual measures, respectively. Figure 6 plots the time-series of these three measures in different measuring periods for S&P 500 index. We find that our uncertainty measure (EUNC1) and the conventional risk measure are stable across different measuring periods, but Izhakian (2020) measure (UNC0) decreases with n and goes to zero as n gets sufficiently large. Thus, Izhakian's measure is null when Δ is with infinite or continuum support (i.e. $n \to +\infty$), which is consistent with Proposition 4.

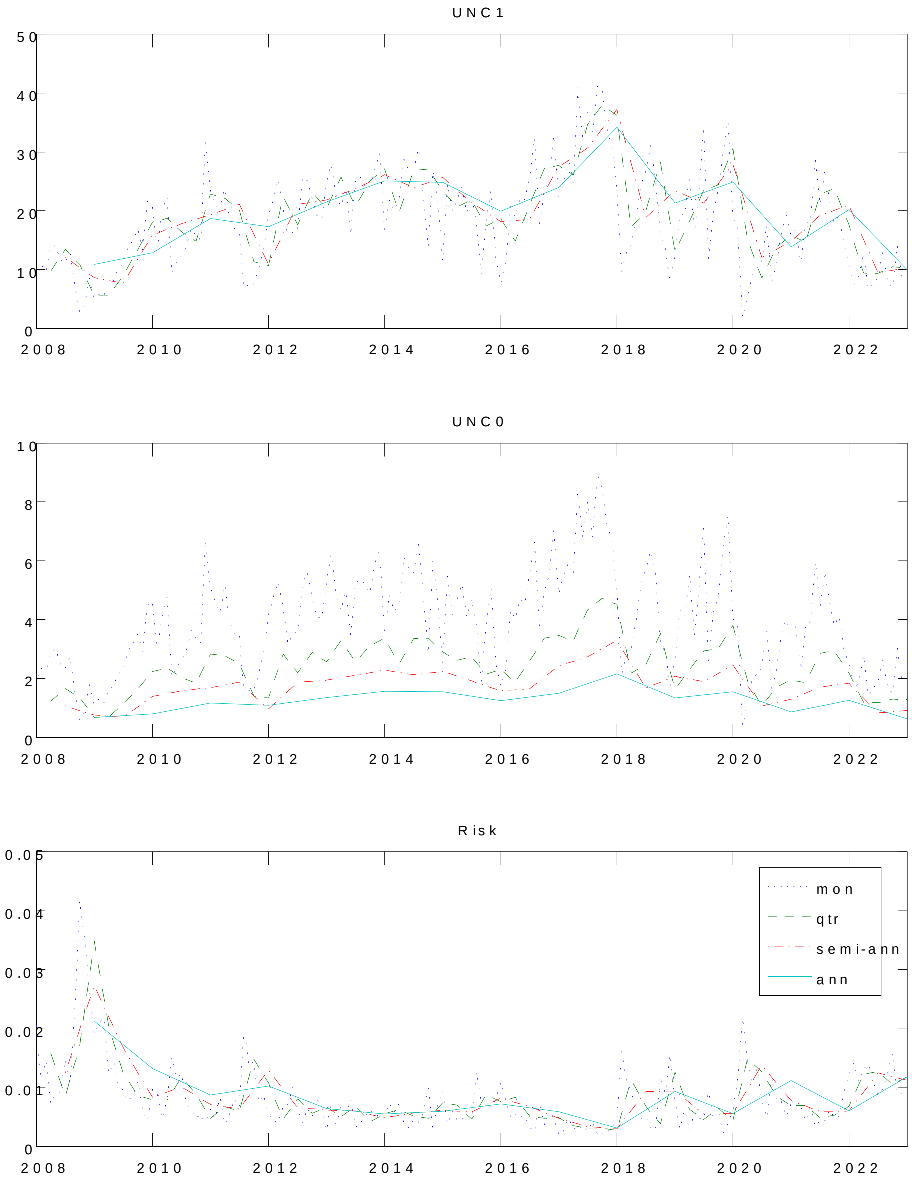

Figure 10. Time-Series of Monthly, Quarterly, Semi-Annual and Annual Uncertainty Measures and Risk of S&P 500.

Next, we compare our empirical uncertainty measure (EUNC1) with Brenner and Izhakian (2018) empirical measure (EUNC0). In the top left panel of Figure 7, we provide the scatter plots of the two uncertainty measures against risk for S&P 500. Both uncertainty measures have a noticeably decreasing and convex relationship with risk. But our measure (EUNC1) monotonically decreases with risk, while Brenner and Izhakian (2018) measure (EUNC0) has a U-shaped relationship with risk. When risk is small, the two uncertainty measures are about the same; however, when risk is large, (EUNC0) becomes spuriously large as it overestimates the uncertainty without appropriately adjust for the range of Δ. Our analysis in the subsection 3.2 implies that the overestimate problem of Brenner and Izhakian (2018) uncertainty measure is more severe when the conventional measure of risk (sample standard deviation) is higher. The time-series plot in the top right panel of Figure 10 show the same pattern over time.

The bottom panels of Figure 11 plot the scattered plot and time-series plot of risk and uncertainty measures of Chinese stock market index CSI 300. The overall sample deviation of Chinese stock market index CSI 300 is higher than that of the US stock market index S&P 500, so the Brenner and Izhakian (2018) uncertainty measure is more likely to overestimate in the Chinese stock market, which is clear when we compare bottom panels with the top panels of Figure 11. Brenner and Izhakian (2018) empirical uncertainty measure is very large in crisis periods such as the 2008 global financial crisis and 2015 China stock market crash. This is counter intuitive, as during the market crisis, the risk is high, but the uncertainty is low when everyone in the market knows the situation is bad.

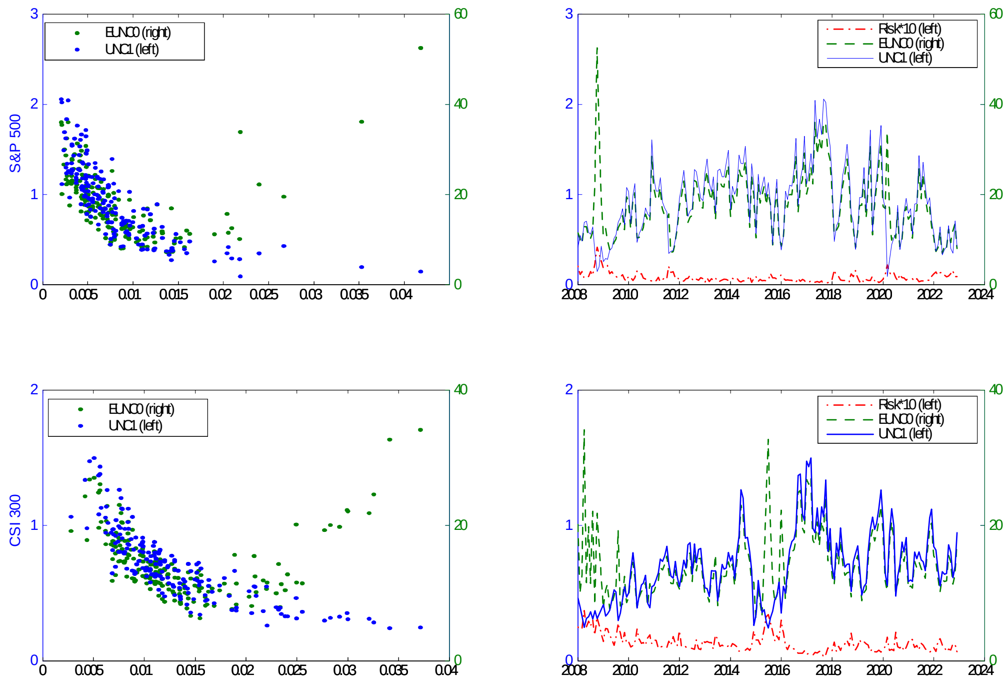

Figure 11. Our Stock Uncertainty Measure vs Brenner and Izhakian's Measure

4.2 Stock Uncertainty and Moments

In Section 3, we show that the rank between two arbitrary rectangle sets of means and standard deviations can be simplified into the assessment of the first, second, fourth and sixth moments $(\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4, \overline{\mathfrak{m}}_6)$. We also show that under the assumption of location-scale family of normal distributions, the composite uncertainty measure (EUNC1) is a consistent measure of the expected variance of the family of normal distributions. Hence, the first, second, fourth and sixth moments should be important factors to determined the composite uncertainty measure, while the third and fifth moments do not contains information in uncertainty.

In Table 4.1 and 4.2, we report the regression results of the stock uncertainty of S&P 500 on the estimates of 1st-6th moments, and their squares, the results are consistent with model implication.

Table 4.1: Regression of Uncertainty on the Moments

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\overline{\mathfrak{m}}_1$	9.463***						0.307	-4.656
	(2.647)						(0.125)	(-1.292)
$\overline{\mathfrak{m}}_2$		-5.565***					-24.042***	-23.831***
		(-14.592)					(-4.981)	(-4.962)
$\overline{\mathfrak{m}}_3$			0.462					1.137
			(1.121)					(1.352)
$\overline{\mathfrak{m}}_4$				-3.183***			31.681***	30.231***
				(-12.569)			(3.434)	(3.286)
$\overline{\mathfrak{m}}_5$					0.121			-0.207
					(0.607)			(-0.600)
$\overline{\mathfrak{m}}_6$						-2.193***	-14.849***	-13.942***
						(-11.585)	(-3.179)	(-2.989)
Constant	18.342***	27.778***	18.452***	27.302***	18.429***	26.873***	27.677***	27.976***
	(30.81)	(36.53)	(30.52)	(32.78)	(30.43)	(31.21)	(36.09)	(36.04)
Adj. R^2	0.032	0.542	0.001	0.467	-0.004	0.427	0.576	0.581
F-test							61.767	42.377

Table 4.2: Regression of Uncertainty on the Moments and Squares of Moments

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\overline{\mathfrak{m}}_1$	8.130**						-5.823***	-5.664**
	(2.461)						(-3.364)	(-2.106)
$\overline{\mathfrak{m}}_1^2$	-64.318***						-3.204	0.308
	(-5.710)						(-0.513)	(0.045)
$\overline{\mathfrak{m}}_2$		-13.136***					-40.577***	-44.301***
		(-16.261)					(-6.748)	(-7.076)
$\overline{\mathfrak{m}}_2^2$		1.203***					2.423**	2.645**
		(10.118)					(2.240)	(2.382)
$\overline{\mathfrak{m}}_3$			0.24					0.49
			(0.700)					(0.816)
$\overline{\mathfrak{m}}_3^2$			-1.405***					-0.359
			(-9.045)					(-1.053)
$\overline{\mathfrak{m}}_4$				-7.039***			33.000***	39.230***
				(-10.935)			(2.779)	(3.225)
$\overline{\mathfrak{m}}_4^2$				0.406***			-0.49	-0.677
				(6.408)			(-0.370)	(-0.502)
$\overline{\mathfrak{m}}_5$					0.174			-0.284
					(1.039)			(-1.142)
$\overline{\mathfrak{m}}_5^2$					-0.320***			0.211*
					(-8.747)			(1.954)
$\overline{\mathfrak{m}}_6$						-4.792***	-12.203*	-15.255**
						(-9.319)	(-1.963)	(-2.407)
$\overline{\mathfrak{m}}_6^2$						0.205***	0.035	0.046
						(5.379)	(0.070)	(0.091)
Constant	20.136***	35.722***	21.465***	33.668***	21.376***	32.652***	36.687***	36.986***
	(31.849)	(35.995)	(35.658)	(27.016)	(35.094)	(24.370)	(39.164)	(39.166)
Adj. R^2	0.178	0.708	0.313	0.565	0.295	0.504	0.803	0.806
F-test							91.943	62.682

Table 4.3: Correlation Coefficients

	$\overline{R}$	risk	unc1	$\overline{\mathfrak{m}}_1$	$\overline{\mathfrak{m}}_2$	$\overline{\mathfrak{m}}_4$	$\overline{\mathfrak{m}}_6$	$\frac{\overline{\mu}-\underline{\mu}}{2}$	$\frac{\overline{\nu}+\underline{\nu}}{2}$
risk	-0.35								
unc1	0.21	-0.74							
$\overline{\mathfrak{m}}_1$	0.99	-0.32	0.19						
$\overline{\mathfrak{m}}_2$	-0.28	0.91	-0.74	-0.25					
$\overline{\mathfrak{m}}_4$	-0.25	0.82	-0.69	-0.22	0.97				
$\overline{\mathfrak{m}}_6$	-0.24	0.77	-0.66	-0.21	0.94	0.99			
$\frac{\overline{\mu}-\underline{\mu}}{2}$	-0.20	0.68	-0.58	-0.17	0.87	0.96	0.98		
$\frac{\overline{\nu}+\underline{\nu}}{2}$	0.10	-0.31	0.30	0.09	-0.54	-0.71	-0.77	-0.84	
$\frac{\overline{\nu}-\underline{\nu}}{2}$	-0.16	0.66	-0.49	-0.12	0.88	0.94	0.95	0.94	-0.82

We note that the second, fourth and sixth moments are highly correlated with correlation coefficients of more than 0.95 as shown in Table 4.3, and regression models in columns (7) and (8) in Table 4.1 and 4.2 are subject to the multicollinearity problem, so $\overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4$,and $\overline{\mathfrak{m}}_6$ are not suitable candidates for the factors in the linear factor models in the context of Knightian uncertainty. Table 4.3 also shows that $\overline{\mathfrak{m}}_1$ and average daily return ($\overline{R}$) are highly correlated (0.99), and $\overline{\mathfrak{m}}_2$ and conventional risk are highly correlated (0.91). It worth noting that uncertainty measure is negatively correlated with risk and $\overline{\mathfrak{m}}_2$. We also note that the risk and $\overline{\mathfrak{m}}_2$ are negatively correlated with estimates of expected return $\overline{R}$ and $\overline{\mathfrak{m}}_1$.

In Section 3, equations (3.2) connect the mean and average of location and scale coefficients, $\frac{\overline{\mu}+\underline{\mu}}{2}$, $\frac{\overline{\mu}-\underline{\mu}}{2}$, $\frac{\overline{\nu}+\underline{\nu}}{2}$ and $\frac{\overline{\nu}-\underline{\nu}}{2}$, with moments, $\overline{\mathfrak{m}}_1, \overline{\mathfrak{m}}_2, \overline{\mathfrak{m}}_4$,and $\overline{\mathfrak{m}}_6$.In particular, the mean of location coefficients $\frac{\overline{\mu}+\underline{\mu}}{2}$is $\overline{\mathfrak{m}}_1$, while the mean of scale coefficients $\frac{\overline{\nu}+\underline{\nu}}{2}$ is variance corrected for the dispersion of location coefficients. Table 4.3 shows that $\frac{1}{2}(\overline{\nu} + \underline{\nu})$ is positively correlated with the mean return, negatively correlated with risk and $\overline{\mathfrak{m}}_2$, which implies that it may be a better risk measure than the variance or $\overline{\mathfrak{m}}_2$.

5 Conclusions

In this paper, we first establish and prove that it is impossible to find a quantitative measure of uncertainty that is independent of preferences for risk and uncertainty in the context of Knightian uncertainty. Then we show that the mean of the first, second, fourth, and sixth moments of asset returns fully characterize the statistical property of a family of normal distributions, in the Knightian-uncertainty context, where asset returns fall into the class of location-scale family of normal distributions. Next, we propose a composite uncertainty measure that is a consistent and reliable measure of the expected variance of a family of probability distributions, and an empirical implementation of the uncertainty measure as an explicit function of means and standard deviations of asset returns. Then, We show that Izhakian (2020) measure is null when the possible means and standard deviations are randomly drawn from a range with infinite or continuum support, and Brenner and Izhakian (2018)'s approach introduces a spurious positive correlation between stock uncertainty and risk when the risk is high.

Lastly, we assess uncertainty of stock market indices, S&P 500 in the U.S. and CSI 300 in China using both the moments and composite uncertainty measure. We find that the time-series and cross-section relationship between the composite stock uncertainty and risk measured by sample standard deviation are nonlinear and negative, contrary to the common belief that the higher the risk, the larger the uncertainty. On the other hand, large swings in the stock market such as the 1987 Black Monday in the U.S. or the 2008 stimulus plan in China cause the ranges of the location of scale to increase dramatically and the impact remains high for long time.

Our findings imply that the asset pricing model in the context of Knightian uncertainty should be fundamentally different from the linear factor models derived in the risk-only context. As the properties of a family of normal distributions are fully characterized by the 1st, 2nd, 4th and 6th moments, the analogy of the one-factor CAPM in the risk-only context may be a nonlinear three-factor model with possible factors as risk, kurtosis, and uncertainty.

A Proof of Proposition 5

Proof. The uncertainty measure $\Psi_n(r)$

$$
\begin{aligned}
\Psi_n(r) &= \int_{-\infty}^{\infty} \overline{\varphi}_n(x) S_n^2(x)\, dx = \frac{1}{n}\sum_{k=1}^{n}\int_{-\infty}^{\infty} \overline{\varphi}_n(x)\left(\varphi_k(x) - \overline{\varphi}_n(x)\right)^2 dx \\
&= \frac{1}{n}\sum_{k=1}^{n}\int_{-\infty}^{\infty} \overline{\varphi}_n(x)\varphi_k^2(x)\, dx - \int_{-\infty}^{\infty} \overline{\varphi}_n^3(x)\, dx \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{\infty} \varphi_i(x)\varphi_k^2(x)\, dx \\
&\quad -\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{\infty} \varphi_i(x)\varphi_j(x)\varphi_k(x)\, dx
\end{aligned}
$$

It is straightforward to show that

$$
\int_{-\infty}^{\infty} \varphi_i(x)\varphi_j(x)\varphi_k(x)\, dx = \frac{\varphi_i(0)\varphi_j(0)\varphi_k(0)}{\varphi_{i,j,k}(0)}
$$

where

$$
\varphi_{i,j,k}(0) \equiv \sqrt{\frac{\frac{1}{\sigma_i^2}+\frac{1}{\sigma_j^2}+\frac{1}{\sigma_k^2}}{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\frac{\mu_i}{\sigma_i^2}+\frac{\mu_j}{\sigma_j^2}+\frac{\mu_k}{\sigma_k^2}}{\sqrt{\frac{1}{\sigma_i^2}+\frac{1}{\sigma_j^2}+\frac{1}{\sigma_k^2}}}\right)^2\right\},
$$

Hence

$$
\begin{aligned}
\Psi_n(r) &= \frac{1}{n^2}\left(\sum_i\sum_k \frac{\varphi_i(0)\varphi_k^2(0)}{\varphi_{i,k,k}(0)} - \frac{1}{n}\sum_i\sum_j\sum_k \frac{\varphi_i(0)\varphi_j(0)\varphi_k(0)}{\varphi_{i,j,k}(0)}\right) \\
&= \frac{1}{n^3}\sum_i\sum_j\sum_k \underbrace{\frac{\varphi_i(0)\varphi_j(0)\varphi_k(0)}{\varphi_{i,j,k}(0)}\left(\frac{\varphi_k(0)}{\varphi_j(0)}\frac{\varphi_{i,j,k}(0)}{\varphi_{i,k,k}(0)} - 1\right)}_{\Psi(i,j,k)} \qquad \text{(A.1)}
\end{aligned}
$$

with $\Psi(i,j,k)$ interpreted as contribution towards ambiguity measure by the ordered triplet $\{i,j,k\}$. ■

It can be easier to make use of scale and Sharpe-ratio bundles $\{(\sigma_i, \rho_i)\}$ with

$$
\rho_{ijk} = \frac{\frac{\rho_i}{\sigma_i}+\frac{\rho_j}{\sigma_j}+\frac{\rho_k}{\sigma_k}}{\sqrt{\frac{1}{\sigma_i^2}+\frac{1}{\sigma_j^2}+\frac{1}{\sigma_k^2}}}.
$$

By Cauchy-Schwarz inequality, we obtain

$$
\rho_{ijk}^2 \le \rho_i^2 + \rho_j^2 + \rho_k^2 \qquad \text{(A.2)}
$$

with equality only when the vectors (ρ_i, ρ_j, ρ_k) and $\left(\frac{1}{\sigma_i}, \frac{1}{\sigma_j}, \frac{1}{\sigma_k}\right)$ are proportional to each other — This is equivalent to $\mu_i = \mu_j = \mu_k$.

Also, for the sake of economic interpretation, we introduce several useful notations:

$$\begin{aligned}\Upsilon_{ijk} &= \exp\left\{\frac{\rho_i^2+\rho_j^2+\rho_k^2-\rho_{ijk}^2}{2}\right\}\\ \boldsymbol{\Xi}_{ijk} &= \frac{\sigma_i^2+\sigma_j^2+\sigma_k^2}{3}\\ \mathbf{V}_{ijk} &= \frac{\left(\sigma_i^2-\boldsymbol{\Xi}_{ijk}\right)^2+\left(\sigma_j^2-\boldsymbol{\Xi}_{ijk}\right)^2+\left(\sigma_k^2-\boldsymbol{\Xi}_{ijk}\right)^2}{3}\end{aligned}$$

in which $\boldsymbol{\Xi}_{ijk}$ and $\mathbf{V}_{ijk}$ measure respectively the average scale of volatility σ^2 and dispersion in the volatility (or, variance of variance) within the triplet. If we treat $\rho_i^2+\rho_j^2+\rho_k^2-\rho_{ijk}^2$, as a "distorted" dispersion in the Sharpe ratios, then Υ_{ijk} as exponential function of the Sharpe ratio dispersion summarizes its impact on uncertainty measure. We have $\Upsilon_{ijk}\geq 1$ with equality when $\mu_i=\mu_j=\mu_k$.

We may write[27]

$$\begin{aligned}\frac{\phi_i(0)\phi_j(0)\phi_k(0)}{\phi_{i,j,k}(0)} &= \frac{1}{2\pi}\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1} \qquad \text{(A.3)}\\ &= \frac{1}{2\sqrt{3}\pi}\frac{\Upsilon_{ijk}^{-1}}{\sqrt{\boldsymbol{\Xi}_{ijk}^2-\frac{1}{2}\mathbf{V}_{ijk}}}\\ \frac{\phi_k(0)\phi_{i,j,k}(0)}{\phi_j(0)\phi_{i,k,k}(0)} &= \frac{\sigma_j}{\sigma_k}\frac{\sigma_{ikk}}{\sigma_{ijk}}\frac{\Upsilon_{ijk}}{\Upsilon_{ikk}} \qquad \text{(A.4)}\\ &= \sqrt{\frac{\boldsymbol{\Xi}_{ijk}^2-\frac{1}{2}\mathbf{V}_{ijk}}{\boldsymbol{\Xi}_{ikk}^2-\frac{1}{2}\mathbf{V}_{ikk}}}\frac{\Upsilon_{ijk}}{\Upsilon_{ikk}}\end{aligned}$$

This yields an expression for $\Psi(i,j,k)$ that is explicitly expressed in terms of dispersion in Sharpe ratios, the average scale and dispersion in volatility:

$$\Psi(i,j,k)=\frac{1}{2\sqrt{3}\pi}\left(\frac{\Upsilon_{ikk}^{-1}}{\sqrt{\boldsymbol{\Xi}_{ikk}^2-\frac{1}{2}\mathbf{V}_{ikk}}}-\frac{\Upsilon_{ijk}^{-1}}{\sqrt{\boldsymbol{\Xi}_{ijk}^2-\frac{1}{2}\mathbf{V}_{ijk}}}\right) \qquad \text{(A.5)}$$

The following observations on the ambiguity measure can be readily obtained:

Proposition 6 *1. Each distinct triplet $\{i,j,k\}$ contributes positively towards $\Psi_n^2(r)$; that is, $\Psi(i,j,k)+\Psi(i,k,j)\geq 0$ with strict inequality unless $j=k$.*

2. The impact of dispersion in Sharpe ratio on $\Psi_n^2(r)$ is not monotone; it is null when there is no dispersion in the location coefficients, i.e. $\mu_i=\mu_j=\mu_k$.

3. When there is no dispersion on σ^2, it yields $\frac{\rho_{ijk}}{\sqrt{3}}=\frac{\rho_i+\rho_j+\rho_k}{3}=\overline{\rho}_{ijk}$, $\Upsilon_{ijk}=e^{\frac{3}{2}V_{ijk}[\rho]}$ with

$$\mathbf{V}_{ijk}[\rho]=\frac{\left(\rho_i-\overline{\rho}_{ijk}\right)^2+\left(\rho_j-\overline{\rho}_{ijk}\right)^2+\left(\rho_j-\overline{\rho}_{ijk}\right)^2}{3}.$$

Moreover, taking ρ_i,ρ_j,ρ_k as given, we have

$$\Psi(i,j,k)=\frac{e^{-\frac{3}{2}V_{ijk}[\rho]}}{2\sqrt{3}\pi\sigma^2}\left(e^{\frac{(\rho_i-\rho_k)(\rho_j-\rho_k)}{3}}-1\right) \qquad \text{(A.6)}$$

that is inversely proportional to σ^2.

[27] We have $\frac{\sigma_i^2\sigma_j^2\sigma_k^2}{\sigma_{ijk}^2}=\sigma_i^2\sigma_j^2+\sigma_j^2\sigma_k^2+\sigma_k^2\sigma_i^2=3\left(\boldsymbol{\Xi}_{ijk}^2-\frac{1}{2}\mathbf{V}_{ijk}\right)$.

4. $\Psi_n^2(r)$ *has its upper and lower bound to be respectively given by*

$$\begin{aligned}\overline{\Psi}_n^2 &= \frac{1}{2\pi n^3}\sum_i\sum_j\sum_k\left(\frac{\sigma_{ijj}}{\sigma_i\sigma_j^2}\Upsilon_{ijk}-\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\right)\\ \underline{\Psi}_n^2 &= \frac{1}{2\pi n^3}\sum_i\sum_j\sum_k\left(\frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-2}-\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\right)\end{aligned}$$

Proof. *Statement 1: We show that each distinct triplet of supports in* Δ *must contribute positively towards the ambiguity measure. To see this, by symmetry, we have*[28]

$$\begin{aligned}&\frac{1}{2}\left(\frac{\sigma_{ijj}}{\sigma_i\sigma_j^2}\Upsilon_{ijj}^{-1}+\frac{\sigma_{ikk}}{\sigma_i\sigma_k^2}\Upsilon_{ikk}^{-1}\right)\\ \geq\ &\frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}}{\sigma_i\sigma_j\sigma_k}\sqrt{\Upsilon_{ijj}^{-1}\Upsilon_{ikk}^{-1}}\geq\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\end{aligned}\tag{A.7}$$

and that it holds with equality only when $j=k$.

Statement 2: By definition, the dispersion in Sharpe ratios is inversely related to that of the volatility. From expressions (A.3)&(A.4) we obtain an negative impact on $\frac{\phi_i(0)\phi_j(0)\phi_k(0)}{\phi_{i,j,k}(0)}$ *from the dispersion on Sharpe-ratios, and impact on* $\frac{\phi_j(0)\phi_{i,j,k}(0)}{\phi_k(0)\phi_{i,j,j}(0)}$ *in either directions. The total impact is thus ambiguous. Particularly, when there is no dispersion in location coefficients, we have* $\rho_{ijk}^2=\rho_i^2+\rho_j^2+\rho_k^2$ *and* $\rho_{ijj}^2=\rho_i^2+2\rho_j^2$. *This yields*

$$\Psi_n^2=\frac{1}{2\pi n^3}\sum_i\sum_j\sum_k\left(\frac{\sigma_{ijj}}{\sigma_i\sigma_j^2}-\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\right)$$

that is invariant in Sharpe-ratios.

Statement 3. If there is no dispersion in σ, *we have* $\sigma_{ijk}^2=\frac{\sigma^2}{3},\rho_{ijk}^2=\frac{(\rho_i+\rho_j+\rho_k)^2}{3}$ *and* $\rho_{ijj}^2=\frac{(\rho_i+2\rho_j)^2}{3}$. *Substitute these into expressions (A.3)&(A.4) to obtain the desired expression for* $\Psi_n^2(r)$.

Statement 4. We look into each element towards the ambiguity measure. With $\rho_{ijk}^2\leq\rho_i^2+\rho_j^2+\rho_k^2$ *and* $\rho_{ijj}^2\leq\rho_i^2+2\rho_j^2$ *we obtain*

$$\begin{aligned}&\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\left(\frac{\frac{\sigma_j}{\sigma_k}\frac{\sigma_{ikk}}{\sigma_{ijk}}\Upsilon_{ikk}^{-1}+\frac{\sigma_k}{\sigma_j}\frac{\sigma_{ijj}}{\sigma_{ijk}}\Upsilon_{ijj}^{-1}}{2}\Upsilon_{ijk}-1\right)\\ \leq\ &\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\left(\frac{\frac{\sigma_j}{\sigma_k}\frac{\sigma_{ikk}}{\sigma_{ijk}}+\frac{\sigma_k}{\sigma_j}\frac{\sigma_{ijj}}{\sigma_{ijk}}}{2}\Upsilon_{ijk}-1\right)\\ =\ &\frac{1}{2}\left(\frac{\sigma_{ikk}}{\sigma_i\sigma_k^2}+\frac{\sigma_{ijj}}{\sigma_i\sigma_j^2}\right)\Upsilon_{ijk}-\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\end{aligned}\tag{A.8}$$

The inequality holds with equality when $\mu_i=\mu_j=\mu_k$ *at which we have* $\rho_{ijk}^2=\rho_i^2+\rho_j^2+\rho_k^2$ *and the r.h.s. equals to*

$$\frac{1}{2}\left(\frac{\sigma_{ikk}}{\sigma_i\sigma_k^2}+\frac{\sigma_{ijj}}{\sigma_i\sigma_j^2}\right)-\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\geq 0.\tag{A.9}$$

[28]

$$\frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}}{\sigma_{ijk}}\exp\left\{\frac{\rho_{ijj}^2+\rho_{ikk}^2-2\rho_{ijk}^2}{4}\right\}\geq 1.$$

To obtain the lower bound, for each term, we have:

$$\begin{aligned}
& \frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\left(\frac{\frac{\sigma_j}{\sigma_k}\frac{\sigma_{ikk}}{\sigma_{ijk}}\Upsilon_{ikk}^{-1}+\frac{\sigma_k}{\sigma_j}\frac{\sigma_{ijj}}{\sigma_{ijk}}\Upsilon_{ijj}^{-1}}{2}\Upsilon_{ijk}-1\right) \\
\geq \; & \frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\left(\frac{\frac{\sigma_j}{\sigma_k}\frac{\sigma_{ikk}}{\sigma_{ijk}}\Upsilon_{ikk}^{-1}+\frac{\sigma_k}{\sigma_j}\frac{\sigma_{ijj}}{\sigma_{ijk}}\Upsilon_{ijj}^{-1}}{2}-1\right) \\
\geq \; & \frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\left(\frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}}{\sigma_{ijk}}\sqrt{\Upsilon_{ijj}^{-1}\Upsilon_{ikk}^{-1}}-1\right) \\
\geq \; & \frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}\left(\frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}}{\sigma_{ijk}}\Upsilon_{ijk}^{-1}-1\right) \\
= \; & \frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-2}-\frac{\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}\Upsilon_{ijk}^{-1}
\end{aligned}$$

with equality for the first inequality when $\mu_i = \mu_j = \mu_k$. The second inequality holds with equality for all $j = k$. When $\mu_i = \mu_j = \mu_k$, the ultimate lower bound is $\frac{\sqrt{\sigma_{ijj}\sigma_{ikk}}-\sigma_{ijk}}{\sigma_i\sigma_j\sigma_k}$. ■

B Proof of Impossibility Theorem for SEU Class with Constant RRA

For r and r' with location-scale coefficient set Δ and Δ', the condition for an unanimous rank between r and r' is $\forall\alpha \leq 1; \forall Q \in \mathcal{M}(\Delta), Q' \in \mathcal{M}(\Delta')$

$$\frac{1}{\alpha}\iint\limits_{\Delta} \exp\left(\alpha\mu + \frac{\alpha^2\sigma^2}{2}\right) dQ(\mu,\sigma) \geq \frac{1}{\alpha}\iint\limits_{\Delta'} \exp\left(\alpha\mu + \frac{\alpha^2\sigma^2}{2}\right) dQ'(\mu,\sigma)$$

which is equivalent to $\forall\alpha \leq 1, \forall(\mu,\sigma) \in \Delta, (\mu',\sigma') \in \Delta'$

$$\frac{1}{\alpha}\exp(\alpha\mu + \frac{\alpha^2\sigma^2}{2}) \geq \frac{1}{\alpha}\exp\left(\alpha\mu' + \frac{\alpha^2\sigma'^2}{2}\right), \tag{B.1}$$

For $0 < \alpha \leq 1$, the condition (B.1) becomes $\forall(\mu,\sigma) \in \Delta, (\mu',\sigma') \in \Delta'$

$$\alpha\mu + \frac{\alpha^2\sigma^2}{2} \geq \alpha\mu' + \frac{\alpha^2\sigma'^2}{2}$$

or equivalently

$$\mu - \mu' \geq \frac{\alpha}{2}\left(\sigma'^2 - \sigma^2\right)$$

When $\alpha \to 0_+$ we obtain

$$\inf\{\mu : (\mu,\sigma) \in \Delta\} \geq \sup\{\mu : (\mu,\sigma) \in \Delta'\}; \tag{B.2}$$

For $\alpha < 0$,the condition (B.1) becomes $\forall(\mu,\sigma) \in \Delta, (\mu',\sigma') \in \Delta'$

$$\alpha\mu + \frac{\alpha^2\sigma^2}{2} \leq \alpha\mu' + \frac{\alpha^2\sigma'^2}{2}$$

or equivalently

$$\mu - \mu' \geq \frac{\alpha}{2}\left(\sigma'^2 - \sigma^2\right)$$

This, together with (B.2), implies[29]

$$\sup\left\{\sigma^2 : (\mu,\sigma) \in \Delta\right\} \leq \inf\left\{\sigma^2 : (\mu,\sigma) \in \Delta'\right\} \tag{B.3}$$

Inequality (B.2) in the opposite direction is obtained when it applies to $-r$ and $-r'$. These together imply that Δ and Δ' must share a common singleton location coefficient μ. In this case, unanimous ranking among SEU with constant RRAs can be reached only when (B.3) are satisfied. These are identical to the set of conditions identified above for the general class of risk averse SEU investors. Thus there exists no quantifiable uncertainty measure that would generate assessments that are consistent with unanimous ranking by those SEU with constant RRAs.

C Proof of Propositions *1,2 and 3*

C.1 Proof of Proposition 1

The location-scale function $u(\mu,\sigma)$ that corresponds to an expected utility u is

$$u(\mu,\sigma) \triangleq \int_{-\infty}^{\infty} u(x)\, d\Phi\left(\frac{x-\mu}{\sigma}\right)$$

for $(\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_+$. For $\Delta = \{(\mu_i,\sigma_i)\}_{i=1}^n$, the expected utilities on Δ are thus summarized by vector of real-value utility functions $[u_1,\cdots,u_n] \in \mathbb{R}^n$ with

$$u_i \triangleq u(\mu_i,\sigma_i) = \int_{-\infty}^{\infty} u(x)\, d\Phi\left(\frac{x-\mu_i}{\sigma_i}\right)$$

We have following the lemma:

Lemma 2 *EUUP defined on Δ with finite supports is*

$$V(r) = \int_{-\infty}^{\infty} u(x)\, d\overline{\Phi}_r(x)$$

where $\overline{\Phi}_r(x)$ is the mean-average cdf.

Proof. The probability distributions on Δ with finite supports $\{(\mu_i,\sigma_i) : i = 1,\cdots,n\}$ is a simplex

$$\mathcal{P} = \{(q_1,....q_n) \in \mathbb{R}^n : \sum_{i=1}^n q_i = 1, q_i \geq 0\}.$$

The EUUP defined as (2.1) on Δ with n finite supports is thus given by

$$\begin{aligned} V(r) &= \int_{\mathcal{P}} \left(\sum_{i=1}^n q_i u_i\right) d\xi(q) = \frac{1}{n}\sum_{i=1}^n u_i \\ &= \frac{1}{n}\sum_{i=1}^n \int_{-\infty}^{\infty} u(x)\, d\Phi_i(x) = \int_{-\infty}^{\infty} u(x)\, d\overline{\Phi}_r(x). \end{aligned}$$

where the second-layer belief ξ is uniformly distributed on $\mathcal{P}$, and $\overline{\Phi}_r(x) \triangleq \frac{1}{n}\sum_{i=1}^n \Phi\left(\frac{x-\mu_i}{\sigma_i}\right)$ is the mean-average c.d.f. The second equation follows from the integration by parts. ■

The statements in Proposition 1 follow by Lemma 2 and by the notions of stochastic dominance.

[29] Suppose $\sigma'^2 < \sigma^2$, then when $\alpha \to -\infty$, to obtain an violation of the inequality.

C.2 Proof of Proposition 2

We start with a well known fundamental representation theorem of characteristic function in probability theory.

Lemma 3 *Two random variables are identically distributed if and only if the corresponding characteristic functions are the same.*

Let $\Theta(\mu,\sigma;x) = \exp(-\mu x + \frac{1}{2}\sigma^2 x^2)$, $x \in \mathbb{R}$ be the characteristic function of normal distribution $\Phi\left(\frac{x-\mu}{\sigma}\right)$, then it is straightforward to show that the characteristic function for the mean-average cdf $\overline{\Phi}_r$ is given by the mean-average characteristic function

$$\overline{\Theta}_r(x) = \frac{1}{n}\sum_{i=1}^{n}\Theta(\mu_i,\sigma_i;x), x \in \mathbb{R}.$$

The equivalence between statements 1 and 2 in Proposition 2 follows by the definition of FSD and Proposition 1. For the equivalence between statement 2 and 3 of the Proposition, we just need to prove the sufficient condition. Suppose condition 2.9 holds. FSD implies condition (2.8) holds for all increasing utility functions. Let $u(x)$ in (2.8) be $\exp(\gamma x)$ and $-\exp(-\gamma x), \gamma \geq 0$, for r and $-r$, respectively. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x)\, d\overline{\Phi}_r(x) &= \int_{-\infty}^{\infty} \exp(\gamma x) d\overline{\Phi}_r(x) \propto \overline{\Theta}_r(-\gamma) \\
\int_{-\infty}^{\infty} u(x)\, d\overline{\Phi}_{-r}(x) &= \int_{-\infty}^{\infty} -\exp(-\gamma x) d\overline{\Phi}_{-r}(x) \propto -\overline{\Theta}_{-r}(\gamma)
\end{aligned}$$

The condition 2.9

$$\overline{\Phi}_r(x) \leq \overline{\Phi}_{r'}(x), \forall x \in \mathbb{R}$$

implies that

$$\overline{\Theta}_r(-\gamma) \geq \overline{\Theta}_{r'}(-\gamma) \text{ and } \overline{\Theta}_r(\gamma) \leq \overline{\Theta}_{r'}(\gamma), \forall \gamma \geq 0. \tag{C.1}$$

Similarly, for $-r$ and $-r'$, the condition 2.9

$$\overline{\Phi}_{-r}(x) \leq \overline{\Phi}_{-r'}(x), \forall x \in \mathbb{R}$$

implies that

$$\overline{\Theta}_{-r}(-\gamma) \geq \overline{\Theta}_{-r'}(-\gamma) \text{ and } \overline{\Theta}_{-r}(\gamma) \leq \overline{\Theta}_{-r'}(\gamma), \forall \gamma \geq 0. \tag{C.2}$$

Note that $\overline{\Theta}_{-r}(-\gamma) = \overline{\Theta}_r(\gamma)$, conditions (C.2) reduce to

$$\overline{\Theta}_r(\gamma) \geq \overline{\Theta}_{r'}(\gamma) \text{ and } \overline{\Theta}_r(-\gamma) \leq \overline{\Theta}_{r'}(-\gamma), \forall \gamma \geq 0 \tag{C.3}$$

Combine conditions (C.1) and (C.3) we have:

$$\overline{\Theta}_r(-\gamma) = \overline{\Theta}_{r'}(-\gamma) \text{ and } \overline{\Theta}_r(\gamma) = \overline{\Theta}_{r'}(\gamma), \forall \gamma \geq 0.$$

Thus, the characteristic functions $\overline{\Theta}_r$ and $\overline{\Theta}_{r'}$ ($\overline{\Theta}_{-r}$ and $\overline{\Theta}_{-r'}$) are identical on $\mathbb{R}$. By Lemma 3, the distributions $\overline{\Phi}_r$ and $\overline{\Phi}_{r'}$ ($\overline{\Phi}_{-r}$ and $\overline{\Phi}_{-r'}$) must be identical with the same supports; that is, $\Delta = \Delta'$.

Condition (3) in Proposition 2 violates the non-degeneracy condition for uncertainty assessment. So, it is impossible to obtain mutual agreeable rank of two arbitrary finite sets $\Delta \neq \Delta'$ among investors with monotone EUUP utility functions.

C.3 Proof of Proposition 3:

Applying Taylor expansion to normal characteristic function $\Theta(\mu, \sigma; x)$ to have

$$\begin{aligned}\Theta(\mu, \sigma; x) &= \sum_{k=0}^{\infty} (-1)^k \frac{\mathfrak{m}_k(\mu, \sigma)}{k!} x^k \\ &= \sum_{k=0}^{K} (-1)^k \frac{\mathfrak{m}_k(\mu, \sigma)}{k!} x^k + o\left(x^K\right),\end{aligned} \tag{C.4}$$

where $\mathfrak{m}_k(\mu, \sigma) = \sum_{s=0}^{k} \binom{k}{s} \mathfrak{m}_s(0,1) \mu^{k-s} \sigma^s$ with

$$\mathfrak{m}_s(0,1) = \begin{cases} 0, & \text{if } s \text{ is odd} \\ \dfrac{2^{-s/2} s!}{(s/2)!}, & \text{if } s \text{ is even} \end{cases}$$

being the s^{th} order moment of standard Gaussian normal distribution. We are particularly interested in the 4th and 6th moments $\mathfrak{m}_4(0,1) = 3, \mathfrak{m}_6(0,1) = 15$.

The equivalence between SSD and unanimous ranking by risk averse expected utility investors are well-documented (Ma and Wong, 2010). Hence, we just need to prove that statement 2 is a sufficient condition for Statement 3.

Similar to proof of Proposition 2, let $u(x)$ in (2.8) be $-\exp(-\gamma x), \gamma \geq 0$, for r and $-r$, then Condition 2 implies $\overline{\Theta}_r(\gamma) \leq \overline{\Theta}_{r'}(\gamma)$ and $\overline{\Theta}_r(-\gamma) \leq \overline{\Theta}_{r'}(-\gamma)$ for all $\gamma \geq 0$, which implies

$$\overline{\Theta}_r(x) \leq \overline{\Theta}_{r'}(x), \forall x \in \mathbb{R}. \tag{C.5}$$

This proves the first part of Statement 3.

For the second part of Statement 3, note that $\exp(\frac{1}{2}\sigma^2 x^2)$ governs the higher order of growth rates for $\exp(-\mu x + \frac{1}{2}\sigma^2 x^2)$ as $x \to \infty$, so condition (C.5) implies that the maximum volatility coefficient σ in Δ must not exceed that in Δ'; otherwise, we may always obtain an violation of inequality (C.5) by choosing x sufficiently large. Hence, condition (b) must hold.

The inequality condition (C.5) for the characteristic function can be explicitly expressed with the location and scale coefficients, i.e. for all $x > 0$,

$$\begin{cases} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x}[\exp(-\mu_i x + \frac{\sigma_i^2 x^2}{2}) - 1] \leq \frac{1}{m} \sum_{j=1}^{m} \frac{1}{x}[\exp(-\mu_j' x + \frac{\sigma_j'^2 x^2}{2}) - 1)] \\ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x}[\exp(\mu_i x + \frac{\sigma_i^2 x^2}{2}) - 1] \leq \frac{1}{m} \sum_{j=1}^{m} \frac{1}{x}[\exp(\mu_j' x + \frac{\sigma_j'^2 x^2}{2}) - 1)] \end{cases} \tag{C.6}$$

Setting $x \to 0_+$ to obtain

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} \mu_i &\leq \frac{1}{m} \sum_{j=1}^{m} \mu_j' \\ \frac{1}{n} \sum_{i=1}^{n} \mu_i &\geq \frac{1}{m} \sum_{j=1}^{m} \mu_j' \end{aligned}$$

which imply

$$\frac{1}{n} \sum_{i=1}^{n} \mu_i = \frac{1}{m} \sum_{j=1}^{m} \mu_j'. \tag{C.7}$$

Thus, condition (a) is satisfied.

Given equality (C.7), the inequalities (C.6) can be rewritten as

$$\begin{cases} \frac{1}{n}\sum_{i=1}^{n}\frac{\exp(-\mu_i x+\sigma_i^2x^2/2)-1+\mu_i x}{x^2} \le \frac{1}{m}\sum_{j=1}^{m}\frac{\exp(-\mu_j' x+\sigma_j'^2x^2/2)-1+\mu_j' x}{x^2} \\ \frac{1}{n}\sum_{i=1}^{n}\frac{\exp(-\mu_i x+\sigma_i^2x^2/2)-1-\mu_i x}{x^2} \le \frac{1}{m}\sum_{j=1}^{m}\frac{\exp(-\mu_j' x+\sigma_j'^2x^2/2)-1-\mu_j' x}{x^2} \end{cases} \tag{C.8}$$

For x small, we may plug $\exp(\mu x+\frac{\sigma^2x^2}{2})-1-\mu x=\frac{\mu^2+\sigma^2}{2}x^2+o\left(x^2\right)$ into the above two inequalities to obtain

$$\frac{1}{n}\sum_{i=1}^{n}\left(\mu_i^2+\sigma_i^2\right)\le\frac{1}{m}\sum_{i=1}^{m}\left(\mu_j'^2+\sigma_j'^2\right). \tag{C.9}$$

Thus, condition (c) is proved.

To prove condition (d), we take Taylor expansion for the mean-average characteristic function

$$\begin{aligned}\overline{\Theta}_r\left(x\right) &= \sum_{k=0}^{2N}(-1)^k\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k\left(\mu_i,\sigma_i\right)\right)\frac{x^k}{k!} \\ &\quad +(-1)^{2N+1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+1}\left(\mu_i,\sigma_i\right)\right)\frac{x^{2N+1}}{(2N+1)!}+o\left(x^{2N+1}\right)\end{aligned}$$

which implies

$$\frac{\overline{\Theta}_r\left(x\right)-\sum_{k=0}^{2N}(-1)^k\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k\left(\mu_i,\sigma_i\right)\right)\frac{x^k}{k!}}{x^{2N+1}}=\frac{\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+1}\left(\mu_i,\sigma_i\right)}{(2N+1)!}+o\left(1\right)$$

The same applies for $\overline{\Theta}_{r'}\left(x\right)$. Suppose condition (2.12) holds for all $K\le 2N$, then inequalities (2.11) reduce to

$$\begin{aligned}&\frac{\overline{\Theta}_r\left(x\right)-\sum_{k=0}^{2N}(-1)^k\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k\left(\mu_i,\sigma_i\right)\right)\frac{x^k}{k!}}{x^{2N+1}} \\ \le\ &\frac{\overline{\Theta}_{r'}\left(x\right)-\sum_{k=0}^{2N}(-1)^k\left(\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_k\left(\mu_j',\sigma_j'\right)\right)\frac{x^k}{k!}}{x^{2N+1}},\forall x>0;\end{aligned} \tag{C.10}$$

$$\begin{aligned}&\frac{\overline{\Theta}_r\left(x\right)-\sum_{k=0}^{2N}(-1)^k\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k\left(\mu_i,\sigma_i\right)\right)\frac{x^k}{k!}}{x^{2N+1}} \\ \ge\ &\frac{\overline{\Theta}_{r'}\left(x\right)-\sum_{k=0}^{2N+1}(-1)^k\left(\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_k\left(\mu_j',\sigma_j'\right)\right)\frac{x^k}{k!}}{x^{2N+1}},\forall x<0.\end{aligned} \tag{C.11}$$

Setting $x\to 0_+$ to inequality (C.10) to obtain

$$-\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+1}\left(\mu_i,\sigma_i\right)\le-\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_{2N+1}\left(\mu_j',\sigma_j'\right); \tag{C.12}$$

and setting $x\to 0_-$ to inequality (C.11) to obtain

$$-\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+1}\left(\mu_i,\sigma_i\right)\ge-\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_{2N+1}\left(\mu_j',\sigma_j'\right). \tag{C.13}$$

Combining the two inequalities (C.12) and (C.13) results in the desired equality

$$\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+1}\left(\mu_i,\sigma_i\right)=\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_{2N+1}\left(\mu_j',\sigma_j'\right).$$

This proves the validity of condition (2.12) for $k = 2N + 1$.

Similarly, for $k = 2N + 2$, with conditions (2.12) hold for all $k = 0, 1, \cdots, 2N + 1$, the inequalities (2.11) can be expressed into

$$\begin{aligned} & \frac{\overline{\Theta}_r(x) - \sum_{k=0}^{2N+1}(-1)^k\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k(\mu_i, \sigma_i)\right)\frac{x^k}{k!}}{x^{2N+2}} \\ \leq \quad & \frac{\overline{\Theta}_{r'}(x) - \sum_{k=0}^{2N+1}(-1)^k\left(\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_k\left(\mu'_j, \sigma'_j\right)\right)\frac{x^k}{k!}}{x^{2N+2}} \end{aligned} \tag{C.14}$$

for all $x \in \mathbb{R}$. Particularly, with

$$\begin{aligned} \frac{\overline{\Theta}_r(x) - \sum_{k=0}^{2N+1}(-1)^k\left(\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_k(\mu_i, \sigma_i)\right)\frac{x^k}{k!}}{x^{2N+2}} &= \frac{\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+2}(\mu_i, \sigma_i)}{(2N+2)!} + o(1) \\ \frac{\overline{\Theta}_{r'}(x) - \sum_{k=0}^{2N+1}(-1)^k\left(\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_k\left(\mu'_j, \sigma'_j\right)\right)\frac{x^k}{k!}}{x^{2N+2}} &= \frac{\frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_{2N+2}\left(\mu'_j, \sigma'_j\right)}{(2N+2)!} + o(1) \end{aligned}$$

for x sufficiently small, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\mathfrak{m}_{2N+2}(\mu_i, \sigma_i) \leq \frac{1}{m}\sum_{j=1}^{m}\mathfrak{m}_{2N+2}\left(\mu'_j, \sigma'_j\right)$$

by setting $x \to 0$ to Eq. (C.14). This proves the validity of inequality (2.13) for $k = 2N + 2$.

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