Optimal Search Auctions with a Deadline*

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Abstract

How to search for potential bidders to allocate a product by a deadline? We fully solve the optimal mechanisms, which can be implemented by a sequence of second-price auctions with properly selected reserve prices and sampling rules. The optimal search for long-lived bidders is characterized by a constant reserve (except for the last period) and contingent sample sizes, and that for short-lived bidders is by decreasing reserves and increasing sample sizes over time. Moreover, a seller with short-lived bidders searches more intensively and accepts lower reserve prices for stopping. We also solve the efficient mechanisms and provide relevant comparative results.

Keywords: search auction; deadline; sample sizes; reserve prices; adaptive search
JEL classification: D44; D82; D83

1 Introduction

Time constraint is a common concern in many search problems. A consumer searches on Amazon for a Christmas gift, a seller searches for buyers to allocate a perishable good, an employer interviews candidates to fill a position by a deadline, a football club has to sign new players by the end of a transfer window, the board of directors (BOD) searches for potential acquirers in a complex and multi-step M&A process, and so on. This article investigates how a seller should search for potential bidders to allocate a product by a deadline. The bidders' product values are unknown to the seller, and they cannot bid if not invited. To invite a bidder, the seller needs to pay a search cost, and she may invite the bidders batch by batch

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in each period. What would be the seller's optimal selling mechanism if the objective is to maximize the expected profit, i.e., the expected product revenue net of gross search costs? One may observe that, due to the presence of a deadline, a one-by-one sequential search may not be optimal; and due to the presence of a search cost, a full-scale simultaneous transaction, e.g., inviting bidders only in one period, may not be optimal either, as the seller would value the inter-temporal release of bidder information.

This setting encompasses many important situations. For example, a well-known puzzle in M&As is that the dominant selling process is non-competitive negotiation, e.g., when selling a company, the BOD usually contacts just one bidder at the beginning (Andrade et al., 2001; Betton et al., 2008). It is against the conventional wisdom that a seller benefits from the competition among bidders as it will raise the bid premium. Some researchers have argued that the M&A process can be thought of as a multi-period search for potential bidders, e.g., if a deal is not agreed at the negotiation stage, the seller will invite more bidders to join the competition in later stages. As a result, the pressure of following-up auctions will drive up the premium of the early negotiation stage (Aktas et al., 2010). This search argument is supported by the empirical evidence that there is no significant difference in bid premiums across negotiation and auction in M&As (Boone and Mulherin, 2007, 2008).

There are many other real-world examples. In academic hiring, UK universities commonly run several rounds of recruitment campaigns and increase their recruiting intensities when the Research Excellence Framework (REF) deadline gets closer. In public procurement, a government agency needs to screen and contact potential contractors, and complete the procurement auction by a deadline. In dating markets, men and women attend speed-dating events regularly and search for potential partners, arguably under age pressures. When exploring price quotes, a customer searches sequentially for good or service quotes offered by potential suppliers, usually under a time constraint.

In this article, we develop a tractable framework to analyse this category of problems in a seller-bidder context. In our model, a seller wants to allocate an indivisible product among a large number of potential bidders within T periods. Bidders are ex-ante homogeneous, with their product values being independent draws from a common distribution. Bidders can only bid if invited. To invite a bidder, the seller needs to pay a constant search cost c, and she needs to decide the sample size (i.e., the number of bidders to invite) and the stage mechanism in each period. The seller's objective is to maximize the expected profit.

We fully characterize the seller's optimal search rules in both cases of *short-lived* and *long-lived* bidders,[1] and show that the optimal mechanisms can be implemented by a sequence of second-price auctions with properly selected reserve prices and sampling rules. For long-lived bidders, it is optimal to set a constant reserve price until the second-last period, but lower it to the optimal reserve price for static auctions in the last period. The optimal sample size is increasing over time and decreasing in the seller's fallback revenue *ceteris paribus*. As the realizations of fallback revenues are random, the sequence of optimal sample sizes is generally a random process. For short-lived bidders, the sequence of optimal reserve prices is decreasing, and that of optimal sample sizes is increasing over time. Both sequences are deterministic and pre-determined, which is different from long-lived bidders. This is because, with short-lived bidders, if the seller declines the current offers and continues to search, her fallback revenue always turns to 0. Therefore, the optimal reserve prices and the optimal sample sizes depend only on the number of remaining periods.

We provide some comparative results between long and short-lived bidders. First, the optimal reserve price is lower for short-lived bidders than for long-lived bidders. This is because a seller with short-lived bidders has a lower fallback revenue (0 indeed) if she continues to search, and hence, she is willing to accept a lower cutoff revenue. We call this *discouragement effect* of short-lived bidders, as it discourages the seller from further searching. Second, conditional on search, a seller with short-lived bidders searches more intensively (i.e., a larger sample size) than one with long-lived bidders. Again, this is because the fallback revenue turns to 0 if the seller continues to search, and a lower fallback revenue increases marginal search revenue and encourages her to search more intensively. We call this *encouragement*

[1] A short-lived bidder only participates in the stage transaction when invited, yet a long-lived bidder, once invited, will stay in the transaction till the deadline. The two cases of short and long-lived bidders are analogous to sequential search with *no* and *full* recall, respectively.

effect of short-lived bidders, as it encourages the seller to search more intensively if she continues to search.

We further consider efficient search rules, where the seller's objective is to maximize the expected welfare which is the winner's expected product value net of gross search costs. Similarly, the efficient mechanisms can be implemented by a sequence of second-price auctions. We demonstrate that an optimal search auction is featured by over-searching, in the sense that the cutoff product value for optimal stopping is higher than that of an efficient auction. It also indicates that the inefficiency of an optimal search auction stems from its inefficient search rule. This result is robust in both cases of long and short-lived bidders and echoes similar results in static search auctions (Szech, 2011; Xu and Li, 2019).

Finally, we show the convergence result that, when $T \to \infty$, the optimal search rules and values will converge to the standard results of the stationary and infinite horizon (SIH) search problems. To be specific, in both cases of long and short-lived bidders, the cutoff revenues for optimal stopping converge to the same constant, the optimal search profits converge to the same value, and the optimal sampling rules are both one-by-one sequential search.

The remainder of this article is organized as follows. Section 2 reviews the literature. Section 3 sets up the model. Section 4 solves the optimal search mechanism with long-lived bidders. Section 5 solves the optimal search mechanism with short-lived bidders. Section 6 further characterizes the efficient search mechanisms in both cases of long and short-lived bidders. Section 7 concludes. All proofs appear in the Appendix.

2 Related Literature

Our work is mainly related to the following strands of literature: sequential search and search with a deadline, search mechanism, negotiation versus auction, sequential auctions and revenue management, and finite horizon decision-making problems.

For classic sequential search problems with *full* recall (so-called Pandora's problem), Gittins (1979) and Weitzman (1979) have fully characterized the optimal search rule. To be specific, Pandora faces a number of closed boxes, each containing a random prize; she can open the boxes sequentially, each at a search cost; her objective is to maximize the expected prize value discovered net of gross search costs. They show that a unique cutoff value can be allocated to each box, at which the searcher is indifferent between keeping that value and inspecting that box. The optimal search rule is simple: i) the searcher should inspect the boxes in the order of descending cutoff values; ii) she should stop searching whenever the value discovered is greater than the highest cutoff values of the remaining unopened boxes.[2]

When the search is bounded by a deadline, a one-by-one sequential search may no longer be optimal. Gal et al. (1981) introduces a deadline into the sequential job search model of Lippman and McCall (1976). With *no* recall, they show that the optimal search rule is featured by decreasing reservation wages and increasing search intensities over time. Morgan (1983) further studies the case of *full* recall and shows that the sequence of optimal search intensities is a stochastic process.[3] Lee and Li (2022) investigate a problem of sequential search with a deadline and continuous search effort. They solve the optimal search rules and characterize the optimal search value and intensity. This article studies the optimal search for strategic bidders by a deadline and with a discrete choice variable of sample size.

Second, our work is closely related to the literature on search mechanisms, where the targets for search are strategic agents rather than non-strategic objects like boxes. McAfee and McMillan (1988) study a procurement problem where a buyer searches for homogeneous long-lived suppliers one-by-one sequentially. They show that the optimal search mechanism is a sequential auction with a constant reserve price. Crémer et al. (2007) examine a search mechanism where a seller searches for heterogeneous long-lived bidders. Using a mechanism design approach, they prove that the optimal search mechanism problem can be reformulated as Pandora's problem. They further show that, in the case of private values and no discounting, the mechanisms can be implemented by a sequence of second-price auctions

[2]Kleinberg et al. (2017) prove that the sequential search problem *à la* Pandora can be reformulated as a static combinatorial optimization problem, and Armstrong (2017) and Choi et al. (2018) further show some important applications of this new approach in consumer search problems.

[3]Morgan and Manning (1985) investigate the problem where a searcher also chooses the number of periods for search, and present some results on the existence and properties of optimal search rules.

with decreasing reserve prices and one-by-one sequential search.[4]

Different from one-by-one sequential search, Lee and Li (2023) study a seller's compound search for homogeneous bidders, where a seller commits to a partition of T bidder-samples and searches across them sequentially. If one takes a bidder-sample as an aggregate bidder, their compound search problem can be recast as Pandora's problem as in Crémer et al. (2007). For long-lived bidders, they show that the optimal stopping cutoff for a bidder-sample is decreasing in its sample size. Therefore, the optimal search rule suggests a search order of decreasing reserve prices and increasing sample sizes over time.

However, the articles mentioned have focused on search mechanisms without an effective time constraint. For example, Lee and Li (2023) study a seller's T period sequential search across T preset bidder-samples, and there is no time constraint indeed. Moreover, their compound search mechanism is in general suboptimal in a T period search problem, as the seller commits to a set of preset bidder-samples. In other words, the seller cannot make full use of the information released from the previous search history. In this article, we forgo the commitment assumption and propose an optimal search mechanism where a seller can make fully contingent search decisions in each period.

Third, we address the persistent debate on the choice between negotiation and auction as an optimal selling mechanism. Bulow and Klemperer (2009) argue that a simultaneous auction is better than a sequential negotiation in a setting of costly entry, where an early entered bidder can make a jump-bid to deter the entries of outside bidders, which may harm the seller.[5] But the empirical evidence does not support their results in general. For example, in M&As, the dominant selling process is negotiation, not auction. This article proposes a

[4] Lauermann and Wolinsky (2016) study a common-value search auction, where an informed buyer searches for short-lived sellers one-by-one sequentially. An invited seller may partially learn the buyer's type through a noisy signal. They find that information aggregation through price is worse in this search auction than in a standard common-value auction. They attribute this failure of information aggregation to a stronger form of the winner's curse that arises with sequential search.

[5] Lu et al. (2021) study how to orchestrate costly information acquisition in an auction with a pre-short-listing stage. Bidders are initially endowed with private signals that are positively correlated to their true values, and a bidder can learn his true value by paying an entry cost. They show that, under a sequential short-listing rule, the seller admits the most efficient remaining bidder in each round, provided that his conditional expected contribution to the virtual surplus is positive.

possible explanation for this puzzle by modelling the selling process in M&As as a seller's adaptive search for bidders by a deadline.

Fourth, our work is connected to the literature on sequential auctions and revenue management. Skreta (2015) investigates optimal sequential auctions with limited commitment, where the same population participates in each round of the auction. Said (2011) studies sequential auctions of multi-unit products with changing populations. Liu et al. (2019) study sequential auctions in the case of limited commitment. Other literature on revenue management includes Board and Skrzypacz (2016) with forward-looking buyers in the case of full commitment, and Dilme and Li (2019), who study revenue management with the arrivals of strategic buyers in the case of no commitment.[6] This article studies an optimal search mechanism by a deadline with a changing population.

Finally, our work also contributes to the literature on finite horizon decision-making problems. Baucells and Zhao (2018) study a continuous time decision-making problem within a finite horizon, and formalize the notion that fatigue accumulates with effort and decays with rest. They show that the optimal effort is of a U-shape over time, which is supported by the empirical evidence from swimming competitions. Du et al. (2022) consider a seller's optimal effort management when she sells a product over a finite horizon with consumers arriving by a Poisson process. Under an all-or-nothing contract, they show that the optimal sales effort is non-monotonic with respect to the remaining time or the outstanding sales volume required to reach the target. We study a sequential search mechanism in the case of discrete time and discrete choice variable of effort level, i.e., sample size.

3 The Model

A (female) seller wants to allocate an indivisible product among a large number of ex-ante homogeneous (male) bidders. She needs to complete the transaction within T periods. A

[6] Zhang (2021) studies the optimal sequence of posted-price and auction in a sequential mechanism, where a population of short-lived bidders enters the market periodically. In each period, the seller chooses between a posted-price and an auction mechanism. He shows that, when there is a deadline and the auction cost is moderate, the optimal mechanism sequence takes the form of posted-prices then auctions.

bidder can not bid if not invited. To invite a bidder, the seller needs to pay a search cost $c > 0$.[7] Let M_t denote the bidder-sample the seller invites in period t, and $N_t := \bigcup_{\tau=0}^{t} M_\tau$ is the union of disjoint bidder-samples the seller has invited till the end of period t, with $N_0 \equiv M_0 \equiv \emptyset$. Denote m_t the cardinality (sample size) of M_t. Both the seller and the bidders are risk-neutral, and we disregard time discounting.

The seller's value of the product is 0. Bidders' product values, X_i's, are independent draws from the same distribution F on $\mathcal{X} = [0, \bar{x}]$, with its density $f > 0$ on $(0, \bar{x})$. F is common knowledge, yet the realization of X_i, denoted by x_i, is the private information of bidder i. We assume F is of increasing failure rate, and hence the virtual value

$$\psi(x) = x - \frac{1 - F(x)}{f(x)}$$

is strictly increasing. Let G be the distribution of the virtual value $V := \psi(X)$ on $[\underline{v}, \bar{v}]$ with $\underline{v} = \psi(0)$ and $\bar{v} = \psi(\bar{x}) = \bar{x}$. It then follows that

$$G(v) = \Pr[V \leq v] = \Pr[\psi(X) \leq v] = F(\psi^{-1}(v)). \tag{1}$$

Denote V^m and X^m the largest order statistics of m independent draws from G and F respectively, which follow the distributions of G^m and F^m, with $V^0 \equiv \underline{v}$ and $X^0 \equiv 0$.

We consider both cases of long and short-lived bidders. A long-lived bidder, once invited, will stay in the transaction till the deadline. Yet a short-lived bidder will only participate in the stage transaction when invited, and then drops out. Apparently, for long and short-lived bidders, the sets of participating bidders in period t are N_t and M_t, respectively.

4 Optimal Search Auctions: Long-lived Bidders

An optimal search mechanism is one that maximizes the seller's expected search profit, i.e., the expected product revenue net of gross search costs. We can derive the optimal mechanism by backward induction. Specifically, at the beginning of period T, suppose the seller has a

[7]Instead of a constant marginal cost c, introducing a non-decreasing marginal cost function $c(m)$ does not change the results qualitatively.

fallback revenue $v \geq 0$ from the set N_{T-1} of already invited bidders, which is reclaimable till the deadline as bidders are long-lived.[8] If the seller stops, she can claim the fallback revenue v. If she continues to search, due to Myerson (1981), the optimal period-T mechanism is a second-price auction with a reserve price $\psi^{-1}(v)$. In this case, the expected auction revenue of inviting m bidders, denote by $R_T(m; v)$, is then

$$R_T(m; v) := \mathbb{E} \max\{v, \psi(X^m)\} = \mathbb{E} \max\{v, V^m\} = v + \int_v^{\bar{v}} (1 - G(z)^m) dz, \tag{2}$$

and the marginal revenue of inviting one more bidder is

$$MR_T(m; v) := R_T(m + 1; v) - R_T(m; v) \quad = \quad \int_v^{\bar{v}} G(z)^m (1 - G(z)) dz, \tag{3}$$

which is strictly decreasing both in m and v, and converges to 0 as $m \to \infty$.

If $MR_T(0; v) \leq c$, the seller will stop and allocate the product among the set N_{T-1} of bidders. We denote the cutoff revenue for optimal stopping by v^0, which is given by

$$MR_T(0; v^0) = \int_{v^0}^{\bar{v}} (1 - G(z)) dz = c. \tag{4}$$

If $v < v^0$, the seller will keep on inviting bidders as long as the marginal revenue is greater than the marginal cost. Denote the optimal sample size, i.e., the optimal number of new bidders to invite, in period T by $m_T^*(v)$. It then follows that

$$m_T^*(v) = \min\{m \in \mathbb{N}_0 : MR_T(m; v) \leq c\}, \tag{5}$$

where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The properties of $MR_T(m; v)$ guarantee the existence of $m_T^*(v)$. Given a fallback revenue v at the beginning of period T, let $\Pi_T(v)$ be the expected profit achieved in an optimal search mechanism. It then follows that

$$\Pi_T(v) = \max\left\{v, \max_{m \in \mathbb{N}_0} \{\mathbb{E} \max\{v, V^m\} - cm\}\right\}.$$

[8] To be concrete, with the set N_{T-1} of participating bidders, the seller can guarantee a fallback revenue

$$v = \max\left\{0, \max_{i \in N_{T-1}} \psi(x_i)\right\} \geq 0$$

using the second-price auction with a reserve price of $\psi^{-1}(0)$.

Constructing the problem recursively, we have the Bellman equation as follows: for $t \leq T$

$$\Pi_t(v) = \max\left\{v, \max_{m\in\mathbb{N}_0}\{\mathbb{E}\Pi_{t+1}(\max\{v, V^m\}) - cm\}\right\}, \tag{6}$$

and $\Pi_{T+1}(v) = v$ as the seller keeps whatever she has after the deadline T. It is easy to show $\Pi_t(v)$ is strictly increasing in v for any $t \leq T$, using a standard induction argument.

Similarly, we define the expected revenue of inviting m bidders in any period $t \leq T$ by

$$R_t(m; v) := \mathbb{E}\Pi_{t+1}(\max\{v, V^m\}), \tag{7}$$

and the corresponding marginal revenue by $MR_t(m; v) := R_t(m + 1; v) - R_t(m; v)$. Lemma 1 below gives some basic properties of the marginal revenue.

Lemma 1. *For any $t \leq T$, the marginal revenue $MR_t(m; v)$ is*

i) strictly positive for any $v < \bar{v}$;

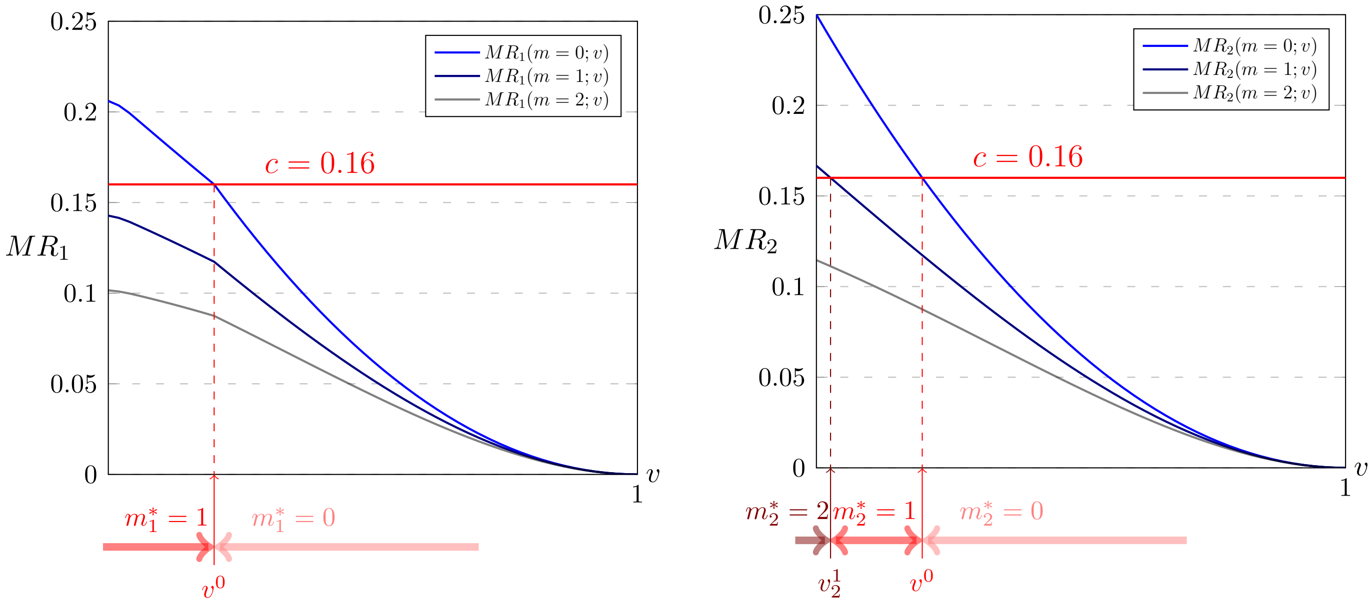
ii) strictly decreasing in v; and

iii) strictly decreasing in m and $\lim_{m\to\infty} MR_t(m; v) = 0$.

These properties are intuitive. For example, ii) simply suggests that, when the fallback revenue v gets higher, it becomes less likely to achieve a higher revenue by inviting one more bidder. Result i) and iii) state that auction revenue increases in the number of bidders yet at a diminishing rate, and in the limit when there are an infinite number of bidders, the marginal benefit of inviting one more bidder is negligible. Example 1 and Figure 1 below illustrate the properties in Lemma 1

Given a fallback revenue v at the beginning of period t, similar to (5), the seller will keep on inviting bidders as long as the marginal revenue $MR_t(m; v)$ is greater than the marginal cost c. The existence of an optimal sample size $m_t^*(v)$ is guaranteed by the properties of $MR_t(m; v)$ in Lemma 1. Solving the problem recursively, Theorem 1 specifies the optimal search rule and the optimal search profit in the case of long-lived bidders.

Figure 1: Optimal search rule $m_t^*(v)$ in Example 1



(a) Period 1 (b) Period 2

Note. $c = 0.16$, $F(x) = x\mathbb{1}(0 \le x \le 1)$, and $T = 2$. In period 1, the optimal number of invitees is $m_1^*(v) = \mathbb{1}(v < v^0)$, where $v^0 = 0.2$. In period 2, the optimal number of invitees is $m_2^*(v) = \mathbb{1}(v < v_2^1) + \mathbb{1}(v < v^0)$, where $v_2^1 \approx 0.0267$.

Theorem 1 (Optimal search for long-lived bidders)**.** *Let v be the seller's fallback revenue at the beginning of period $t \le T$. The optimal sample size $m_t^*(v)$ in period t is given by*

$$m_t^*(v) = \min\{m \in \mathbb{N}_0 : MR_t(m; v) \le c\}, \tag{8}$$

where the marginal revenue $MR_t(m; v)$ is recursively determined by

$$MR_t(m; v) = \int_v^{\bar{v}} G(z)^m (1 - G(z)) \prod_{\tau \ge t+1} G(z)^{m_\tau^*(z)} dz. \tag{9}$$

It is optimal to stop in period t if $m_t^*(v) = 0$. The optimal search profit $\Pi_t(v)$ is

$$\Pi_t(v) = \mathbb{E}\max\{v, V_t^*\} = v\mathcal{G}_t(v) + \int_v^{\bar{v}} z d\mathcal{G}_t, \tag{10}$$

where V_t^* is the continuation search profit achieved by following the optimal search mechanism from period t on, and its distribution function $\mathcal{G}_t(v)$ is recursively defined by

$$\mathcal{G}_t(v) = G(v)^{m_t^*(v)} \mathcal{G}_{t+1}(v) = \prod_{\tau \ge t} G(v)^{m_\tau^*(v)}. \tag{11}$$

Remark 1. It is clear from Lemma 1 that the optimal sample size $m_t^*(v)$ given by (8) is uniquely determined and decreasing in v. Therefore, for each $m = 0, 1, \dots, m_t^*(0) - 1$, there exists a unique cutoff revenue, denoted by v_t^m, that solves $MR_t(m; v) = c$. When $v \geq v_t^0$, $m_t^*(v) = 0$ and it is optimal to stop as shown in Bellman equation (6). Importantly, Theorem 1 also suggests that, in any period $t \leq T$, the cutoff revenue for optimal stopping $v_t^0 = v^0$ which is constant over time.[9] We then have the following explicit expression of the optimal sample sizes in any period $t \leq T$

$$m_t^*(v) = \sum_{m=1}^{m_t^*(0)} \mathbb{1}(v < v_t^{m-1}) = \begin{cases} 0 & \text{if } v \geq v^0 \\ 1 & \text{if } v_t^1 \leq v < v^0 \\ 2 & \text{if } v_t^2 \leq v < v_t^1 \\ \vdots & \\ m_t^*(0) & \text{if } v < v_t^{m_t^*(0)-1}. \end{cases} \tag{12}$$

Remark 2. The marginal revenue $MR_t(m, v)$ is decreasing in v by Lemma 1 and increasing in t by (9) as $G(z)^{m_\tau^*(z)} \leq 1$. It follows that the optimal sample size $m_t^*(v)$ is decreasing in v and increasing in t. Therefore, other things being equal, the seller will invite the smallest number of bidders at the beginning. The result explains why negotiation can be a dominant selling process in many important markets, such as M&As. Furthermore, for long-lived bidders, the time-invariant property of cutoff revenue v^0 for optimal stopping also explains why there is no significant difference in bid premiums across negotiation and auction, i.e., the seller has the same reserve price for stopping no matter if it is in the early negotiation stage or the later auction stages.

Remark 3. By (10), the optimal search profit $\Pi_t(v)$ is increasing and convex in v and decreasing in t. To be specific:

i) For $v \in [0, v^0)$, $\Pi_t(v)$ is strictly convex, and $\Pi_t(v) > v$ and $\Pi_t(v) > \Pi_{t+1}(v)$

ii) For $v \in [v^0, \bar{v}]$, $\Pi_t(v) = v$.

[9]For instance, in period $T-1$, if the seller's fallback revenue v is greater than or equals to v_T^0, then it would be her last period of search, as she will stop searching in period T given her next period fallback revenue will be no less than v. The proof of Theorem 1 formally shows this property by induction.

In other words, the optimal search profit $\Pi_t(v)$ is larger if the fallback revenue v, which she can reclaim in the case of stopping, is larger, or if it is in an early period (smaller t) such that the seller will have more opportunities of search to increase her expected profit. One may also observe that the distributions of V_t^* are ordered in terms of first order stochastic dominance, i.e., $\mathcal{G}_{t+1}(v) \geq \mathcal{G}_t(v)$, which implies that $\Pi_t(v) > \Pi_{t+1}(v)$.

Sequential Second-Price Auction Implementation

The optimal mechanisms with long-lived bidders, as specified in Theorem 1, can be implemented by a sequence of second-price auctions with properly selected reserve prices $\{r_t\}_{t=1}^T$ and sampling rules $\{m_t^*(v)\}_{t=1}^T$. To be specific,

- At the beginning of any period $t \leq T$, the seller's fallback revenue

$$v = \max\left\{0, \psi^{-1}\left(\max_{i \in N_{t-1}} x_i\right)\right\}$$

  is given by the auction outcome in period $t-1$ with the set N_{t-1} of invited bidders.

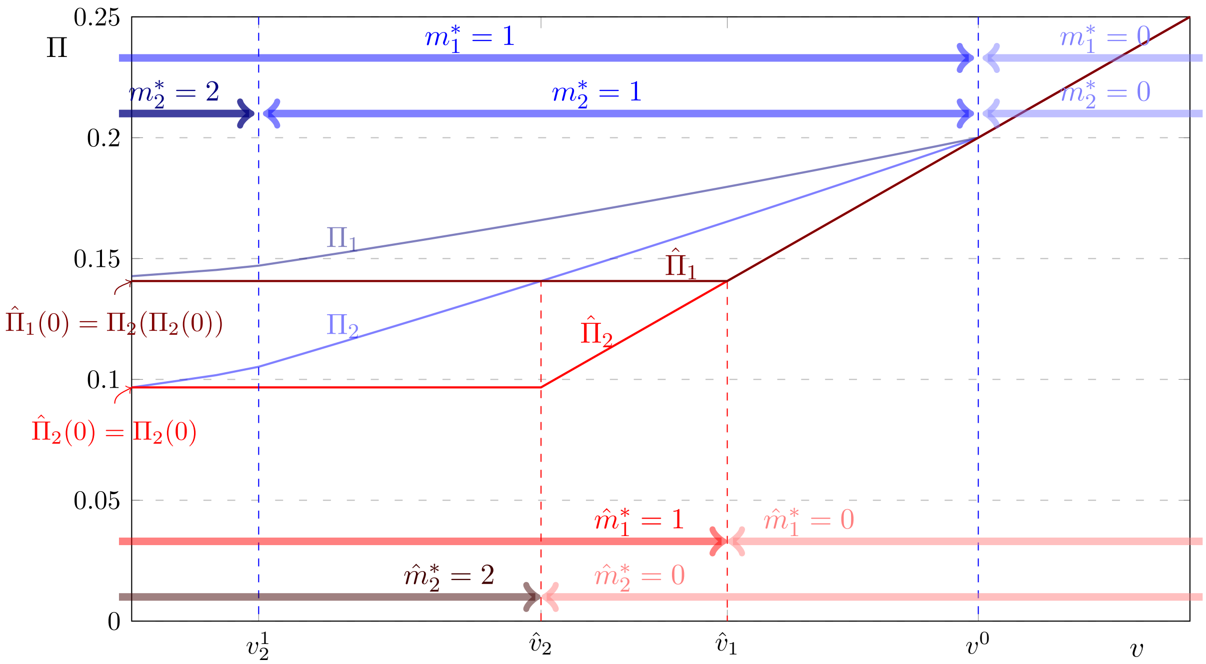
- Given v, the seller invites $m_t^*(v)$ new bidders in period t and runs a second-price auction with a reserve price of

$$r_t = \begin{cases} \psi^{-1}(v^0) & \text{if } t < T \\ \psi^{-1}(0) & \text{if } t = T. \end{cases}$$

  Note that any invited bidder will participate and to bid his true product value is a weakly dominant strategy, regardless of when he has been invited.

A key feature of the optimal search auction is that the reserve price r_t remains at the same constant level of $\psi^{-1}(v^0)$ through the periods of $t = 1, \cdots, T-1$, and drops to the optimal reserve price for static auctions, i.e., $\psi^{-1}(0)$, only in the last period T. Therefore, at the end of any period $t < T$, if any bid from the set N_t of bidders is greater than the reserve $r_t = \psi^{-1}(v^0)$, the seller can guarantee a revenue higher than v^0 and it is optimal for her to stop and allocate the product among the set N_t of bidders. In the last period T, the optimal reserve price drops to $r_T = \psi^{-1}(0)$, as 0 is the revenue that the seller can guarantee

Figure 2: Optimal search rules and profits



Note. $c = 0.16$, $F(x) = x\mathbb{1}(0 \le x \le 1)$, and $T = 2$. Π_t is the search profit with long-lived bidders from the optimal search rule m_t^* (presented in blue); whereas $\hat{\Pi}_t$ is that with short-lived bidders from the corresponding optimal search rule $\hat{m}_t^*$ (presented in red).

by retaining the product. It is also interesting to observe the fallback revenue v will only affect the optimal sample size $m_t^*(v)$, yet not the reserve price r_t for optimal stopping.

At the end of this section, we provide a uniform example of two-period search auction. It also shows how to construct the optimal search auction for long-lived bidders by a deadline.

Example 1 (Optimal search auction with long-lived bidders)**.** Consider $c = 0.16$, $F(x) = x\mathbb{1}(0 \le x \le 1)$, and $T = 2$. The virtual value $\psi(x) = 2x - 1$ has the distribution $G(v) = \frac{1}{2}(v+1)\,\mathbb{1}(-1 \le v \le 1)$. Backward induction yields the following optimal search rule.

- In the last period $t = 2$, the marginal search revenue is

$$MR_2(m; v) = \int_v^1 G(z)^m(1 - G(z))dz = \frac{2^{m+2} - (1+v)^{m+1}(m+3-v(m+1))}{2^{m+1}(m+1)(m+2)},$$

  which is illustrated in Figure 1(b) for $m = 0, 1, 2$. As $MR_2(2; 0) \approx 0.1146 < c$, the optimal sample size should not be greater than 2 from (12). For $m = 0, 1$, the solution to $MR_2(m; v) = c$ gives the cutoff values $v^0 = v_2^0 = 0.2$ and $v_2^1 \approx 0.02667$. The optimal sample size in period $t = 2$ is then

$$m_2^*(v) = \mathbb{1}(v < v_2^1) + \mathbb{1}(v < v^0). \tag{13}$$

Given $m_2^*(v)$, we have $\mathcal{G}_2(v) = G(v)^{m_2^*(v)}$ and the optimal search profit in period $t = 2$

$$\Pi_2(v) = \mathbb{E}\max\{v, V_2^*\} = v\mathcal{G}_2(v) + \int_v^{\bar{v}} z d\mathcal{G}_2. \tag{14}$$

- In the first period $t = 1$, the marginal search revenue, by (9), is

$$MR_1(m; v) = \int_v^1 G(z)^{m+m_2^*(z)}(1 - G(z))dz.$$

Using (13), we have the following piecewise expressions of $MR_1(m; v)$:

i) if $v < v_2^1$,

$$MR_1(m; v) = \int_v^{v_2^1} G(z)^{m+2}(1-G(z))dz + \int_{v_2^1}^{v^0} G(z)^{m+1}(1-G(z))dz + \int_{v^0}^1 G(z)^m(1-G(z))dz;$$

ii) if $v_2^1 \leq v < v^0$,

$$MR_1(m; v) = \int_v^{v^0} G(z)^{m+1}(1 - G(z))dz + \int_{v^0}^1 G(z)^m(1 - G(z))dz; \text{ and}$$

iii) if $v \geq v^0$,

$$MR_1(m; v) = \int_v^1 G(z)^m(1 - G(z))dz,$$

which is illustrated in Figure 1(a), for $m = 0, 1, 2$. Observe that for any m and any $v \geq v^0$, $MR_2(m, v) = MR_1(m, v)$, which suggests that the cutoff revenue for optimal stopping is v^0 which does change over time. If $v < v^0$, the optimal sample size $m_2^*(v) \geq 1$ in period 2 makes $MR_1(m; v) < MR_2(m; v)$. That is, the marginal search revenue is increasing over time, and the seller searches more intensively as the deadline approaches. This deadline effect of $m_1^*(v) \leq m_2^*(v)$ is illustrated by Figure 2 (presented in blue) or comparing Figure 1(a) and Figure 1(b). As $MR_1(1; 0) \approx 0.1428 < c$, the optimal sample size is at most 1, and it follows

$$m_1^*(v) = \mathbb{1}(v < v^0).$$

Given $m_1^*(v)$ and $m_2^*(v)$, the continuation search profit V_1^* has the following distribution

$$\mathcal{G}_1(v) = G(v)^{m_1^*(v)+m_2^*(v)} = \begin{cases} G(v)^3 & \text{if } v < v_2^1 \\ G(v)^2 & \text{if } v_2^1 \leq v < v^0 \\ G(v)^0 = 1 & \text{if } v \geq v^0. \end{cases}$$

Given an initial fallback revenue of 0, the seller's optimal search profit is then

$$\Pi_1(0) = \mathbb{E}\max\{0, V_1^*\} = \int_0^1 z d\mathcal{G}_1 = v^0 - \int_{v_2^1}^{v^0} G(z)^2 dz - \int_0^{v_2^1} G(z)^3 dz \approx 0.1427.$$

The optimal search profits for each period $t = 1, 2$ are plotted in Figure 2 (in blue).

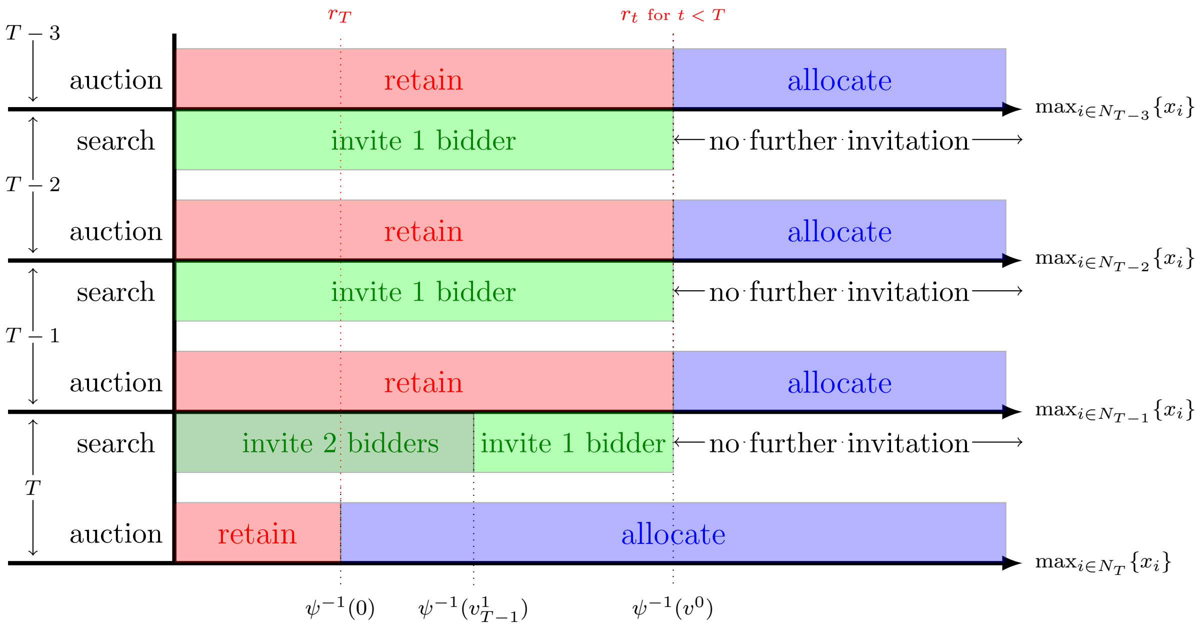
The optimal search profit can be implemented by the following sequence of second-price auctions. Given a fallback revenue 0 at the beginning of period $t = 1$, the seller invites one bidder (bidder 1) and runs a second-price auction with a reserve price $\psi^{-1}(v^0) = 0.6$. Bidder 1 will bid truthfully. If bidder 1's bid is greater than or equals to $\psi^{-1}(v^0)$, the seller then allocate the product to him at the reserve price $\psi^{-1}(v^0)$. Otherwise, the seller continues to search in period $t = 2$ with the following optimal sample size $m_2^*(v)$:

- invite two more bidders if bidder 1's bid is lower than $\psi^{-1}(v_2^1) = 0.5133$;
- invite one more bidder if bidder 1's bid is between $\psi^{-1}(v_2^1) = 0.5133$ and $\psi^{-1}(v^0) = 0.6$.

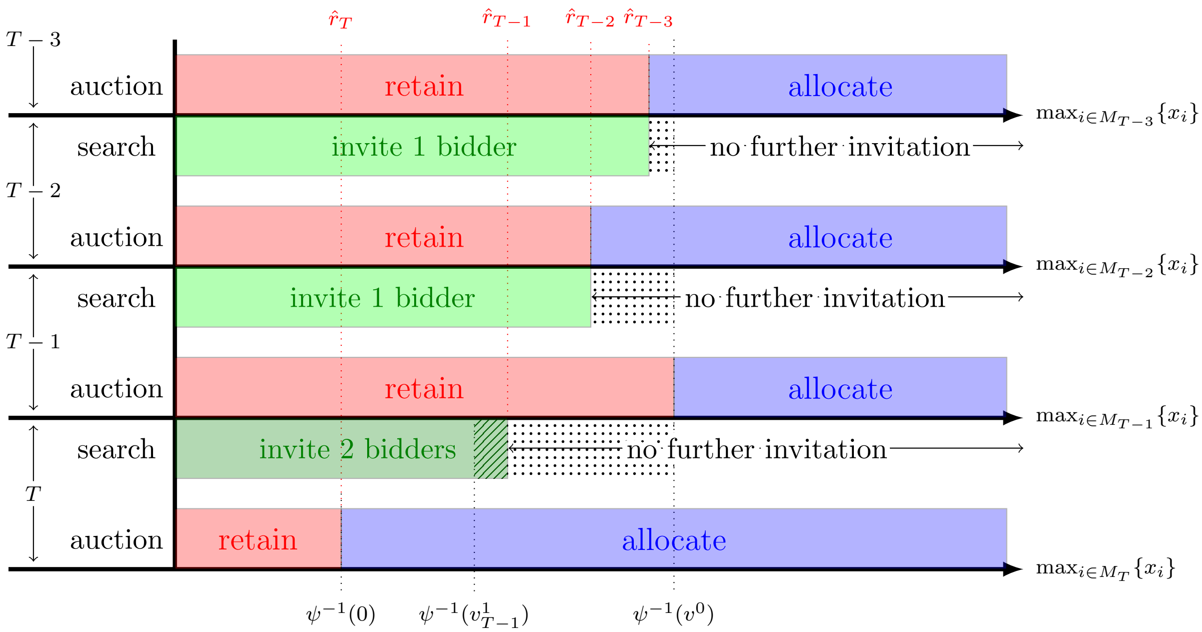
In period 2, the seller runs a second-price auction with a reserve price of $\psi^{-1}(0) = 0.5$.

The two-period example can be extended to any finite T-period problems. Note that, in any finite period problem, we have $MR_t(m; v) \leq MR_{t+1}(m; v)$ and hence $m_t^*(v) \leq m_{t+1}^*(v)$. In this particular case of $c = 0.16$ and $F(x) = x\mathbb{1}(0 \leq x \leq 1)$, we have $m_{T-1}^*(0) = 1$. It then follows that in any finite period problem, $m_t^*(0) = 1$ for any $t < T$, i.e., the optimal sampling rule is one-by-one invitation in any period $t < T$. We can similarly construct a sequence of second-price auctions to implement the optimal search profit. Specifically, at the beginning of any period $t < T$, if the highest ever bid is smaller than $\psi^{-1}(v^0) = 0.6$, the seller then invites one more bidder in period t and asks him to bid, who will bid his true value. At the beginning of the last period T, the optimal sampling rule is given as in the above two-period example and the optimal reserve price is $r_T = \psi^{-1}(0) = 0.5$.

Figure 3: Optimal Search Auctions with a Deadline



(a) Long-lived Bidders: The reserve price r_t is constant except for the last period T, as the cutoff for stopping is fixed with long-lived bidders.



(b) Short-lived Bidders: The reserve price $\hat{r}_t$ is determined by the cutoff for optimal stopping in the next period, and is decreasing over time. Compared to (a), the dotted areas capture the discouragement effect of short-lived bidders, and the green slashed area in period T highlights the encouragement effect.

This optimal search auction is illustrated in Figure 3(a), where the vertical axes represents different periods and the horizontal one is the highest bid $\max_{i\in N_t}\{x_i\}$ at the end of each period t. Observe that the optimal reserve price r_t remains at the constant $\psi^{-1}(v^0)$ throughout all the periods $t < T$ and drops to $\psi^{-1}(0)$ only in the last period T. In our example, the optimal search auction runs as follows. At the beginning of any period $t < T$, if the highest bid of the set N_{t-1} of participating bidders in period $t-1$ is greater than r_{t-1}, the seller then allocates the product to the winning bidder in N_{t-1}; otherwise, the seller invites one new bidder and runs an auction with the reserve price r_t. Particularly, at the beginning of the last period T, if the highest bid of the set N_{T-1} of bidders is greater than r_{T-1}, the seller then stops and allocates the product; otherwise, she will runs an auction with a new reserve price r_T by inviting either one or two more new bidders, depending on the value of $\max_{i\in N_{T-1}}\{x_i\}$, as shown in the bottom block of Figure 3(a).

5 Optimal Search Auctions: Short-lived Bidders

For short-lived bidders, we can similarly derive the optimal search mechanism by backward induction. At the beginning of period T, suppose the seller has a fallback revenue v from the set M_{T-1} of bidders. If she stops, she can claim the revenue v. If she continues to search, her fallback revenue turns to 0 as bidders are short-lived, and she no longer has an opportunity to search after the deadline T. Therefore, the optimal period-T mechanism is a second-price auction with a reserve price $\psi^{-1}(0)$, as her reservation revenue is 0. For short-lived bidders, the expected revenue of inviting m bidders in period T is thus

$$\mathbb{E}\max\{0, V^m\} = \int_0^{\bar{v}} (1 - G(z)^m)dz = R_T(m; 0),$$

where the last equality is from (2). Note that $R_T(m; 0)$ is the period-T search revenue for long-lived bidders with a fallback revenue 0, and the marginal revenue of inviting one more bidder is then $MR_T(m; 0)$, as given in (3). Hence, for short-lived bidders, if the seller continues to search in period T, the optimal sample size is $m_T^*(0)$ which is independent of the fallback revenue v at the beginning of period T.

For short-lived bidders, let $\hat{v}_T$ be the continuation search profit in period T, and

$$\hat{v}_T := \max_{m\in\mathbb{N}_0}\{\mathbb{E}\max\{0,V^m\} - cm\} = R_T(m_T(0);0) - c\cdot m_T(0) = \Pi_T(0).$$

In this case, the optimal search profit in period T, denoted by $\hat{\Pi}_T(v)$, is then

$$\hat{\Pi}_T(v) = \max\left\{v, \max_{m\in\mathbb{N}_0}\{\mathbb{E}\max\{0,V^m\} - cm\}\right\} = \max\{v, \hat{v}_T\}.$$

Constructing the problem recursively, we have the following Bellman equation: for $t \leq T$

$$\hat{\Pi}_t(v) = \max\left\{v, \max_{m\in\mathbb{N}_0}\{\mathbb{E}\hat{\Pi}_{t+1}\left(\max\{0,V^m\}\right) - cm\}\right\}, \tag{15}$$

and $\hat{\Pi}_{T+1}(v) = v$ obviously. Note that, for short-lived bidders, the new state variable at the beginning of period $t+1$ is the realization of $\max\{0,V^m\}$.

Theorem 2 gives the optimal search rule for short-lived bidders, which is featured by decreasing cutoff values $\hat{v}_t$ for optimal stopping and increasing sample sizes $\hat{m}_t^*$ over time.

Theorem 2 (Optimal search for short-lived bidders)**.** *Let v be the seller's fallback revenue at the beginning of period $t \leq T$. It is optimal for the seller to stop searching in period t if and only if $v \geq \hat{v}_t$, where the cutoff value $\hat{v}_t$ for stopping is recursively determined by*

$$\hat{v}_t = \Pi_T(\hat{v}_{t+1}) = \Pi_T(\Pi_T(\hat{v}_{t+2})) = \cdots = \underbrace{\Pi_T\circ\cdots\circ\Pi_T}_{T-t\ times}(\hat{v}_T) = \underbrace{\Pi_T\circ\cdots\circ\Pi_T}_{T-t+1\ times}(0), \tag{16}$$

with $\hat{v}_{T+1} \equiv 0$. Moreover, $\hat{v}_{t+1} < \hat{v}_t < v^0$ and $\lim_{T\to\infty}\hat{v}_1 = v^0$. The optimal sample sizes $\hat{m}_t^*$ in period t is given by

$$\hat{m}_t^* = m_T^*(\hat{v}_{t+1})\mathbb{1}(v < \hat{v}_t), \tag{17}$$

and $\hat{m}_t^*$ is increasing over time. The optimal search profit is

$$\hat{\Pi}_t(v) = \max\{v, \hat{v}_t\}. \tag{18}$$

Remark 4. The recursive relation (16) reveals an interesting link between the optimal search for short-lived and long-lived bidders, copied as follows

$$\hat{v}_t = \max_{m\in\mathbb{N}_0}\left\{\mathbb{E}\max\{\hat{v}_{t+1},V^m\} - mc\right\} = \Pi_T(\hat{v}_{t+1}).$$

Note that $\hat{v}_t$ is the continuation search profit in period t, which is achieved by choosing the optimal sample size $\hat{m}_t^*$ in period t and then following the optimal search rule till the deadline. Equation (16) establishes a recursive relation between $\hat{v}_t$ and $\hat{v}_{t+1}$ through the optimal search profit function Π_T. To be specific, it states that, when a searcher with short-lived bidders decides to search in period t, she behaves as if she is searching for long-lived bidders in the last period T, yet with a fallback revenue $\hat{v}_{t+1}$.

Remark 5. It is intuitive that the continuation search profit $\hat{v}_t$ is decreasing over time, as the searcher has fewer search opportunities to improve her payoff when the deadline gets closer. It also implies that the optimal sample size $\hat{m}_t^*$ is increasing over time, as the marginal search revenue $MR_T(m; \hat{v}_{t+1})$ gets larger when $\hat{v}_{t+1}$ decreases over time.

The monotone properties of the optimal search rule fit many real-world observations. For example, in M&As, if bidders are short-lived, a seller will contact the smallest number of bidders and has the highest reserve price at the beginning. Therefore, it may explain why negotiation is the dominant selling process in M&As. In academic recruiting, UK universities generally increase their recruitment intensities when the REF deadline approaches. Naturally, job candidates on a waiting list may no longer be available with time going on, and therefore can be thought of short-lived candidates. Our results may explain the dynamics of recruitment intensities observed in the UK academic markets.

Remark 6. For short-lived bidders, the sequences of optimal cutoffs for stopping and optimal sample sizes are both deterministic and predetermined. This is because, when the seller continues to search, her fallback revenue always turns to 0, and therefore, a declined fallback value will no affect her decisions in the following periods. Rather, both sequences are determined by the number of remaining periods. This is different from the case of long-lived bidders, where the sequence of optimal sample sizes is a random process, as the optimal sample size $m_t^*(v)$ depends on the realization of the fallback value v, which is random.

Remark 7. Theorem 2 also provides an algorithm for deriving the sequences of $\{\hat{v}_t\}_{t=1}^T$ and $\{\hat{m}_t^*\}_{t=1}^T$. Specifically, in period T, the continuation search profit $\hat{v}_{T+1} = 0$, and we can solve for the optimal sample size $\hat{m}_T^* = m_T^*(0)$ using (5). With $\hat{m}_T^*$, we then can calculate the continuation search profit of $\hat{v}_T$ using (16). Moving one period earlier, in period $t = T - 1$, we can again derive the optimal sample size $\hat{m}_{T-1}^* = m_T^*(\hat{v}_T)$ using (5), and then get the continuation search profit $\hat{v}_{T-1}$ using (16). Continuing with the process, we then fully solve the optimal search rule for short-lived bidders.

Sequential Second-Price Auction Implementation

The optimal mechanisms with short-lived bidders, as specified in Theorem 2, can be implemented by a sequence of second-price auctions with properly selected reserve prices $\{\hat{r}_t\}_{t=1}^T$ and sample sizes $\{\hat{m}_t^*\}_{t=1}^T$. To be specific,

- At the beginning of any period $t \leq T$, the seller's fallback revenue

$$v = \max\left\{0, \psi^{-1}\left(\max_{i \in M_{t-1}} x_i\right)\right\}$$

  is given by the auction outcome in period $t - 1$ with the set M_{t-1} of bidders.

- If v is greater than the cutoff value $\hat{v}_t$, then the seller stops and claims the revenue v by allocating the product to the winning bidder in M_{t-1}.

- Otherwise, the seller invites $\hat{m}_t^* = m_T^*(\hat{v}_{t+1})$ new bidders in period t and runs a second-price auction with a reserve price of $\hat{r}_t = \psi^{-1}(\hat{v}_{t+1})$.

Again, in the proposed second-price auction with a reserve price, it is a weakly dominant strategy for a bidder to participate if invited and to bid his true product value. Continuing with the uniform Example 1, Example 2 below shows how to derive the optimal search rule for short-lived bidders and how to implement the optimal mechanism by a sequence of second-price auctions.

Example 2 (Optimal search auction with short-lived bidders)**.** Consider the same setting of Example 1, yet the bidders are now short-lived.

- In the last period $T = 2$, if the seller continues to search, she will choose the optimal sample size $\hat{m}_2^* = m_2^*(\hat{v}_{T+1} = 0) = 2$ by (13) in Example 1. From (16), the continuation search profit $\hat{v}_2 = \Pi_2(0) \approx 0.0967$ using equation (14). Hence, the seller's optimal search profit with a fallback revenue v at the beginning of period T is

$$\hat{\Pi}_2(v) = \max\{v, \hat{v}_2\}. \tag{19}$$

- In the first period $t = 1$, if she continues to search, she will choose the optimal sample size $\hat{m}_1^* = m_2^*(\hat{v}_2) = 1$ by (13) again. The continuation search profit $\hat{v}_1 = \Pi_2(\hat{v}_2) \approx 0.1407$ using equation (14). Hence, the seller's optimal search profit with a fallback revenue 0 at the beginning of period 1 is

$$\hat{\Pi}_1(0) = \max\{0, \hat{v}_1\} = \hat{v}_1.$$

The optimal sample sizes $\hat{m}_t^*$ and the search profits $\hat{\Pi}_t$ are plotted in Figure 2 (in red).

The optimal search profit can be obtained by the following sequential second-price auction. At the beginning of the period $t = 1$, the seller's fallback revenue $v = 0$, which is smaller than the continuation search profit $\hat{v}_1 \approx 0.1407$. The seller will then invite $\hat{m}_1^* = m_2^*(\hat{v}_2) = 1$ bidder and run a second-price auction with a reserve price $\hat{r}_1 = \psi^{-1}(\hat{v}_2) \approx 0.5703$. If the bidder's bid $x_1 \geq \hat{r}_1$, the seller then allocates the product to him at price $\hat{r}_1$; otherwise, the seller continues to search in period 2, by inviting $\hat{m}_2^* = m_2^*(0) = 2$ new bidders and running a second-price auction with a reserve price of $\hat{r}_2 = \psi^{-1}(0) = 0.5$.

The two-period example can be easily extended to any finite T-period problems. As shown above, one can recursively derive the optimal sample size $\hat{m}_t^*$ and continuation search profit $\hat{v}_t$ from the last period T. Note that the optimal sample size $\hat{m}_t^*$ is increasing over time, and $\hat{m}_{T-1}^* = 1$ in this example. It is then follows that, at the beginning of any period $t < T$, if the seller decides to search, it is optimal to invite just one bidder. Second, the cutoff revenue for optimal stopping $\hat{v}_t$ is strictly decreasing over time. The monotone properties of optimal sample sizes and the reserve prices are illustrated in Figure 3(b).

Similarly, for short-lived bidders, we can implement the optimal search profit by a sequence of second-price auctions, as illustrated in Figure 3(b). Specifically, the optimal reserve price in any period t is $\hat{r}_t = \psi^{-1}(\hat{v}_{t+1})$, and an invited bidder will bid his true product value. As illustrated in Figure 3(b), at the beginning of any period t, if the highest bid $\max_{i \in M_{t-1}}\{x_i\}$ of the period $t-1$ participating bidders is higher than the reserve price $\hat{r}_{t-1}$, the seller then allocates the product to the winning bidder in M_{t-1}; otherwise, she will continue by inviting $\hat{m}_t^*$ new bidders in period t and run an auction at the reserve price $\hat{r}_t$. The seller runs the auctions in this way till the last period T. Figure 3(b) also illustrates that $\hat{r}_t$ is decreasing and $\hat{m}_t^*$ is increasing over time.

Long-lived vs. short-lived bidders

We here present some comparative results between long-lived and short-lived bidders. With short-lived bidders, a seller cannot reclaim previously declined offers as those bidders already drop out of the transaction. This induces two seemingly opposite effects on a seller's optimal search decisions. On the one hand, a seller with short-lived bidders is more likely to stop and accept a lower offer than one with long-lived bidders. That is, for short-lived bidders, the cutoff revenue for optimal stopping is lower than that for long-lived bidders, e.g., $\hat{v}_t < v^0$. We term this as discouragement effect, i.e., it discourages the seller from further searching by accepting a lower fallback revenue. On the other hand, conditional on search, a seller with short-lived bidders will search more intensively than one with long-lived bidders. This is because the seller's fallback revenue turns to 0 if she continues to search, and a lower fallback revenue will increase the marginal search revenue and hence the optimal sample size. We term this effect as encouragement effect which encourages the seller to search more intensively. Corollaries 1 and 2 formally summarize the two effects.

Corollary 1 (Discouragement Effect)**.** *A seller with short-lived bidders stops at a lower cutoff revenue than one with long-lived bidders in any period $t = 1, \cdots, T$. To be specific,*

$$v^0 > \hat{v}_1 > \hat{v}_2 > \cdots > \hat{v}_T > 0.$$

Corollary 2 (Encouragement Effect)**.** *Conditional on searching, a seller with short-lived bidders searches more intensively than one with long-lived bidders in any period* $t = 1, \cdots, T$*. That is, for any* $v < \hat{v}_t$*, we have* $\hat{m}_t^*(v) \geq m_t^*(v)$*.*

We can also compare the optimal search profit with long-lived and short-lived bidders. Intuitively, as a seller can reclaim a long-lived bidder's previous offer, long-lived bidders are more valuable, and hence, the optimal search profit with long-lived bidders is greater than that with short-lived bidders. We call the difference between them as the value of bidder longevity, which is shown to be positive and single-peaked.

Corollary 3 (Value of Bidder Longevity)**.** *The value of* $\Pi_t(v) - \hat{\Pi}_t(v)$ *is*

i) non-negative for all v *and* t*;*

ii) strictly increasing in $0 \leq v \leq \hat{v}_t$ *and strictly decreasing in* $\hat{v}_t \leq v \leq v^0$*.*

Example 3 (Optimal search: long vs. short-lived bidders)**.** Figure 2 summarizes the optimal search rules and profits for both cases of long and short-lived bidders, as derived in Example 1 and Example 2. The horizontal axes is the fallback revenue v, and the vertical one is the search profit. First, we observe that a seller with short-lived bidders has lower cutoffs for optimal stopping than one with long-lived bidders, i.e., $\hat{v}_2 < \hat{v}_1 < v^0$. This demonstrates the discouragement effect (Corollary 1), that is, a seller with short-lived bidders is willing to accept a lower reserve price and then stop. Second, we also observe that, conditional on searching, a seller with short-lived bidders searches more intensively. Particularly, when $v_2^1 < \hat{v}_2$, $\hat{m}_2^* = 2$ for short-lived bidders is strictly greater than $m_2^*(v) = 1$ for long-lived bidders. This reveals the encouragement effect (Corollary 2). Third, Figure 2 also illustrates that the value of bidder longevity is always positive and single-peaked. Particularly, when $v \geq v^0$, $\Pi_t(v) = \hat{\Pi}_t(v)$ as the seller always stops and keeps the fallback revenue v. When $v < v^0$, we have $\Pi_t(v) > \hat{\Pi}_t(v)$ for any t and any v, except for $\Pi_T(0) = \hat{\Pi}_T(0)$. Moreover, $\Pi_t(v) - \hat{\Pi}_t(v)$ is single-peaked, achieving its maximum at $\hat{v}_t$ (Corollary 3).

Remark 8. The results of Corollary 1 and 2 are also illustrated in Figure 3. By comparing Figure 3(a) and 3(b), first, we find that the optimal auction reserve prices for long-lived bidders are greater than those of short-lived bidders in each period, i.e., $r_t > \hat{r}_t$ for $t < T$ and $r_T = \hat{r}_T$ for $t = T$. Specifically, the dotted areas in Figure 3(b) highlight the discouragement effect (Corollary 1) that a seller with short-lived bidders is willing to accept a lower reserve price and stop in each period. Second, conditional on searching, we find that a seller with short-lived bidders will search more intensively than one with long-lived bidders, i.e., $\hat{m}_t^* > m_t^*(v)$ if $v < \hat{v}_t$. The green slashed area in period T in Figure 3(b) highlights the encouragement effect (Corollary 2) by showing that $\hat{m}_T^* = 2 > m_T(v)^* = 1$ for $v_{T-1}^1 < v < \hat{v}_T$.

A Convergence Result

Theorems 1 and 2 give the optimal search rules and values for long and short-lived bidders. When $T \to \infty$, we will show that the difference between the two cases vanishes and both converge to the stationary and finite horizon (SIH) sequential search problem.

We first consider the optimal search auction with short-lived bidders. Theorem 2 gives that the optimal cutoff value for stopping $\lim_{T\to\infty} \hat{v}_1 = v^0$. Therefore, when $T \to \infty$, for any finite t, the optimal cutoff values $\hat{v}_t = v^0$, and the optimal sample size, from (17),

$$\hat{m}_t^* = \mathbb{1}(0 \leq v < v^0),$$

that is, one-by-one sequential search is optimal. The optimal search profit is hence

$$\hat{\Pi}_t(v) = \max\{v, v^0\}.$$

We next consider the optimal search auction with long-lived bidders. When $T \to \infty$, (11) gives $\lim_{T\to\infty} \mathcal{G}_{t+1} = 0$ and hence the optimal search profit with long-lived bidders is

$$\Pi_t(v) = \max\{v, v^0\},$$

and it is achieved by one-by-one sequential search. Therefore, when $T \to \infty$, both cases converge to the optimal search rule and profits of SIH search problems.

6 Efficient Search Auctions

In many situations, the objective of a seller is not to maximize profit but social welfare. This section turns to the efficient search for bidders by a deadline, where a seller's objective is to maximize the expected welfare, which is the expected product value of the winning bidder net of gross search costs. Now the seller cares about a bidder's product value x rather than the revenue $v = \psi(x)$ she can secure from a truthful bidder.

We will derive the efficient search rules and values for both long and short-lived bidders. Similarly, the efficient mechanisms can be implemented by a sequence of second-price auctions with appropriately selected reserve prices and sampling rules. We will show that an optimal search auction is featured by over-searching in both cases of long and short-lived bidders, in terms of either higher cutoff values for stopping or greater sample sizes than those in an efficient search auction. Therefore, the inefficiency of an optimal search auction can stem from its inefficient search rule.

Long-lived Bidders

When the seller's objective is to maximize welfare, she cares about the winning bidder's product value rather than the product revenue. Let x be the fallback product value at the beginning of period t, i.e., the highest product value revealed by the N_{t-1} bidders, and $W_t(x)$ be the net search welfare achieved by following an efficient search rule from period t on.[10] Similar to (6), the Bellman equation for an efficient search problem is as follows

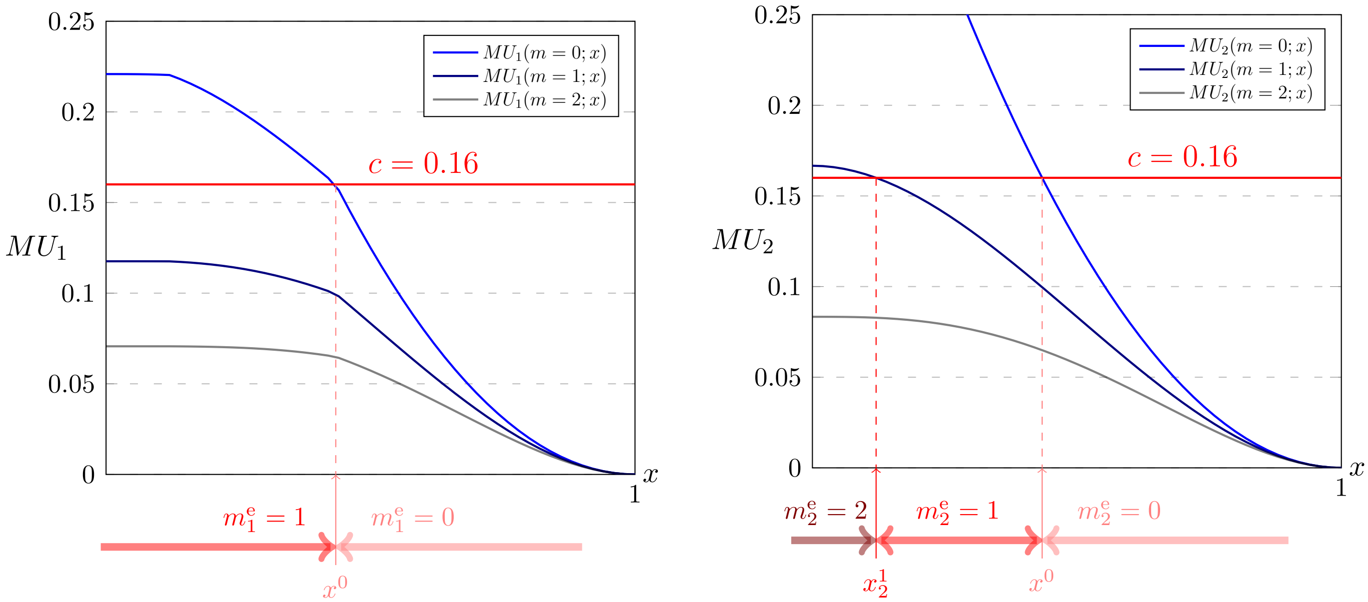
$$W_t(x) = \max\left\{x, \max_{m\in\mathbb{N}_0} \left\{\mathbb{E}W_{t+1}\left(\max\left\{x, X^m\right\}\right) - mc\right\}\right\}, \tag{20}$$

where X^m is the highest product value from a sample of m bidders, with $W_{T+1}(x) = x$. We define the expected search welfare of inviting m bidders in period t by

$$U_t(m; x) := \mathbb{E}W_{t+1}\left(\max\left\{x, X^m\right\}\right), \tag{21}$$

[10] We use x to denote the fallback product value in order to differentiate it from v for the revenue-maximizing seller.

Figure 4: Efficient search rule $m_t^{\mathrm{e}}(x)$ in Example 4



(a) Period 1

(b) Period 2

Note. $c = 0.16$, $F(x) = x\mathbb{1}(0 \le x \le 1)$, and $T = 2$. In period 1, the efficient sample size is $m_1^{\mathrm{e}}(x) = \mathbb{1}(x < x^0)$, where $x^0 = 0.4343$. In period 2, the efficient sample size is $m_2^{\mathrm{e}}(x) = \mathbb{1}(x < x_2^1) + \mathbb{1}(x < x^0)$, where $x_2^1 \approx 0.1204$.

and the marginal search welfare $MU_t(m; x) := U_t(m + 1; x) - U_t(m; x)$. It's obvious that $MU_t(m; x)$ is strictly decreasing in m and converges to 0 when $m \to \infty$, and is strictly decreasing in x. Let $m_t^{\mathrm{e}}(x)$ denote the efficient sample size in period t that maximizes the expected net search welfare $U_t(m; x) - mc$, and we have

$$m_t^{\mathrm{e}}(x) = \min \left\{ m \in \mathbb{N}_0 : MU_t(m; x) \le c \right\}. \tag{22}$$

As before, for each $0 \le m < m_t^{\mathrm{e}}(0)$, there exists a unique cutoff product value x_t^m that solves $MU_t(m; x) = c$. Moreover, the cutoff x_t^0 for stopping is constant for all $t \le T$, and hence, we denote $x^0 = x_t^0$ which is the unique solution to

$$MU_T(0; x^0) = \int_{x^0}^{\bar{x}} (1 - F(z))dz = c. \tag{23}$$

Similarly, we have the following efficient search rule and net welfare for long-lived bidders.

Proposition 1 (Efficient Search for Long-lived Bidders). *Let x be the seller's fallback product value at the beginning of period t. The efficient sample size $m_t^e(x)$ is given by*

$$m_t^e(x) = \min\{m \in \mathbb{N}_0 : MU_t(m;x) \leq c\}, \tag{24}$$

where the marginal search welfare $MU_t(m;x)$ is recursively determined by

$$MU_t(m;x) = \int_x^{\bar{x}} F(z)^m(1-F(z)) \prod_{\tau \geq t+1} F(z)^{m_\tau^e(z)} dz, \tag{25}$$

and it is efficient to stop if $m_t^e(x) = 0$. The maximum net search welfare $W_t(x)$ is

$$W_t(x) = \mathbb{E}\max\{x, X_t^e\} = x\mathcal{F}_t(x) + \int_x^{\bar{x}} zd\mathcal{F}_t, \tag{26}$$

where X_t^e is the continuation net welfare achieved by following the efficient search mechanism from period t on, and its distribution function $\mathcal{F}_t(x)$ is recursively defined by

$$\mathcal{F}_t(x) = F(x)^{m_t^e(x)}\mathcal{F}_{t+1}(x) = \prod_{\tau \geq t} F(x)^{m_\tau^e(x)}. \tag{27}$$

Remark 9. The property of $MU_t(m;x)$ guarantees that, for each $m = 0, 1, \ldots, m_t^e(0) - 1$, there exists a unique cutoff product value x_t^m that solves $MU_t(m;x) = c$. When $x \geq x_t^0$, $m_t^e(x) = 0$ and it is efficient to stop. As before, we can show $x_t^0 = x^0$ for all $t = 1, \cdots, T$, and therefore, there is a time-invariant cutoff product value for efficient stopping. Similar to (12), we provide the following explicit expression of $m_t^e(x)$ in period t:

$$m_t^e(x) = \sum_{m=1}^{m_t^e(0)} \mathbb{1}(x < x_t^{m-1}) = \begin{cases} 0 & \text{if } x \geq x^0 \\ 1 & \text{if } x_t^1 \leq x < x^0 \\ 2 & \text{if } x_t^2 \leq x < x_t^1 \\ \vdots & \\ m_t^e(0) & \text{if } x < x_t^{m_t^e(0)-1}. \end{cases} \tag{28}$$

Remark 10. The maximum welfare with long-lived bidders, as specified in Proposition 1, can be obtained by a sequence of second-price auctions with reserve prices $\{r_t^e\}_{t=1}^T$ and sampling rules $\{m_t^e(x)\}_{t=1}^T$. Again, for long-lived bidders, a key feature of the efficient search auction is that the efficient reserve prices are time-invariant throughout all the periods $t < T$ and drops to 0 in the last period T. To be specific,

$$r_t^e = \begin{cases} x^0 & \text{if } t < T \\ 0 & \text{if } t = T. \end{cases}$$

Example 4 below shows how to construct an efficient search auction using the results of Proposition 1. Particularly, we are interested in comparing the stopping cutoffs and sampling rules between the two cases of optimal and efficient search for long-lived bidders.

Example 4 (Efficient search auction for long-lived bidders)**.** Consider Example 1 again, yet we now examine an efficient, not an optimal, search auction for long-lived bidders.

- In the last period $t = 2$, the marginal search welfare is

$$MU_2(m;x) = \int_x^1 F(z)^m(1 - F(z))dz = \frac{1 + (1+m)x^{m+2} - (m+2)x^{m+1}}{(m+1)(m+2)},$$

  which is plotted in Figure 4(b). As $MU_2(2;0) = 1/16 < c$, the efficient sample size can not be greater than 2. For $m = 0, 1$, solving $MR_2(m;x) = c$ gives the cutoff values $x^0 = x_2^0 \approx 0.4343$ and $x_2^1 \approx 0.1204$. From (24), the efficient sample size is

$$m_2^{\mathrm{e}}(x) = \mathbb{1}(x < x_2^1) + \mathbb{1}(x < x^0). \tag{29}$$

  We have $\mathcal{F}_2(x) = F(x)^{m_2^{\mathrm{e}}(x)}$, and the maximum net welfare in period $t = 2$ is

$$W_2(x) = \mathbb{E}\max\{x, X_2^{\mathrm{e}}\} = x\mathcal{F}_2(x) + \int_x^{\bar{x}} z d\mathcal{F}_2. \tag{30}$$

- In the first period $t = 1$, the marginal search welfare is

$$MU_1(m;x) = \int_x^1 F(z)^m(1 - F(z))\mathcal{F}_2(z)dz,$$

  which is illustrated in Figure 4(a). It's interesting to observe that, for and $x \geq x^0$, $MU_2(m,x) = MU_1(m,x)$ for any m, and therefore, there is a time-invariant cutoff x^0 for efficient stopping. If $x < x^0$, the positive sample size in period $t = 2$, e.g., $m_2^{\mathrm{e}}(x) \geq 1$, decreases the marginal search welfare in period 1 as $\mathcal{F}_2(x) < 1$, and hence, $MU_1(m,x) < MU_2(m,x)$. As $MU_1(1;0) = \int_0^{x_2^1} z^3(1-z)dz + \int_{x_2^1}^{x^0} z^2(1-z)dz + \int_{x^0}^1 z(1-z)dz \approx 0.1428 < c$, the efficient sample size in period $t = 1$ is at most 1, and

$$m_1^{\mathrm{e}}(x) = \mathbb{1}(x < x^0).$$

Given $m_1^{\rm e}(x)$ and $m_2^{\rm e}(x)$, the continuation net search welfare $X_1^{\rm e}$ has the distribution

$$\mathcal{F}_1(x) = F(x)^{m_1^{\rm e}(x)+m_2^{\rm e}(x)} = \begin{cases} F(x)^3 & \text{if } x < x_2^1 \\ F(x)^2 & \text{if } x_2^1 \le x < x^0 \\ F(x)^0 = 1 & \text{if } x \ge x^0. \end{cases}$$

The maximum net search welfare at the beginning of period $t = 1$ is

$$W_1(0) = \mathbb{E}\max\{0, X_1^{\rm e}\} = \int_0^1 z d\mathcal{F}_1 = x^0 - \int_{x_2^1}^{x^0} F(z)^2 dz - \int_0^{x_2^1} F(z)^3 dz \approx 0.4075.$$

The maximum net search welfare can be obtained by the following sequential second-price auction. At the beginning of period $t = 1$, as the fallback value $0 < x^0$, the seller invites one bidder (bidder 1) and runs a second-price auction with a reserve price x^0. Bidder 1 will bid truthfully. If the bidder's bid is greater than $0 < x^0 \approx 0.4343$, the seller then allocates the product to him at the price x^0. Otherwise, the seller moves to period $t = 2$ with the following sampling rule:

- invite two more bidders if bidder 1's bid is lower than $x_2^1 \approx 0.1204$;
- invite one more bidder if bidder 1's bid is between $x^0 \approx 0.4343$ and $x_2^1 \approx 0.1204$.

In period 2, the seller runs a standard second-price auction with a reserve price 0.

Short-lived Bidders

Similar to Theorem 2, we can derive the efficient search rule and welfare for short-lived bidders. Let $\hat{x}_t$, $\hat{m}_t^e$ and $\hat{W}_t(x)$ respectively denote the cutoff value for efficient stopping, the efficient sample size, and the maximum net search welfare in the case of short-lived bidders. Proposition 2 below gives the results.

Proposition 2 (Efficient Search for Short-lived Bidders)**.** *Let x be the seller's fallback product value at the beginning of period $t \le T$. It is efficient for the seller to stop in period t if and only if $x \ge \hat{x}_t$, where the cutoff value $\hat{x}_t$ is recursively determined by*

$$\hat{x}_t = W_T(\hat{x}_{t+1}) = W_T(W_T(\hat{x}_{t+2})) = \cdots = \underbrace{W_T \circ \cdots \circ W_T}_{T-t \text{ times}}(\hat{x}_T) = \underbrace{W_T \circ \cdots \circ W_T}_{T-t+1 \text{ times}}(0). \qquad (31)$$

Moreover, $\hat{x}_{t+1} < \hat{x}_t < x^0$ *and* $\lim_{T\to\infty} \hat{x}_1 = \hat{x}$*. The efficient sample sizes* $\hat{m}^e_t(x)$ *is given by*

$$\hat{m}^e_t(x) = m^e_T(\hat{x}_{t+1})\mathbb{1}(x < \hat{x}_t), \tag{32}$$

and $\hat{m}^e_t(x)$ *is increasing over time, i.e.,* $\hat{m}^e_t \leq \hat{m}^e_{t+1}$*. The maximum net search welfare is*

$$\hat{W}_t(x) = \max\{x, \hat{x}_t\}. \tag{33}$$

Example 5 derives the efficient search auction with short-lived bidders. We are interested in comparing the efficient search rules across different cases, i.e., between short and long-lived bidders, and between efficient and optimal searches.

Example 5 (Efficient search auction with short-lived bidders)**.** Let us reconsider Example 4, yet now with short-lived bidders. Again we solve the problem by backward induction.

- At the beginning of the last period $t = 2$, if the seller continues to search, she will choose a sample size $\hat{m}^e_2 = m^e_2(0) = 2$ as given in Example 4 and the continuation net search welfare is $\hat{x}_2 = W_2(0) \approx 0.3467$ from (30). The maximum net search welfare at the beginning of period $t = 2$ is hence $\hat{W}_2(x) = \max\{x, \hat{x}_2\}$.
- At the beginning of period $t = 1$, if the seller continues to search, she will choose a sample size $\hat{m}^e_1 = m^e_2(\hat{x}_2) = 1$, and the continuation net search welfare is $\hat{x}_1 = W_2(\hat{x}_2) \approx 0.4001$ again from (30). The maximum net search welfare at the beginning of period $t = 1$ is hence $\hat{W}_1(0) = \max\{0, \hat{x}_1\}$.

The efficient mechanisms can be implemented by the following sequential second-price auction. At the beginning of period $t = 1$, as the fallback value 0 is smaller than $\hat{x}_1 \approx 0.4001$, the seller then invites $\hat{m}^e_1 = 1$ bidder (bidder 1) and runs a second-price auction with a reserve price $\hat{x}_2 \approx 0.3467$. If the bid is higher than $\hat{x}_2$, the seller then allocates the product to bidder 1 at the price $\hat{x}_2$. Otherwise, the seller continues to invite $\hat{m}^e_2 = 2$ bidders in period $t = 2$ and run a second-price auction with a zero reserve price.

Optimal vs. Efficient Search Auctions

The optimal search rules are different from the efficient ones, in both cases of long and short-lived bidders. Particularly, in terms of product value, we can show that the optimal cutoff value for stopping is higher than the efficient one, and therefore, there would be over-searching in an optimal search auction. This result echoes the similar result in static auctions, where the product value associated with the optimal reserve, i.e., $\psi^{-1}(0)$, is greater than one with the efficient reserve, i.e., 0. Our results then extend the static result to the more general case of sequential search auctions with a deadline.

Corollary 4 states that, for long-lived bidders, the cutoff product value for optimal stopping is higher than that for efficient stopping. As a results, a profit-maximizing seller would over search bidders than a welfare-maximizing seller.

Corollary 4. *For long-lived bidders, the cutoff product value for optimal stopping is greater than that for efficient stopping, i.e.,* $v^0 > \psi(x^0)$*.*

For short-lived bidders, we have the similar result that the optimal sample size in period t is greater than the efficient one, if their continuation search values are equal. Therefore, the over-searching result for an optimal search auction also holds for short-lived bidders.

Corollary 5. *For short-lived bidders, the optimal sample size in period* t *is greater than the efficient one, i.e.,* $\hat{m}_t^* \geq \hat{m}_t^e$*, if their continuation search values are equal in the sense that* $\hat{v}_{t+1} = \psi(\hat{x}_{t+1})$*.*

7 Conclusions

This article studies a seller's optimal search for strategic bidders by a deadline. The setting encompasses many real-world problems, such as M&A selling processes, government procurements, multi-round recruiting campaigns, online search for price quotes, and so on. We propose a tractable model to analyse this category of problems, and fully characterize the optimal search rules in both cases of long-lived and short-lived bidders.

We show that the optimal mechanisms can be implemented by a sequence of second-price auctions with properly selected reserve prices and sampling rules. For long-lived bidders, it is optimal to set a constant reserve price throughout all the periods $t < T$ but lower it to the optimal reserve price for static auctions in the last period T. The optimal sample size is increasing over time and decreasing in the seller's fallback revenue *ceteris paribus*. As the realizations of bidder values are random, the sequence of optimal sample sizes is a random process. For short-lived bidders, the sequence of optimal reserve prices is decreasing, and that of optimal sample sizes is increasing over time. Both sequences are deterministic and pre-determined, which is different from long-lived bidders.

We provide some comparative results between long and short-lived bidders. First, the optimal reserve price is lower for short-lived bidders than for long-lived bidders. Therefore, a seller with short-lived bidders is more likely to stop by accepting a lower reserve price, and we call this *discouragement effect* of short-lived bidders. Second, conditional on searching, a seller with short-lived bidders will search more intensively than one with long-lived bidders, and we term this *encouragement effect* of short-lived bidders.

We further study the efficient search for bidders, where the seller maximizes the expected product value of the winning bidder net of gross search costs. The efficient search auctions that implement the efficient mechanisms are qualitatively similar to the optimal ones, yet with different reserve prices and quantitatively differentiated sampling rules. We demonstrate that an optimal search auction is featured by over-searching in the sense of a higher reserve price or a larger search intensity than an efficient search auction. Therefore, the inefficiency of an optimal search auction may stem from its inefficient search rule.

This article makes important contributions to the literature on search mechanisms. The existent literature has mostly focused on the search for long-lived bidders without a time constraint and also largely neglected the case of short-lived bidders. This article introduces a finite deadline to a search mechanism and examines both cases of long and short-lived bidders. More importantly, the results can be applied to a large variety of real-world problems where a decision-maker searches for strategic agents with a deadline.

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Appendix: Omitted Proofs

Proof of Lemma 1. It follows from (7) that

$$MR_t(m;v) = \mathbb{E}\Pi_{t+1}\left(\max\left\{v, V^{m+1}\right\}\right) - \mathbb{E}\Pi_{t+1}\left(\max\left\{v, V^{m}\right\}\right).$$

Noting that V's are independent draws, we have $\max\{v, V^{m+1}\} = \max\{v, \max\{V^m, V\}\} = \max\{\max\{v, V^m\}, V\}$. As Π_{t+1} is strictly increasing, we have

$$\begin{aligned}&\Pi_{t+1}\left(\max\{\max\{v, V^m\}, V\}\right) - \Pi_{t+1}\left(\max\left\{v, V^m\right\}\right)\\ &= \begin{cases}\Pi_{t+1}(V) - \Pi_{t+1}(\max\{v, V^m\}) & \text{if } V \geq \max\{v, V^m\}\\ 0 & \text{if } V < \max\{v, V^m\}.\end{cases}\end{aligned}$$

As a result, an alternative expression of the marginal revenue is

$$MR_t(m;v) = \mathbb{E}\max\{\Pi_{t+1}(V) - \Pi_{t+1}(\max\{v, V^m\}), 0\},$$

which directly implies the results (i)-(iii) in the Lemma. □

Proof of Theorem 1. Assume $MR_T(0,0) > c$, otherwise the problem becomes trivial as the seller never invites any bidder. Let $\pi_t^*(v) := \max_{m\in\mathbb{N}_0}\{R_t(m;v) - cm\}$. We prove the results by backward induction.

Step 1: For the last period T, substituting $\Pi_{T+1}(v) = v$ into (7), the expected revenue is

$$R_T(m;v) = \mathbb{E}\max\left\{v, V^m\right\} = v + \int_v^{\bar{v}}(1 - G(z)^m)dz, \tag{34}$$

and $MR_T(m;v) = \int_v^{\bar{v}} G(z)^m(1 - G(z))dz$. Let $v^0 \equiv v_T^0$, which is the unique value satisfying $MR_T(0, v^0) = \int_{v^0}^{\bar{v}}(1 - G(z))dz = c$. We have $0 < v^0 < \bar{v}$ as $MR_T(0, \bar{v}) < c < MR_T(0, 0)$. As $\pi_T^*(v)$ is a contraction on $[0, v^0]$,[11] there exists a unique fixed point on $[0, v^0]$ such that

[11] For any $0 \leq v < v' \leq v^0 < \bar{v}$,

$$\begin{aligned}\pi_T^*(v') - \pi_T^*(v) &= R_T(m_T^*(v'); v') - cm_T^*(v') - [R_T(m_T^*(v); v) - cm_T^*(v)]\\ &\leq R_T(m_T^*(v'); v') - cm_T^*(v') - [R_T(m_T^*(v'); v) - cm_T^*(v')]\\ &= R_T(m_T^*(v'); v') - R_T(m_T^*(v'); v)\\ &= \int_v^{v'} G(z)^{m_T^*(v')}dz \leq G(v')^{m_T^*(v')} \leq G(v^0) < 1.\end{aligned}$$

$\pi_T^*(v) = v$ due to the Banach fixed point theorem.[12] In fact, this fixed point is v^0 as $m_T^*(v) = 0$ for $v \geq v^0$, and $\pi_T^*(v) = v$. Therefore, the optimal search profit in period T is

$$\Pi_T(v) = \max\{v, \pi_T^*(v)\} = \begin{cases} v & \text{if } v \in [v^0, \bar{v}] \\ \pi_T^*(v) = R_T(m_T^*(v); v) - cm_T^*(v) & \text{if } v \in [0, v^0), \end{cases}$$

which is obtained by the optimal search rule described in (8).

We next show that $\Pi_T(v)$ is convex and can be represented as (10). First, as $\partial R_T(m; v)/\partial v = G(v)^m$ is increasing in v, $R_T(m; v) - cm$ is convex in v for any given m; and $\pi_T^*(v)$, as the maximum of a family of convex functions, is hence also convex in v.[13] Second, the optimal profit $\Pi_T(v)$ is the maximum of two convex functions. It is then convex and absolutely continuous, and its derivative $\Pi_T'(v)$ is defined almost everywhere.[14] As such, $\Pi_T(v)$ can be represented by a definite integral of its derivative,[15] i.e.,

$$\Pi_T(v) = \Pi_T(\bar{v}) - \int_v^{\bar{v}} \Pi_T'(z)dz = \bar{v} - \int_v^{\bar{v}} \Pi_T'(z)dz,$$

where we apply $\Pi_T(\bar{v}) = \bar{v}$. As $\partial R_T(m; v)/\partial v = G(v)^m$, applying envelop theorem gives[16]

$$\Pi_T'(v) = \begin{cases} 1 & \text{if } v \in [v^0, \bar{v}] \\ G(v)^{m_T^*(v)} & \text{if } v \in [0, v^0), \end{cases}$$

which is equivalent to $\mathcal{G}_T(v)$ as given in (11). The optimal search profit $\Pi_T(v)$ is then represented by (10). Note that $\mathcal{G}_T(x) = \Pi_T'(x)$ is increasing in x as $\Pi_T(x)$ is convex.

Step 2: For $t < T$, as an induction hypothesis, assume

$$\Pi_{t+1}(v) = \bar{v} - \int_v^{\bar{v}} \mathcal{G}_{t+1}(z)dz, \tag{35}$$

where $\mathcal{G}_{t+1}(v)$ is given by (11) and is increasing in x. Integrating by parts then gives

$$\begin{aligned} R_t(m; v) &= \mathbb{E}\Pi_{t+1}(\max\{v, V_m\}) \\ &= \Pi_{t+1}(v)G(v)^m + \int_v^{\bar{v}} \Pi_{t+1}(z)dG(z)^m \end{aligned}$$

[12]See Ok (2007, p.176).
[13]See Aliprantis and Border (2006, p.187).
[14]See Varberg and Roberts (1973, p.4,5) and Royden and Fitzpatrick (2010, p.124,131,132).
[15]See Royden and Fitzpatrick (2010, p.125).
[16]See Simon and Blume (1994, Theorem 19.4).

$$= \quad \bar{v} - \int_v^{\bar{v}} G(z)^m \mathcal{G}_{t+1}(z) dz,$$

and $MR_t(m; v) = \int_v^{\bar{v}} G(z)^m(1 - G(z))\mathcal{G}_{t+1}(z)dz$. At the cutoff revenue v^0, we still have $MR_t(0; v^0) = c$ as $\mathcal{G}_{t+1}(v) = 1$ for $v \geq v^0$. Again, $\pi_t^*(v)$ is a contraction on $[0, v^0]$, on which v^0 is the unique fixed point. Therefore, for $t < T$, the optimal search profit is

$$\Pi_t(v) = \max\{v, \pi_t^*(v)\} = \begin{cases} v & \text{if } v \in [v^0, \bar{v}] \\ \pi_t^*(v) = R_{t+1}(m_t^*(v); v) - cm_t^*(v) & \text{if } v \in [0, v^0), \end{cases}$$

which is obtained by the optimal search rule described in (8).

Next, we have $\partial R_{t+1}(m; v)/\partial v = G(v)^m \mathcal{G}_{t+1}(v)$, which is increasing in v. As in **Step 1**, $R_{t+1}(m; v) - cm$ is convex in v for any given m, and hence $\pi_t^*(v)$ is also convex in v. The optimal profit $\Pi_t(v)$, as the maximum of two convex functions, is convex and absolutely continuous. Hence, its derivative $\Pi_t'(v)$ is defined almost everywhere and $\Pi_t(v)$ can be represented by

$$\Pi_t(v) = \Pi_t(\bar{v}) - \int_v^{\bar{v}} \Pi_t'(z)dz = \bar{v} - \int_v^{\bar{v}} \Pi_t'(z)dz,$$

where we apply $\Pi_t(\bar{v}) = \bar{v}$. As $\partial R_{t+1}(m; v)/\partial v = G(v)^m \mathcal{G}_{t+1}(v)$, envelop theorem yields

$$\Pi_t'(v) = \begin{cases} 1 & \text{if } v \in [v^0, \bar{v}] \\ G(v)^{m_t^*(v)} \mathcal{G}_{t+1}(v) & \text{if } v \in [0, v^0), \end{cases}$$

which is equivalent to $\mathcal{G}_t(v)$ given in (11). □

Proof of Theorem 2. We solve the search problem (15) by backward induction. For $t \leq T$, define

$$\hat{v}_t := \max_{m \in \mathbb{N}_0} \left\{ \mathbb{E}\hat{\Pi}_{t+1}\left(\max\{0, V^m\}\right) - mc \right\}, \tag{36}$$

which is the continuation search profit of following an optimal search rule from period t on, and $\hat{v}_{T+1} \equiv 0$. Substituting (36) into (15) then gives (18), i.e., $\hat{\Pi}_t(v) = \max\{v, \hat{v}_t\}$. It's clear that the seller stops searching in period t if and only if $v \geq \hat{v}_t$.

Starting from the last period T, substituting $\hat{\Pi}_{T+1}(v) = v$ into (36) gives

$$\hat{v}_T = \max_{m \in \mathbb{N}_0} \left\{ \mathbb{E} \max\{0, V^m\} - mc \right\} = \Pi_T(0) = \Pi_T(\hat{v}_{T+1}) > 0,$$

where we apply $\hat{v}_{T+1} = 0$. The strict inequality is from the fact that, for $v \in [0, v^0)$, $v < \Pi_T(v) < v^0$ (Remark 3). It then confirms (16) and that $v^0 > \hat{v}_t > \hat{v}_{t+1}$ for $t = T$. Suppose the induction hypotheses hold for periods $t+1, \cdots, T$. In period t, we have

$$\hat{v}_t = \max_{m \in \mathbb{N}_0} \left\{ \mathbb{E}\hat{\Pi}_{t+1}\left(\max\{0, V^m\}\right) - mc \right\} = \max_{m \in \mathbb{N}_0} \left\{ \mathbb{E}\max\{\hat{v}_{t+1}, V^m\} - mc \right\} = \Pi_T(\hat{v}_{t+1}),$$

where the second equality is from (18) and the fact that $\hat{v}_{t+1} > 0$, e.g.,

$$\hat{\Pi}_{t+1}\left(\max\{0, V^m\}\right) = \max\left\{\max\{0, V^m\}, \hat{v}_{t+1}\right\} = \max\left\{\hat{v}_{t+1}, V^m\right\}.$$

Again $\hat{v}_{t+1} < \hat{v}_t < v^0$, and the induction hypotheses hold for any period $t \leq T$. Furthermore, by monotone convergence theorem and that $\Pi_T(v^0) = v^0$, we have $\lim_{T\to\infty} \hat{v}_1 = v^0$.

We next solve for the optimal sample size $\hat{m}_t^*$ when the seller continues to search. Similarly, for the payoff function on the RHS of (36), we define the search revenue

$$\hat{R}_t(m; v) := \mathbb{E}\hat{\Pi}_{t+1}(\max\{0, V^m\}) = R_T(m, \hat{v}_{t+1}).$$

The maximizer $\hat{m}_t^*$ is then given by

$$\hat{m}_t^* = \min\left\{m \in \mathbb{N}_0 : MR_T(m, \hat{v}_{t+1}) \leq c\right\} = m_T^*(\hat{v}_{t+1}).$$

As $\hat{v}_t > \hat{v}_{t+1}$, we have $m_T^*(\hat{v}_t) \leq m_T^*(\hat{v}_{t+1})$, which implies that $\hat{m}_t^* \leq \hat{m}_{t+1}^*$. □

Proof of Corollary 1. The proof is already given in the proof of Theorem 2. □

Proof of Corollary 2. When $v < \hat{v}_t$, it is optimal to search in both cases of short and long-lived bidders. The result is then clear by comparing Theorem 1 and Theorem 2. □

Proof of Corollary 3. i) It is easy to show that $\Pi_T(v) \geq \hat{\Pi}_T(v)$. Then the result $\Pi_t(v) \geq \hat{\Pi}_t(v)$ can be recursively derived from the last period T, using the Bellman equations of (6) and (15) and that $\Pi_{t+1}(v) \geq \hat{\Pi}_{t+1}(v)$. ii) Note that $\hat{\Pi}_t(v) = \max\{v, \hat{v}_t\}$ by (18), and $\Pi_t(v) = \mathbb{E}\max\{v, V_t^*\}$ where V_t^* has the distribution $\mathcal{G}_t(v) = \prod_{\tau \geq t} G(v)^{m_\tau^*(v)}$ by (11). The single-peak result then comes from comparing the slopes of $\hat{\Pi}_t(v)$ and $\Pi_t(v)$. □

Proof of Proposition 1. Let $w_t^*(x) := \max_{m\in\mathbb{N}_0}\{U_t(m;x) - cm\}$. We prove the results by backward induction.

Step 1: Let $t = T$. Substituting $W_{T+1}(x) = x$ into (21), the expected search welfare is

$$U_T(m;x) = \mathbb{E}\max\{x, X^m\} = x + \int_x^{\bar{x}} (1 - F(z)^m)dz,$$

and $MU_T(m;x) = \int_x^{\bar{x}} F(z)^m(1 - F(z))dz$. Denote $x^0 < \bar{x}$ the unique value satisfying $MU_t(0;x^0) = \int_{x^0}^{\bar{x}}(1 - F(z))dz = c$. Note $w_T^*(x)$ is a contraction on $[0, x^0]$, and there is a unique fixed point on $[0, x^0]$ such that $w_T^*(x) = x$. In fact this fixed point is x^0, as $m_T^{\mathrm{e}}(x) = 0$ for $z \geq x^0$ and $w_t^*(x) = x$. Therefore, the net search welfare function is

$$W_T(x) = \max\{x, w_T^*(x)\} = \begin{cases} x & \text{if } x \in [x^0, \bar{x}], \\ w_T^*(x) = U_T(m_T^{\mathrm{e}}(x);x) - cm_T^{\mathrm{e}}(x) & \text{if } x \in [0, x^0), \end{cases}$$

which is obtained by the efficient search rule described in (24).

We next show that $W_T(x)$ is convex and can be represented as (26). First, as $\partial U_T(m;x)/\partial x = F(x)^m$ is increasing in x, $U_T(m;x)$ is convex in x for any given m. $w_T^*(x)$, as the maximum of a family of convex functions, is hence convex in x. Second, the net search welfare $W_T(x)$ is the maximum of two convex functions. It is then convex and absolutely continuous, and its derivative $W_T'(x)$ is defined almost everywhere, and $W_T(x)$ can be represented by a definite integral of its derivative, i.e.,

$$W_T(x) = W_T(\bar{x}) - \int_x^{\bar{x}} W_T'(z)dz = \bar{x} - \int_v^{\bar{v}} \Pi_T'(z)dz,$$

where we apply $W_T(\bar{x}) = \bar{x}$. As $\partial w_T(m;x)/\partial x = F(x)^m$, applying envelop theorem gives

$$W_T'(x) = \begin{cases} 1 & \text{if } x \in [x^0, \bar{x}], \\ F(x)^{m_T^{\mathrm{e}}(x)} & \text{if } x \in [0, x^0), \end{cases}$$

which is equivalent to $\mathcal{F}_T(x)$ as given in (27). The net search welfare $W_T(x)$ is then represented by (26). Note that $\mathcal{F}_T(x) = W_T'(x)$ is increasing in x as $W_T(x)$ is convex.

Step 2: For $t < T$, as an induction hypothesis, assume

$$W_{t+1}(x) = \bar{x} - \int_x^{\bar{x}} \mathcal{F}_{t+1}(z)dz,$$

where $\mathcal{F}_{t+1}(z)$ is given by (27) and is increasing in x. Integrating by parts then gives

$$U_t(m;x) = \bar{x} - \int_x^{\bar{x}} F(z)^m \mathcal{F}_{t+1}(z)dz,$$

and $MU_t(m;x) = \int_x^{\bar{x}} F(z)^m(1-F(z))\mathcal{F}_{t+1}(z)dz$. For the cutoff value x^0, we still have $MU_t(0;x^0) = c$ as $\mathcal{F}_{t+1}(z) = 1$ for $x \geq x^0$. Again, x^0 is the unique fixed point on $[0, x^0]$ such that $w_t^*(x^0) = x^0$. Therefore, for $t < T$, the net search welfare is

$$W_t(x) = \max\{x, w_t^*(x)\} = \begin{cases} x & \text{if } x \in [x^0, \bar{x}], \\ w_t^*(x) = U_t(m_t^{\mathrm{e}}(x);x) - cm_t^{\mathrm{e}}(x) & \text{if } x \in [0, x^0), \end{cases}$$

which is obtained by the efficient search rule described in (24).

Next, $\partial U_t(m;x)/\partial x = F(x)^m \mathcal{F}_{t+1}(x)$, which is increasing in x. As in **Step 1**, $w_t(m;x)$ is convex in x for any given m, and hence $w_t^*(x)$ is also convex in x. The search value $W_t(x)$, as the maximum of two convex functions, is convex and absolutely continuous. Hence, its derivative $W_t'(x)$ is defined almost everywhere and $W_t(x)$ can be represented by

$$W_t(x) = W_t(\bar{x}) - \int_x^{\bar{x}} W_t'(z)dz = \bar{x} - \int_x^{\bar{x}} W_t'(z)dz,$$

where we apply $W_t(\bar{x}) = \bar{x}$. As $\partial w_t(m;x)/\partial x = F(x)^m \mathcal{F}_{t+1}(x)$, envelop theorem yields

$$W_t'(x) = \begin{cases} 1 & \text{if } x \in [x^0, \bar{x}], \\ F(x)^{m_t(x)} \mathcal{F}_{t+1}(x) & \text{if } x \in [0, x^0), \end{cases}$$

which is equivalent to $\mathcal{F}_t(x)$ given in (27). □

Proof of Proposition 2. The proof is analogous to that of Theorem 2 and Proposition 1, and is hence omitted here. □

Proof of Corollary 4. v^0 and x^0 are given by $\int_{v^0}^{\bar{v}}[1-G(z)]dz = \int_{x^0}^{\bar{x}}[1-F(z)]dz = c$, where $\bar{v} = \bar{x}$. Suppose $v^0 \leq \psi(x^0)$. We then have

$$\begin{aligned} \int_{v^0}^{\bar{v}}[1-G(z)]dz &\geq \int_{\psi(x^0)}^{\bar{x}}[1-G(z)]dz = \int_{\psi(x^0)}^{\bar{x}}[1-F(\psi^{-1}(z))]dz \\ &= \int_{x^0}^{\bar{x}}[1-F(x)]\psi'(x)dx > \int_{x^0}^{\bar{x}}[1-F(x)]dx, \end{aligned}$$

where we replace $z = \psi(x)$ and the last inequality is from the fact that $\psi'(x) > 1$, which results in a contradiction. It then must be that $v^0 > \psi(x^0)$. □

Proof of Corollary 5. We know that the optimal and efficient sample sizes are given by

$$\hat{m}_t^* = \min\{m \in \mathbb{N}_0 : MR_T(m, \hat{v}_{t+1}) \le c\} \text{ and } \hat{m}_t^e = \min\{m \in \mathbb{N}_0 : MU_T(m, \hat{x}_{t+1}) \le c\},$$

respectively. To prove that $\hat{m}_t^* \ge \hat{m}_t^e$ when $\hat{v}_{t+1} = \psi(\hat{x}_{t+1})$, it suffices to show that $MR_T(m, \hat{v}_{t+1}) \ge MU_T(m, \hat{x}_{t+1})$. We have

$$\begin{aligned} MR_T(m, \hat{v}_{t+1}) &= \int_{\hat{v}_{t+1}}^{\bar{v}} G(z)^m(1 - G(z))dz = \int_{\psi(\hat{x}_{t+1})}^{\bar{x}} G(z)^m(1 - G(z))dz \\ &= \int_{\hat{x}_{t+1}}^{\bar{x}} F(x)^m(1 - F(x))\psi'(x)dx \ge MU_T(m; \hat{x}_{t+1}), \end{aligned}$$

where we substitute $z = \psi(x)$ and the last inequality is from the fact that $\psi'(x) > 1$. $\square$