Credit Rationing in Unsecured Debt Markets*

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Abstract

We revisit the welfare properties of competitive credit markets with endogenous debt limits that arise from limited commitment. We introduce a novel concept of constrained efficiency to encompass economies where market interest rates influence default values and participation constraints. We explore this concept through a baseline economy featuring heterogeneous agents with cyclical endowments. The analysis shows that the efficiency of market equilibria and welfare outcomes are sensitive to the nature of default punishments. It suggests that laissez-faire debt limits, while supportive of risk-sharing, might not always optimize welfare. The study illustrates how credit rationing—tightening future debt limits—can enhance current risk-sharing and overall welfare. This occurs because such interventions influence market interest rates and the default option's value, thereby affecting agents' present consumption and saving choices, and enabling a more efficient allocation of resources.

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1 Introduction

We revisit welfare properties of competitive credit markets with debt limits that endogenously arise from limited commitment, as analyzed in the foundational works of Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000). Our point of departure is an infinite-horizon general equilibrium model in which agents with heterogeneous income shocks trade debt contracts and may choose to default on their debts. This lack of commitment endogenously gives rise to state- and agent-specific debt limits (akin to an individual's debt limit on the credit card). With rational lenders, these debt limits should be tight enough to prevent default. However, with a competitive credit market, the limits should be loose enough to allow as much risk-sharing as possible. In other words, our laissez-faire baseline is a general equilibrium with *not-too-tight* debt limits, as pioneered by Alvarez and Jermann (2000). This framework has been proven to be valuable for studying the welfare implications, as well as the asset pricing, business cycles, and consumption/wealth inequality implications of incomplete financial markets, where the incompleteness is microfounded by limited commitment (Alvarez and Jermann 2001, Krueger and Perri 2006, Azariadis et al. 2016).

Our first contribution is to show that the nature of the default punishment is pivotal in determining the efficiency of competitive equilibrium outcomes. Specifically, efficiency relies on the assets agents can trade upon default. In settings where defaulting agents are entirely excluded from trading activities (effectively, financial autarky), laissez-faire allocations exhibit finite-time efficiency. This essentially means that Pareto improvements are infeasible when the scope of the social planner's redistributive actions is confined to a finite number of periods.[1] However, analyzing the efficiency properties of equilibria becomes considerably

[1]Standard Pareto efficiency assumes that a planner can enhance welfare by undertaking feasible redis-

more intricate in settings where the default punishment is less severe than financial autarky.

Specifically, inspired by the works of Bulow and Rogoff (1989), Krueger and Uhlig (2006), Krueger and Perri (2006) and Hellwig and Lorenzoni (2009), we study the practically relevant case where defaulters (akin to individuals with poor credit scores) are excluded from borrowing but retain the ability to save.[2] In this context, existing concepts of constrained efficiency, which typically involve specifying exogenous outside values (such as the continuation value in autarky), become inapplicable. This is because when a defaulter can save in the financial market, the value of default naturally and endogenously depends on the prevailing interest rates in the market. It is precisely this dependence that might overturn the first welfare theorem. To see this, we should first carefully define constrained efficiency, a task that proves quite challenging.

In standard textbooks, an efficient allocation is defined as a feasible allocation (where total consumption does not exceed total endowments) that is not Pareto dominated by another feasible allocation. With limited commitment, where the risk of default restricts the set of consumption and asset allocations, efficiency is similarly defined, with the additional requirement that a feasible allocation must satisfy participation constraints. These participation constraints are straightforward when the default punishment is autarky: at every state, each agent must prefer their consumption allocation to the consumption in autarky.

However, when the default punishment does not preclude agents from saving in the financial market, the value of the default option depends on the market interest rate. Con-

tributions across the infinite timeline of the economy. This criterion sets a high bar, as competitive markets with sequential trading often fail to curtail welfare-improving reallocations in the distant future without explicit limits on price behavior in the long term. Notably, in situations where the consequence of default is financial autarky, Bloise and Reichlin (2011) highlighted that laissez-faire equilibria could become Pareto inefficient if interest rates remain persistently low. In settings with full commitment, the equilibrium prices naturally adhere to a transversality condition, eliminating the possibility of long-term mutual gains from trade and thus safeguarding Pareto efficiency. However, in contexts of limited commitment, without imposing additional constraints, equilibrium prices are not guaranteed to fulfill this critical condition.

[2]In the US, no law prevents a defaulter from saving at banks, and in many cases, states and federal laws can make it difficult or costly for creditors to garnish bank accounts (Warren et al. 2020).

sequently, the definition of feasibility must be appropriately adjusted to specify the prices or interest rates that are integral to formulating the participation constraint. As a result, we introduce a novel concept of feasibility for the planner, where prices (implicitly constructed by consumption) accompany allocations, and the default option in the participation constraint depends on these prices. Specifically, the planner sets prices equal to the highest marginal rates of substitution, as it happens in any self-enforcing equilibrium.[3] However, in contrast to laissez-faire market prices, which are determined by taking the maximum among agents with nonbinding participation constraints, the planner selects the maximal marginal rate of substitution among all agents. In doing this, the planner, unlike individuals, internalizes the broader economic implications of saving and borrowing decisions—which are influenced by consumption choices—on asset prices, default values, and, ultimately, participation constraints.

This paper's main contribution is to show that finite-time efficiency, defined according to our novel definition of feasibility, might not prevail in settings where agents can save after default. The starting point of our analysis amounts to establishing a relationship between finite-time efficiency and a planner's problem. This connection enables us to introduce novel optimality conditions that illustrate how changes in prices impacting the default value–the pecuniary externality–can lead to finite-time inefficiencies in laissez-faire equilibrium allocations. Furthermore, we offer an implementability insight: if a laissez-faire allocation fails to meet the necessary conditions for finite-time efficiency, then it is possible to identify a superior Pareto allocation among those supported by self-enforcing competitive equilibria with nonnegative debt limits. Moreover, this Pareto superior allocation differs from the laissez-faire allocation only for finitely many periods. This is a strong inefficiency result that aligns closely with practical scenarios where policy interventions last a certain number of periods because long-term commitments are infeasible.

[3]This property of the pricing kernel in models with heterogeneous agents is crucial, as it allows these models to generate pricing kernels that could be sufficiently volatile to help explain behaviors of asset prices and wealth accumulation in the data (Constantinides and Duffie 1996, Alvarez and Jermann 2001, Heathcote et al. 2009, Rampini and Viswanathan 2010, Cao 2018).

The welfare analysis suggests that imposing restrictions on private credit beyond what is traditionally deemed necessary might yield a finite-time Pareto improvement. To clarify this point, we examine a baseline economy where two agents face endowment uncertainty exclusively in the initial period, followed by deterministic cyclical endowments thereafter. Within this framework, we single out a symmetric Markovian laissez-faire equilibrium characterized by zero interest rates and binding debt limits in the high-income state.

Our analysis delivers two main findings. First, we analytically characterize the optimality conditions of the planner's problem and compare them with the ones of the laissez-faire equilibrium. There, we establish a sufficient parameter condition under which the laissez-faire allocation is finite-time inefficient and can be Pareto dominated by another feasible allocation. This alternative allocation corresponds to an equilibrium outcome with debt limits that are self-enforcing (i.e., sufficiently tight to prevent default) but are possibly *too tight*. This means that maximal risk sharing might not always be the planner's optimal strategy, indicating that policy interventions that tighten or ration laissez-faire debt limits could enhance welfare. Second, we formally construct a credit rationing intervention in this economy. Our findings demonstrate that imposing too-tight debt limits in some future periods can raise the ex-ante social welfare. Such interventions are interpreted as a simplified form of regulatory or prudential policies to constrain financial leverage.

Intuitively, even though all agents are fully rational and forward-looking, they fail to internalize how changes in the severity of credit restrictions affect equilibrium prices and, most crucially, feedback on the value of the default option. Specifically, tightening the debt constraints at some period τ reduces (current) risk-sharing and leads to an increase in bond prices or, equivalently, a decrease in the implied interest rates at that period. In our model, where default induces exclusion from credit markets, such a policy intervention could diminish the attractiveness of defaulting in periods $t \leqslant \tau$ by making it more expensive to smooth consumption over time through savings alone. Consequently, the extent of risk-sharing implemented at the previous periods $t < \tau$ might increase, paving the way for a potential Pareto improvement: the benefits of the enlarged risk-sharing opportunities at periods $t < \tau$ may compensate for the costs of facing tighter constraints at period τ.[4] This is the essence of the mechanism we explore in this paper.

A critical part of our analysis is to clarify the feedback effects of such policy interventions on equilibrium prices and the default option. It is essential to note that the proposed financial market intervention does not equate to altering the penalties for defaulting, which remain constant throughout our discussion. Instead, the reallocation is achieved by tightening borrowing limits to levels lower than the loosest self-enforcing ones.

In our model, where debt is fully unsecured, a laissez-faire equilibrium—with maximal debt limits—necessarily features a zero interest rate. One may be tempted to attribute the inefficiencies we discovered to low interest rates. We argue that this conjecture is false by demonstrating that Pareto improvement is robust to more stringent default punishments that give rise to laissez-faire equilibria with positive interest rate.

Specifically, we extend our policy experiment to a setting where default triggers not only exclusion from credit, but also deadweight losses to endowment.[5] To single out a symmetric Markovian laissez-faire equilibrium in this modified setting poses additional challenges that we overcome by providing a novel and intuitive decomposition of not-too-tight debt limits into a fundamental and a bubble component.[6] This characterization is particularly useful for our analysis. It significantly streamlines the computation of laissez-faire equilibria, eliminating the complexities of determining the fixed-point debt limits.

[4]Debt constraints at periods $t < \tau$ remain the loosest compatible with repayment incentives. However, repayment incentives might be stronger in these periods due to increased bond prices (and the associated fall in the default value) at time τ. This, in turn, can increase the amount of available liquidity at $t < \tau$ and support more risk-sharing.

[5]In the context of consumer and corporate defaults, they represent legal repercussions such as recourse and seized collateral.

[6]The fundamental component equates to the present value of endowment losses and the bubble component stands for the amount of credit that agents can perpetually rollover. Although this decomposition parallels the characterization result of Hellwig and Lorenzoni (2009) in our modified setting, our result is not a straightforward extension of their work. It relies on novel insights with no counterparts in models without output losses.

We find that irrespective of the magnitude of endowment loss, the laissez-faire equilibrium consistently features positive interest rates. We demonstrate that subject to the same policy intervention—namely, tightening the debt limits at a specific period—the resulting new equilibrium still yields a consumption allocation that Pareto dominates the laissez-faire allocation. This confirms that inefficiencies in markets for unsecured credit are not inherently tied to low interest rates.

Related Literature. The study of competitive credit markets with limited commitment dates back to Kehoe and Levine (1993) and Kocherlakota (1996). When there are multiple commodities and default does not preclude agents from participating in spot markets, constrained efficiency may not be achieved due to the inability of private contracts to account for their impact on relative prices and the default option. The logic there parallels the discussions in economies with incomplete markets, where the redistribution of asset holdings—via induced price changes—alters the spanning properties of the limited assets (Hart 1975, Stiglitz 1982, Geanakoplos and Polemarchakis 1986). Similar conclusions have been drawn in contexts where contracting is subject to private information (Greenwald and Stiglitz 1986).

However, the single-commodity model we examine lacks both spot markets and private information, so this mechanism is absent. Furthermore, studies by Alvarez and Jermann (2000), Bloise and Reichlin (2011), Bloise et al. (2013) and Bloise (2020) show that laissez-faire equilibria are finite-time efficient when default leads to autarky. Our analysis diverges by establishing that finite-time inefficiency can arise in single-commodity economies when debt enforcement is based on a weaker form of exclusion (i.e., one-sided exclusion) from financial markets. Changes in the severity of credit restrictions induce price changes in bond markets. These price changes, in turn, affect the value of default and, therefore, the extent of risk sharing, potentially improving efficiency. This source of inefficiency is absent in the model by Alvarez and Jermann (2000) since the default's value remains unaffected by changes in bond prices.

Our work is related to a well-developed literature studying the emergence of pecuniary externalities in production economies with collateral constraints. Gromb and Vayanos (2002) show that both distributive and collateral externalities can emerge due to market segmentation. Lorenzoni (2008), Bianchi (2011), Bianchi and Mendoza (2011) and Jeanne and Korinek (2019) show how optimal borrowing decisions at the individual level can lead to overborrowing at the social level and potentially cause fire sales, asset prices to decline, and the economy's borrowing capacity to shrink. Dávila and Korinek (2018) explore pecuniary externalities in dynamic environments subject to reduced-form, price-dependent collateral constraints. They distinguish between distributive and collateral externalities and show that these two types can be quantified using sufficient intuitive statistics. In all of these studies, a change in the level of investment induces resource reallocation through capital accumulation. A planner can mitigate this market failure by reducing aggregate investment ex-ante and, therefore, the size of the asset sales during downturns. In contrast, this channel is absent in our pure exchange framework as aggregate resources are fixed, and only their distribution is subject to change. As our findings suggest, within our model's context, the planner selects from among all feasible self-enforcing equilibria, utilizing the adjustment of debt constraints as the exclusive mean for reallocating resources.

Our paper also relates to the work of Gottardi and Kubler (2015), which studies the effects of unexpected mandatory savings in an economy where agents can only take long positions on contingent trees, the work of Guerrieri and Lorenzoni (2017), which studies the effects of unexpected credit contractions in a heterogeneous-agent incomplete-market model with exogenous borrowing limits, and the work of Aguiar et al. (2023), which studies Pareto-improving fiscal policies in this environment when the interest rate on the government bond is below the growth rate. Our research distinguishes itself from these works by placing limited commitment and the endogenous nature of borrowing limits as key drivers of inefficiencies in debt markets.

The plan of the paper is as follows. Section 2 introduces the model environment and defines self-enforcing and not-too-tight debt limits. Section 3 defines finite-time efficiency, identifies necessary optimality conditions, and shows that Pareto superior allocations can be implemented as self-enforcing competitive equilibria. Section 4 applies the results of

Section 3 to a baseline economy. Section 5 incorporates endowment losses upon default and establishes that our findings regarding market inefficiencies do not depend on the premise of interest rates being lower than growth rates. Section 6 concludes. The online appendix provides detailed proofs, technical arguments, and additional discussions.

2 General Model

Consider an infinite-horizon endowment economy with a single nonstorable consumption good at each date. Time and uncertainty are both discrete. We use an event tree Σ to describe the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$, and for each date $t \in \{0,1,2,\dots\}$, there is a finite set $S^t \subseteq \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. The notation $s^{t+1} \succ s^t$ specifies that s^{t+1} is a successor of s^t. The event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$.[7] The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} : s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree starting at event s^t is then given by:

$$\Sigma(s^t) := \bigcup_{\tau \geqslant 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$.

There is a finite set I of household types, each consisting of a unit measure of identical, infinitely lived agents who consume a single perishable good. Preferences over (nonnegative) consumption processes $c = (c(s^t))_{s^t \succeq s^0}$ are represented by the lifetime expected and discounted utility:

$$U(c) := \sum_{t \geqslant 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t)),$$

[7] Formally, σ is a mapping from $\Sigma \setminus \{s^0\}$ to Σ such that $\sigma(S^{t+1}) = S^t$ for every $t \geqslant 0$. We pose $\sigma^{(1)} := \sigma$ and $\sigma^{(\tau+1)} := \sigma \circ \sigma^{(\tau)}$ for every $\tau \geqslant 1$.

where $\beta \in (0,1)$ is the discount factor, $\pi(s^t)$ is the agent's subjective belief of state s^t, and $u : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ is a utility function that is strictly increasing, strictly concave, continuously differentiable on $(0,\infty)$, and satisfies Inada's condition $\lim_{\varepsilon\to 0}[u(\varepsilon) - u(0)]/\varepsilon = \infty$. Given an event s^t, we denote by $U(c|s^t)$ the lifetime continuation utility conditional on s^t, as defined by:

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geqslant 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau})),$$

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t. Agents' endowments are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \succeq s^0}$ the process of positive endowments $y^i(s^t) > 0$ of a representative agent of type i. For notational convenience, we have written the primitives as if agents' preferences and beliefs are homogeneous. However, our arguments remain valid when agents have heterogeneous preferences and beliefs, and the only necessary change is to replace (u, β, π) with (u^i, β^i, π^i).

2.1 Asset Markets with Self-Enforcing Debt Constraints

At any event s^t, agents can issue and trade state-contingent one-period bonds, each one promising to pay one unit of the consumption good contingent on the realization of a successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ represent the price, at event s^t, of the bond contingent on event s^{t+1}.[8] Agent i's bond holdings are $a^i = (a^i(s^t))_{s^t \succeq s^0}$, where $a^i(s^t) \leqslant 0$ is a liability, and $a^i(s^t) \geqslant 0$ is a claim. Each agent's debt is observable and subject to certain (state-contingent, finite and) nonnegative debt limits $D^i = (D^i(s^t))_{s^t \succeq s^0}$.[9] Given an initial bond holding $a^i(s^0)$ and debt limits D^i, we denote by $B^i(D^i, a^i(s^0)|s^0)$ the budget set of an agent who never defaults. It consists of all pairs (c^i, a^i) of consumption and bond holdings

[8]More precisely, for each successor event $s^{t+1} \succ s^t$, the yield of the bond traded at event s^t is given by the reciprocal of its price, i.e., $1/q(s^{t+1})$.

[9]In this paper, we focus on nonnegative debt limits, thereby refraining from imposing mandatory savings that would compel individuals to accumulate assets.

satisfying the following budget flows and debt constraints: for all $s^t \succeq s^0$,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leqslant y^i(s^t) + a^i(s^t), \tag{2.1}$$

and

$$a^i(s^{t+1}) \geqslant -D^i(s^{t+1}), \quad \text{for all } s^{t+1} \succ s^t. \tag{2.2}$$

We naturally restrict attention to allocations where the initial asset holdings clear the market, i.e., $\sum_{i \in I} a^i(s^0) = 0$, and satisfy the debt constraints, i.e., $a^i(s^0) \geqslant -D^i(s^0)$ for each i. Similarly, contingent on an event s^τ, we let $B^i(D^i, x|s^\tau)$ be the set of all plans (c^i, a^i) satisfying restrictions (2.1) and (2.2) at every successor node $s^t \succeq s^\tau$ with initial claim $a^i(s^\tau) = x$. Denote the contingent value function at event s^τ, when agent i starts with financial wealth x, by $V^i(D^i, x|s^\tau)$. It is defined as the largest continuation utility $U(c^i|s^\tau)$ among all budget feasible plans $(c^i, a^i) \in B^i(D^i, x|s^\tau)$. When $x = a^i(s^\tau)$, this will be the equilibrium value, i.e., the payoff to each agent i along the equilibrium path following any event s^τ.

So far, debt limits are arbitrary. We next move to the endogenous determination of the debt limits, which are a critical determinant of equilibrium allocations and payoffs. Debt limits represent the maximal amount of debt that borrowers can issue. In general equilibrium, they also represent the maximal amount of liquidity (or storage of value) that savers can access. We follow Alvarez and Jermann (2000) and provide a microfoundation for the level of debt limits by assuming that agents have limited commitment. We consider an environment where agents cannot commit to their financial contracts and may opt for default. We denote by $V^i_{\text{def}}(q|s^t)$ agent i's value of the default option at event s^t, and we impose that the debt limits reflect the fact that repayment is always individually rational.[10] Specifically, we say that debt limits D^i are *self-enforcing* if debtors prefer to repay even the maximum debt allowed, i.e.,

$$V^i(D^i, -D^i(s^t)|s^t) \geqslant V^i_{\text{def}}(q|s^t), \quad \text{for all } s^t \succeq s^0. \tag{2.3}$$

[10] The continuation value $V^i(D^i, x|s^t)$ is affected by the process of asset prices $(q(s^\tau))_{s^\tau \succ s^t}$, although, for simplicity, we do not specifically outline this connection. On the other hand, for the default value, we explicitly highlight its dependence on asset prices due to its pivotal role in identifying "feasible" actions for a social planner. For further elaboration, please refer to Section 3.

We say that D^i are *not too tight* if (2.3) always holds with equality, i.e., borrowers are indifferent between repaying and defaulting:

$$V^i(D^i, -D^i(s^t)|s^t) = V^i_{\text{def}}(q|s^t), \quad \text{for all } s^t \succeq s^0. \tag{2.4}$$

Given future debt limits $(D^i(s^\tau))_{s^\tau \succ s^t}$, the level $D^i(s^t)$ satisfying (2.4) is the largest self-enforcing debt limit contingent on event s^t. We say that D^i are *too tight* if they are self-enforcing and (2.3) holds with strict inequality at some event $s^t \succ s^0$.

Definition 2.1. Given a family of default value functions $(V^i_{\text{def}})_{i\in I}$, we call a *self-enforcing equilibrium* $(q, (c^i, a^i, D^i)_{i\in I})$ a collection of state-contingent bond prices q, a consumption allocation $(c^i)_{i\in I}$, a bond holdings allocation $(a^i)_{i\in I}$, and a family of debt limits $(D^i)_{i\in I}$ such that:

(a) the plan (c^i, a^i) of agent i is optimal among budget feasible plans in $B^i(D^i, a^i(s^0)|s^0)$;

(b) the debt limits D^i of agent i satisfy the self-enforcing condition (2.3);

(c) markets clear: $\sum_{i\in I} c^i = \sum_{i\in I} y^i$ and $\sum_{i\in I} a^i = 0$.

When the debt limits of all agents satisfy condition (2.4), we use the term *not-too-tight equilibrium*. It is reasonable to expect that in a competitive market, competition among lenders should naturally lead them to offer as much credit as possible without violating borrowers' incentive to repay. Hence, we will also use the term *laissez-faire equilibrium* as a synonym for not-too-tight equilibrium. Similarly, when the debt limits are too tight, we use the term *too-tight equilibrium*.

The default value is the key object determining the not-too-tight debt limits. We consider a framework where all assets are seized upon default, and debtors lose access to credit while retaining the ability to save (by purchasing other people's debt). Consequently, the value of default for any agent i at any event s^t is given by:

$$V^i_{\text{def}}(q|s^t) = V^i(0, 0|s^t) := \sup\{U(c^i|s^t) \,:\, (c^i, a^i) \in B^i(0, 0|s^t)\}. \tag{2.5}$$

The condition (2.4) then reads as follows:

$$V^i(D^i, -D^i(s^t)|s^t) = V^i(0, 0|s^t), \quad \text{for all } s^t \succeq s^0. \tag{2.6}$$

The assumption that agents can save after default follows Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009) but contrasts with Alvarez and Jermann (2000) who assume that defaulting agents can neither borrow nor save. This weaker form of punishment captures the idea that it is much easier for market participants to coordinate on not purchasing the claims issued by a borrower with a "bad reputation" than to enforce an outright ban from financial markets. Most importantly, Krueger and Uhlig (2006) provided microfoundations for this default punishment by analyzing dynamic equilibrium risk-sharing contracts between profit-maximizing intermediaries and agents facing income uncertainty. They showed that our default value coincides with the endogenously determined outside option in the model they study.

2.2 Optimality Conditions

For a given (strictly positive) consumption allocation $c = (c^i)_{i\in I}$, we define the marginal rate of substitution for agent i at event s^t, denoted as $q^i(c^i|s^t)$, by

$$q^i(c^i|s^t) := \beta\pi(s^t|\sigma(s^t))\frac{u'(c^i(s^t))}{u'(c^i(\sigma(s^t)))}.$$

The price $q^\star(c)$ implied by the allocation $c = (c^i)_{i\in I}$ is defined as the highest marginal rate of substitution across all agents:

$$q^\star(c|s^t) := \max_{i\in I} q^i(c^i|s^t).$$

Consider a self-enforcing equilibrium $(q, (c^i, a^i, D^i)_{i\in I})$. The Principle of Optimality implies that $U(c^i|s^t) = V^i(D^i, a^i(s^t)|s^t)$. As debt limits are self-enforcing, we deduce that the consumption process c^i satisfies the participation constraints:

$$U(c^i|s^t) \geqslant V^i_{\text{def}}(q|s^t), \quad \text{for every } s^t \succeq s^0.$$

Euler equations imply that $q(s^t) \geqslant q^i(c^i|s^t)$ with equality if the debt constraint is not binding: $a^i(s^t) > -D^i(s^t)$, or, equivalently, $U(c^i|s^t) > V^i(D^i, -D^i(s^t)|s^t)$. This implies that $q(s^t) \geqslant q^\star(c|s^t)$. If all agents face binding debt constraints at event s^t, then it follows that $a^i(s^t) = D^i(s^t) = 0$ for each agent i.[11] This condition leads to $V^i(D^i, 0|s^t) = V^i(0, 0|s^t)$, implying that $D^i(s^\tau) = 0$ for all successor events $s^\tau \succeq s^t$, and effectively implementing autarky within the subtree $\Sigma(s^t)$ and setting $V^i_{\text{def}}(q|s^\tau) = U(y^i|s^\tau)$ for every subsequent event $s^\tau \succeq s^t$. Under these circumstances, for any $s^\tau \succ s^t$, the price $q(s^\tau)$ may be replaced by the autarkic price $q^\star(y|s^\tau) = q^\star(c|s^\tau)$ when defining the default value contingent on event s^t.[12] Given that debt constraints persist across the subtree $\Sigma(s^t)$, the pair (c^i, a^i) maintains its optimality even when substituting q with $q^\star(c)$. The following proposition encapsulates these findings:

Proposition 2.1. *Given a self-enforcing equilibrium $(q, (c^i, a^i, D^i)_{i\in I})$, the default value $V^i_{\text{def}}(q|s^t)$ equals $V^i_{\text{def}}(q^\star(c)|s^t)$, the participation constraint $U(c^i|s^t) \geqslant V^i_{\text{def}}(q^\star(c)|s^t)$ holds true, and the adjusted family $(q^\star(c), (c^i, a^i, D^i)_{i\in I})$ remains a self-enforcing equilibrium. Additionally, if debt limits are not-too-tight, we have $q^i(c^i|s^t) = q^\star(c|s^t)$ when agent i's participation constraint is not binding.*

Given this result, we can assume (without loss of generality) that the equilibrium prices take the form $q^\star(c)$, where c represents the implemented consumption allocation.

2.3 Weak and Exact Rollover

Hellwig and Lorenzoni (2009) proved that not-too-tight debt limits necessarily form a bubble that captures the possibility of exactly rolling over debt indefinitely. Formally, a process D^i of debt limits is not too tight (for the default punishment described by (2.5)) if,

[11] Given market clearing, the equality $0 = \sum_{i\in I} a^i(s^t) = -\sum_{i\in I} D^i(s^t)$ holds. As $D^i(s^t) \geqslant 0$, we conclude $D^i(s^t) = 0$.

[12] It's important to notice that only the prices $q(s^\tau)$ at strict successor events $s^\tau \succ s^t$ are relevant for determining $V^i_{\text{def}}(q|s^t)$.

and only if:

$$D^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \text{for all } s^t \succ s^0. \tag{2.7}$$

It is straightforward to verify that if there is a strict inequality in (2.7), then debt limits are self-enforcing and too tight.

Proposition 2.2. *Fix an event $s^\tau \in \Sigma$ and assume that:*

$$D^i(s^t) \leqslant \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \textit{for all } s^t \succeq s^\tau \tag{2.8}$$

with a strict inequality in at least one successor event $s^t \succeq s^\tau$. Then $V^i(D^i, -D^i(s^\tau)|s^\tau) > V^i(0, 0|s^\tau)$.

The characterization of not-too-tight and too-tight debt limits, framed in terms of the exact or weak rollover, facilitates the computation of equilibria. It sidesteps the typical complexities associated with the fixed-point process of identifying self-enforcing debt limits: as the default values depend on prices (given that defaulting agents retain the ability to save), they are influenced by equilibrium allocations and, consequently, the debt limits themselves. The usefulness will become clear when we conduct our policy intervention experiment in Section 4.

3 Pareto Criteria and Efficiency

In models where financial frictions stem from limited commitment, the prevailing view is that borrowing should adhere to the most lenient debt constraints that align with the financial friction. Not-too-tight debt limits should emerge organically in a competitive credit market, where competition among lenders would allow borrowers to take on the maximum debt that still aligns with repayment incentives. This perspective is bolstered by the notion that such debt limits facilitate optimal risk-sharing. While this may hold in a partial equilibrium context with static prices, its validity becomes dubious in general equilibrium scenarios where prices and debt limits are endogenously determined.

Hellwig and Lorenzoni (2009) demonstrated that in a laissez-faire equlibrium, not-too-tight debt limits must be bubbly. Our research investigates potential equilibrium alternatives to the conventional *laissez-faire* paradigm. We question whether it's possible to move beyond bubbly equilibria and identify an equilibrium characterized by *too-tight* debt limits that Pareto outperforms the laissez-faire equilibrium.

To motivate our investigation, we first introduce welfare metrics by considering different types of Pareto improvements available to a social planner who faces the same self-enforcing participation constraints as the markets due to the lack of commitment. We shall start by defining feasibility in our context. We formulate a social planner's problem that mimics the "primal approach" to optimal policy analysis: the planner chooses consumption allocations $(c^i)_{i\in I}$ subject to:

(a) resource constraint: $\sum_{i\in I} c^i(s^t) = \sum_{i\in I} y^i(s^t)$ for all $s^t \succeq s^0$;

(b) participation constraints: $U(c^i|s^t) \geqslant V^i_{\mathrm{def}}(q|s^t)$ for all $s^t \succeq s^0$;

(c) implementability constraints: $q(s^t) = q^\star(c|s^t)$ for all $s^t \succ s^0$.

A consumption allocation is feasible when it satisfies properties (a), (b), and (c).[13]

Since asset prices remain market determined, the private agents' Euler equations for assets enter the social planner's problem as an implementablity constraint. Thus, the planner does not set asset prices. Still, it does internalize how saving/borrowing decisions (indirectly determined by consumption choices) affect asset prices, the default value, and participation constraints.

The most commonly utilized metric for evaluating the welfare attributes of a specific feasible allocation is the Pareto criterion: it stipulates that every agent's expected utility at time zero, discounted over time, weakly increases, with at least one agent experiencing a strict improvement. A given feasible allocation $(c^i)_{i\in I}$ is deemed (constrained) efficient if the social planner cannot identify another feasible allocation $(\tilde{c}^i)_{i\in I}$ that Pareto dominates

[13]Proposition 2.1 shows that consumption allocations supported by a self-enforcing competitive equilibrium are feasible.

$(c^i)_{i\in I}$.[14] Efficiency is a demanding welfare criterion as it presupposes that the planner can enhance welfare through a balanced redistribution extended across the infinite horizon of the economy. Competitive markets with sequential trading fail to enforce sufficient constraints to avert welfare-improving adjustments in the distant future unless there are specific limitations on the long-term behavior of prices.[15] A weaker Pareto criterion consists in restricting the social planner's redistributions to a limited number of periods. Specifically, a feasible allocation $(c^i)_{i\in I}$ is considered *finite-time efficient* if there is no finite horizon $T \geqslant 1$ for which there exists an alternative feasible allocation $(\tilde{c}^i)_{i\in I}$ that Pareto dominates $(c^i)_{i\in I}$, while respecting $\tilde{c}^i(s^t) = c^i(s^t)$ for all s^t when $t > T$. The impossibility of achieving Pareto improvements within a finite horizon represents a milder constraint. Thus, demonstrating the finite-time inefficiency of a feasible allocation signifies a stronger assertion than merely showing it is not Pareto efficient.

In Alvarez and Jermann (2000), where autarky is the default punishment, consumption allocations implemented by laissez-faire equilibria are finite-time efficient.[16] The important contribution of this paper is to analyze whether this efficiency result remains valid in the environment where saving is allowed upon default. We start by recalling the standard connection between efficiency and social planner programs.

Lemma 3.1. *If a feasible allocation $c = (c^i)_{i\in I}$ is finite-time efficient, then the pair $(q^\star(c), c)$ solves the social planner program that maximizes welfare $\sum_{i\in I} U(\tilde{c}^i|s^0)$ among all pairs $(\tilde{q}, (\tilde{c}^i)_{i\in I})$ that satisfy the Pareto constraint $U(\tilde{c}^i|s^0) \geqslant U(c^i|s^0)$ in addition to feasibility constraints (a), (b), and (c), and c and $\tilde{c}$ only differ for finitely many periods.*

From the above result, we can derive the necessary optimality conditions for weak efficiency. For every event s^t, let $I^\star(s^t) = \{i \in I\colon q^\star(c|s^t) = q^i(c^i|s^t)\}$ denote the set of agents having the largest marginal rate of substitution at event s^t given the allocation $(c^i)_{i\in I}$. Fix

[14]Meaning that $U(\tilde{c}^i|s^0) \geqslant U^i(c^i|s^0)$ for all agents i, with a strict improvement for at least one agent.

[15]In scenarios where the default punishment is autarky, Bloise and Reichlin (2011) demonstrated that a laissez-faire equilibrium could be inefficient if interest rates remain sufficiently low indefinitely.

[16]See Bloise et al. (2013) and Bloise (2020).

some arbitrary finite horizon $T \geqslant 1$ and $\varepsilon > 0$. Denote by $\mathcal{P}(T,\varepsilon)$ the set of pairs $(\tilde{q}, (\tilde{c}^i)_{i\in I})$ satisfying conditions (a) and (b), together with, for every event s^t and successors $s^{t+1} \succ s^t$,

(i) if $t > T$, then $\tilde{c}^i(s^t) = c^i(s^t)$ and $\tilde{q}(s^{t+1}) = q^\star(c|s^{t+1})$;

(ii) if $t \leqslant T$, then $|\tilde{c}^i(s^t) - c^i(s^t)| < \varepsilon$ and $|\tilde{q}(s^{t+1}) - q^\star(c|s^{t+1})| < \varepsilon$;

(iii) $\tilde{q}(s^t) = q^i(\tilde{c}^i|s^t)$ for every $i \in I^\star(s^t)$.

By construction, $(q^\star(c), c)$ belongs to $\mathcal{P}(T,\varepsilon)$. Choosing $\varepsilon > 0$ small enough, we can verify that all pairs $(\tilde{q}, \tilde{c})$ in $\mathcal{P}(T,\varepsilon)$ also satisfy the feasibility constraint (c).[17] Utilizing Lemma 3.1, we infer that the pair $(q^\star(c), c)$ constitutes a solution to the social planner's program aimed at maximizing the aggregate social welfare $\sum_{i\in I} U(\tilde{c}^i|s^0)$ carried out over all pairs $(\tilde{q}, (\tilde{c}^i)_{i\in I})$ within the set $\mathcal{P}(T,\varepsilon)$ that satisfy the Pareto constraints $U(\tilde{c}^i|s^0) \geqslant U(c^i|s^0)$.[18] Consequently, this leads to the establishment of the following optimality conditions.

Corollary 3.1 (Optimality Conditions)**.** *Finite-time efficiency of a feasible allocation $c = (c^i)_{i\in I}$ yields the following optimality conditions:*[19]

$$\begin{aligned}\mu(s^t) = \beta^t \pi(s^t) u'(c^i(s^t)) \left[\lambda^i + \xi^i(s^0) + \ldots + \xi^i(s^t)\right] \\ + A(c^i(s^t)) \left[\chi^i(s^t) q(s^t) - \sum_{s^{t+1}\succ s^t} \chi^i(s^{t+1}) q(s^{t+1})\right],\end{aligned} \tag{3.1}$$

and[20]

$$\sum_{i\in I} \chi^i(s^t) = \sum_{i\in I} \sum_{r=0}^{t} \beta^r \pi(s^r) \xi^i(s^r) \frac{\partial V^i_{\text{def}}(\cdot|s^r)}{\partial q(s^t)}(q); \tag{3.2}$$

[17]Indeed, fix an event s^t and an agent i. If $i \in I^\star(s^t)$, then $\tilde{q}(s^t) = q^i(\tilde{c}^i|s^t)$. If $q^\star(c|s^t) > q^i(c^i|s^t)$, then for $\varepsilon > 0$ small enough, we also have $\tilde{q}(s^t) > q^i(\tilde{c}^i|s^t)$ due to the continuity of u'. This proves that $\tilde{q}(s^t) = q^\star(\tilde{c}|s^t)$.

[18]Incorporating the Pareto constraints $U(\tilde{c}^i|s^0) \geqslant U(c^i|s^0)$ alongside the feasibility constraints renders the selection of positive welfare weights $\nu^i > 0$ in defining the social welfare function $\sum_{i\in I} \nu^i U(\tilde{c}^i|s^0)$ non-essential. For the sake of clarity in presentation, we set $\nu^i = 1$ for each agent i.

[19]For any $x > 0$, we let $A(x) := -u''(x)/u'(x)$ denote the Arrow–Pratt measure of risk-aversion at the consumption level x.

[20]By convention, we pose $\xi^i(s^{T+1}) = 0$ for every $s^{T+1} \in S^{T+1}$.

where $(\lambda^i)_{i\in I}$ is a vector of nonnegative welfare weights, $\mu(s^t)$ is the Lagrange multiplier for the market clearing condition (a), $\xi^i(s^t) \geqslant 0$ the Lagrange multiplier for the participation constraint (b), and $\chi^i(s^t)$ the Lagrange multiplier for the implementability constraint (c).[21] *Moreover, Lagrange multipliers are not all equal to zero and satisfy: $\xi^i(s^t) = 0$ if $U(c^i|s^t) > V^i_{\text{def}}(q|s^t)$, and $\chi^i(s^t) = 0$ if $q^i(c^i|s^t) < q^\star(c|s^t)$.*[22]

If a feasible consumption allocation fails to satisfy conditions (3.1) and (3.2), the social planner can make a feasible redistribution in finite time that is Pareto improving. If the initial consumption is implemented by a laissez-faire equilibrium with strictly positive debt limits, then the alternative Pareto superior allocation can be chosen among those supported by a self-enforcing competitive equilibrium (with nonnegative debt limits).

Theorem 3.1 (Implementation)**.** *Consider a laissez-faire equilibrium $(q, (c^i, a^i, D^i)_{i\in I})$ with strictly positive debt limits. If the optimality conditions (3.1) and (3.2) cannot be satisfied for some Lagrange multipliers, then there exists a self-enforcing equilibrium $(\tilde{q}, (\tilde{c}^i, \tilde{a}^i, \tilde{D}^i)_{i\in I})$ with nonnegative debt limits such that $\tilde{c}$ Pareto dominates c. Moreover, $\tilde{c}$ can be chosen as close as desired to c and such that it differs from c for finitely many periods only.*

Consider a consumption allocation $c = (c^i)_{i\in I}$ implemented by a self-enforcing competitive equilibrium $(q, (c^i, a^i, D^i)_{i\in I})$. Following Proposition 2.1, we can assume that $q = q^\star(c)$. It is straightforward to construct Lagrange multipliers such that

$$\mu(s^t) = \beta^t \pi(s^t) u'(c^i(s^t)) \left[\lambda^i + \xi^i(s^0) + \ldots + \xi^i(s^t)\right]. \tag{3.3}$$

Indeed, pose $\mu(s^0) := 1$ and define recursively $\mu(s^t) := q^\star(c|s^t)\mu(\sigma(s^t))$ for every $s^t \succ s^0$. We let $\lambda^i := 1/u'(c^i(s^0))$ and $\xi^i(s^0) := 0$. Define $\xi^i(s^t)$ recursively as follows

$$\xi^i(s^t) := \frac{q^\star(c|s^t) - q^i(c^i|s^t)}{q^i(c^i|s^t)}[\lambda^i + \xi^i(s^0) + \ldots + \xi^i(\sigma(s^t))].$$

We have $\xi^i(s^t) \geqslant 0$. Moreover, if debt limits are not-too-tight, then $q^\star(c|s^t) = q^i(c^i|s^t)$ when the participation constraint is slack. This implies that the constructed ξ^i satisfies the

[21] We can set $\chi^i(s^t) := 0$ for every $i \notin I^\star(q)$ (i.e., when $q^i(c^i|s^t) < q^\star(c|s^t)$).

[22] As the right-hand side of (3.2) is nonpositive, we must have $\sum_{i\in I} \chi^i(s^t) \leqslant 0$.

complementary slackness condition $\xi^i(s^t)[U(c^i|s^t) - V^i_{\text{def}}(q|s^t)] = 0$ of Corollary 3.1. However, in the optimality condition (3.1), we encounter the additional pecuniary term

$$\chi^i(s^t)q(s^t) - \sum_{s^{t+1} \succ s^t} \chi^i(s^{t+1})q(s^{t+1}),$$

which captures the impact of saving and borrowing decisions on the default values. Given the relation in (3.2), it's plausible that this term does not vanish, especially when the default value is sensitive to asset prices. In the forthcoming section, we underscore the relevance of Corollary 3.1 and Theorem 3.1 by exploring a baseline economy and providing specific conditions on primitives (risk aversion, impatience, and dispersion of endowments) under which this pecuniary term is significant. Specifically, we show that the optimality conditions (3.1) and (3.2) are unattainable at a laissez-faire equilibrium of this economy.

4 Tightening Debt Constraints

We show that a policy intervention to tighten debt constraints can lead to Pareto improvements over the laissez-faire allocation. We interpret these interventions as a streamlined representation of regulatory or prudential policies crafted to curtail financial leverage.

What is the intuition for our (constrained) inefficiency result? As the value of default depends on market prices, there is a pecuniary externality that agents do not internalize in a competitive environment. In particular, we show that constraining agents' borrowing capacity at a period τ reduces the credit volume, increases bond prices, or, equivalently, lowers the implied interest rates. This adjustment in market prices exerts a negative feedback on the attractiveness of defaulting in periods $t \leqslant \tau$, as it now becomes more expensive to smooth consumption over time by saving only. Debt constraints at the previous periods $t < \tau$ remain the loosest compatible with repayment incentives. Nevertheless, the increase in bond prices (and the associated fall in the value of default) at τ can make debt repayment a more appealing option, potentially increasing liquidity at these earlier times. This increment in liquidity can support more risk-sharing before period τ. A Pareto improvement is attainable when this enhanced risk-sharing at $t < \tau$ sufficiently offsets the diminished risk-sharing at period τ, leading to an overall net gain in welfare.

To illustrate this intuition most straightforwardly, we focus on a baseline economy. There are two agents facing uncertainty only during the initial period. The economy is thereafter deterministic, in which every other period, agents' endowments switch from a high value to a low value. Within this setting, we perform the following exercise. We first construct a Markov laissez-faire equilibrium $(q, (c^i, a^i, D^i)_{i \in I})$, in which after the realization of uncertainty, the economy settles in a cyclical and symmetric steady-state equilibrium where debt limits are not too tight. Second, we identify sufficient conditions on primitives such that the laissez-faire equilibrium cannot be the solution to the planner's problem. Third, we construct an explicit intervention implementing another competitive equilibrium $(\tilde{q}, (\tilde{c}^i, \tilde{a}^i, \widetilde{D}^i)_{i \in i})$, supported by the same allocation of initial financial claims, but with debt constraints being too tight at a single period τ. We then show that the consumption allocation $(\tilde{c}^i)_{i \in I}$ Pareto dominates the consumption allocation $(c^i)_{i \in I}$ of the laissez-faire equilibrium.

4.1 Baseline Economy

There are two agents $I = \{a, b\}$ who enter the market with an identical endowment $y_0 > 0$ and no financial claims (i.e., $a^a(s^0) = a^b(s^0) = 0$). There is uncertainty only at the initial period $t = 0$, described by two possible states $z^a \neq z^b$. After realizing the state z^i, the economy becomes deterministic where agents' endowments switch between a high value y_{H} and a low value y_{L} with $y_{\text{H}} > y_{\text{L}}$. Realizing state z^i means that the agent i starts with the high endowment at $t = 1$. The beliefs are homogeneous, with each agent assigning the probability $\pi_{\text{H}} = 1/2$ ($\pi_{\text{L}} := 1 - \pi_{\text{H}}$) of getting the high (low, respectively) endowment at $t = 1$. Since there is uncertainty only at the initial period, we simplify notation by writing a generic process $(x(s^t))_{s^t \succeq s^0}$ as follows: $x(s^0) = x_0$ and $x(s^t) = x_t(z)$ if $s^t \succeq (s^0, z)$ with $z \in \{z^a, z^b\}$.[23] The representation of the event tree is as in Figure 4.1 below.

[23]The event tree Σ can be formally defined as follows: $S^0 := \{s^0\}$ and for every $t \geqslant 1$, $S^t = \{(z^a, t), (z^b, t)\}$. The binary relation $\succ$ is defined as follows: $(z, 1) \succ s^0$ and $(z, \tau) \succ (\zeta, t)$ when $z = \zeta$ and $\tau > t$. Let $s^t = (z, t)$ for some $t \geqslant 1$. Agent i's beliefs are defined by $\pi^i(s^t) = \pi_{\text{H}}$ if $y^i(z, 1) = y_{\text{H}}$ and $\pi^i(s^t) = \pi_{\text{L}}$ if $y^i(z, 1) = y_{\text{L}}$.

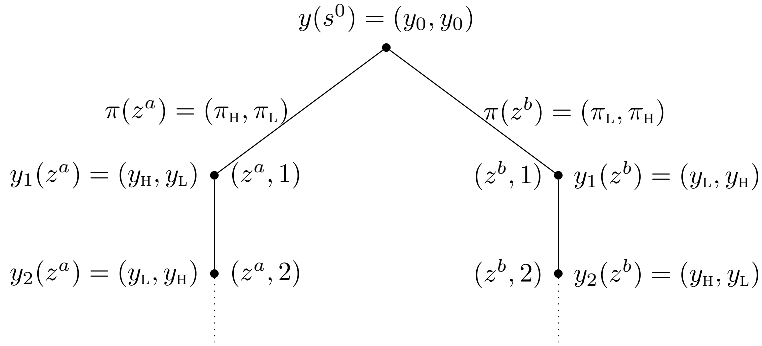


Figure 4.1: Event tree and endowments.

For future reference, we point out that the *symmetric first-best allocation* of this economy obtains when both agents consume their endowment at $t = 0$ and the perfect risk-sharing consumption level $c^{\mathrm{fb}} := (y_{\mathrm{H}} + y_{\mathrm{L}})/2$ at every period $t \geqslant 1$.

4.2 Laissez-Faire Equilibrium

For this baseline economy we single out a symmetric Markov equilibrium as follows: at period $t = 0$, both agents borrow up to the debt limit d^{lf} against their high-income state and save contingent on their low-income state. After resolving the uncertainty at period $t = 1$, the economy settles in a cyclical steady-state where the low-income agent borrows up to the not-too-tight debt limit d^{lf} while the high-income agent saves.

Proposition 4.1. *Assume interest rates at autarky are negative, i.e., $\beta u'(y_{\mathrm{L}}) > u'(y_{\mathrm{H}})$. Then, there exists an equilibrium $(q, (c^i, a^i, D^i)_{i \in I})$ with not-too-tight debt limits where for each $z \in \{z^a, z^b\}$ and every $i \in I$:*

(i) debt limits equal $D^i_t(z) = d^{\mathrm{lf}}$, for $t \geqslant 1$;

(ii) consumption is: $c^i_0 = y_0$ at $t = 0$, $c^i_t(z) = c^{\mathrm{lf}}_{\mathrm{H}}$ if $y^i_t(z) = y_{\mathrm{H}}$, and $c^i_t(z) = c^{\mathrm{lf}}_{\mathrm{L}}$ if $y^i_t(z) = y_{\mathrm{L}}$, for $t \geqslant 1$;

(iii) net asset positions are $a^i_t(z) = -d^{\mathrm{lf}}$ (i.e., the debt limit binds) if $y^i_t(z) = y_{\mathrm{H}}$, and $a^i_t(z) = d^{\mathrm{lf}}$ if $y^i_t(z) = y_{\mathrm{L}}$, for $t \geqslant 1$;

(iv) prices are given by $q_1(z) = \beta\pi_{\mathrm{L}} u'(c^{\mathrm{lf}}_{\mathrm{L}})/u'(y_0)$ *and* $q_{t+1}(z) = q^{\mathrm{lf}} = 1$*, for* $t \geqslant 1$.

The proof of the proposition is standard (and, therefore, omitted). Debt limits are trivially not-too-tight as $d^{\mathrm{lf}} = q^{\mathrm{lf}} d^{\mathrm{lf}}$. The equilibrium net trade $x^{\mathrm{lf}} := (1 + q^{\mathrm{lf}})d^{\mathrm{lf}} = 2d^{\mathrm{lf}}$ is determined by the first order condition:

$$1 = q^{\mathrm{lf}} = \beta \frac{u'(c^{\mathrm{lf}}_{\mathrm{L}})}{u'(c^{\mathrm{lf}}_{\mathrm{H}})} = \beta \frac{u'(y_{\mathrm{L}} + x^{\mathrm{lf}})}{u'(y_{\mathrm{H}} - x^{\mathrm{lf}})}.$$

At $t = 1$, the continuation utilities are $U^{\mathrm{lf}}_{\mathrm{H}} = [u(y_{\mathrm{H}} - x^{\mathrm{lf}}) + \beta u(y_{\mathrm{L}} + x^{\mathrm{lf}})]/(1-\beta^2)$ for the high-income agent and $U^{\mathrm{lf}}_{\mathrm{L}} = [u(y_{\mathrm{L}} + x^{\mathrm{lf}}) + \beta u(y_{\mathrm{H}} - x^{\mathrm{lf}})]/(1-\beta^2)$. As the equilibrium is symmetric, both agents share the same time 0 expected utility $U^{\mathrm{lf}}_0 := u(y_0) + (\beta/2)(U^{\mathrm{lf}}_{\mathrm{H}} + U^{\mathrm{lf}}_{\mathrm{L}})$.

We conclude by introducing some notation that will be useful in the following sections. To this end, we let $A_{\mathrm{H}} := -u''(c^{\mathrm{lf}}_{\mathrm{H}})/u'(c^{\mathrm{lf}}_{\mathrm{H}})$ and $A_{\mathrm{L}} := -u''(c^{\mathrm{lf}}_{\mathrm{L}})/u'(c^{\mathrm{lf}}_{\mathrm{L}})$ denote the Arrow–Pratt measure of risk-aversion at the laissez-faire consumption levels $c^{\mathrm{lf}}_{\mathrm{H}}$ and $c^{\mathrm{lf}}_{\mathrm{L}}$, respectively. When u is the CRRA utility function with risk-aversion coefficient $\gamma > 0$, the contingent consumption and the net trade values satisfy:

$$c^{\mathrm{lf}}_{\mathrm{H}} = \alpha_{\mathrm{H}}(y_{\mathrm{H}} + y_{\mathrm{L}}), \quad c^{\mathrm{lf}}_{\mathrm{L}} = \alpha_{\mathrm{L}}(y_{\mathrm{H}} + y_{\mathrm{L}}) \quad \text{and} \quad x^{\mathrm{lf}} = \alpha_{\mathrm{L}} y_{\mathrm{H}} - \alpha_{\mathrm{H}} y_{\mathrm{L}}$$

where the coefficients α_{H} and α_{L} are given by:

$$\alpha_{\mathrm{H}} = \frac{\beta^{-1/\gamma}}{1 + \beta^{-1/\gamma}} \quad \text{and} \quad \alpha_{\mathrm{L}} = \frac{1}{1 + \beta^{-1/\gamma}}.$$

Letting $\zeta := y_{\mathrm{H}}/y_{\mathrm{L}}$ denote the income dispersion, we also deduce that:

$$A_{\mathrm{H}} x^{\mathrm{lf}} = \gamma \frac{\beta^{1/\gamma}\zeta - 1}{\zeta + 1} \quad \text{and} \quad A_{\mathrm{L}} x^{\mathrm{lf}} = \gamma \frac{\zeta - \beta^{-1/\gamma}}{\zeta + 1}.$$

Finally, observe that interest rates at autarky are negative if, and only if, $\beta^{1/\gamma}\zeta > 1$.

4.3 Inefficiency

In what follows, we demonstrate that the allocations and asset prices of the laissez-faire equilibrium are incompatible with the planner's optimality conditions. To this purpose, we assume that the laissez-faire consumption allocation and asset prices are indeed the solutions of the system of conditions (3.1)–(3.2) for some Lagrange multipliers. We then argue that those Lagrange multipliers supporting the implied optimality conditions must be equal to zero for a large set of parameters. The main challenge in this construction is the computation of Lagrange multipliers $\chi_t(z) := \sum_{i\in I} \chi^i(z^t)$ associated with the asset price choice. This is because their characterization relates to the derivative of the default value. We refer to Appendix D for the detailed arguments that lead to the following characterization.

Proposition 4.2. *For any event $s^t = (t, z)$, the Lagrange multiplier $\chi_t(z)$ in (3.2) satisfies:*

$$\chi_t(z) = \begin{cases} 0, & \text{if } t = 1; \\ -\beta^{t-1}\pi_{\text{H}} u'(c_{\text{H}}^{\text{lf}}) x^{\text{lf}} \sum_{k=0}^{n-1} \xi_{2k+1}^{h_{2k+1}}(z), & \text{if } t = 2n \geqslant 2 \text{ is even;} \\ -\beta^{t-1}\pi_{\text{L}} u'(c_{\text{H}}^{\text{lf}}) x^{\text{lf}} \sum_{k=1}^{n} \xi_{2k}^{h_{2k}}(z), & \text{if } t = 2n+1 \geqslant 3 \text{ is odd,} \end{cases} \tag{4.1}$$

where h_r is the agent with high income at event (r, z), and $\xi_r^{h_r}(z)$ is the Lagrange multiplier of the participation constraint at event (r, z).

We subsequently use the expressions of $\chi_t(z)$ from (4.1) to write down a simpler version of the social planner's optimality conditions (3.1). To establish our contradiction, it is sufficient we analyze the FOCs of the first four periods $t \in \{0, 1, 2, 3\}$.

At $t = 0$. The Lagrange multiplier χ_0^i does not exist. Recall that

$$q_1(z) = q_1 := \beta\pi_{\text{L}} \frac{u'(c_{\text{L}}^{\text{lf}})}{u'(y_0)}, \quad \text{for every } z \in \{z^a, z^b\}.$$

As $y_1^i(z^i) = y_{\text{H}}$, this implies that $\chi_1^i(z^i) = 0$ and $\chi_1^i(z^j) = \chi_1(z)$ for $j \neq i$. The participation constraint at $t = 0$ is slack. This implies that $\xi_0^i = 0$ for each i.

The optimality conditions corresponding to the choice of c_0^i then reads:

$$u'(y_0)\lambda^i - A_0\chi_1(z^j)q_1(z^j) = \mu_0,$$

where $A_0 := -u''(y_0)/u'(y_0)$. Since $\chi_1(z) = 0$ for all $z \in \{z^a, z^b\}$, we deduce that:

$$u'(y_0)\lambda^i = \mu_0. \tag{4.2}$$

This implies that agents must be treated symmetrically, i.e., $\lambda^i = \lambda^j =: \lambda$. Multiplying all Lagrange multipliers by a strictly positive number if necessary, we can assume, without any loss of generality, that $\lambda = \mu_0/u'(y_0) \in \{0, 1\}$.

At $t = 1$. Fix an agent i and consider the event at $t = 1$ corresponding to the realization of shock z^i, i.e., agent i has high income. As agent i's participation constraint is binding, condition (3.1) reads as follows:

$$\beta\pi_{\mathrm{H}} u'(c^{\mathrm{lf}}_{\mathrm{H}}) \left[\lambda + \xi^i_1(z^i)\right] - A_{\mathrm{H}}\chi_2(z^i)q^{\mathrm{lf}} = \mu_1(z^i),$$

where we use the fact that $\chi^i_2(z^i) = \chi_2(z^i)$ and $\chi^i_1(z^i) = 0$.

For agent j (the low-income agent), the participation is not binding, i.e., $\xi^j_1(z^i) = 0$, so condition (3.1) reads as follows:

$$\beta\pi_{\mathrm{L}} u'(c^{\mathrm{lf}}_{\mathrm{L}})\lambda + A_{\mathrm{L}}\chi_1(z^i)q^{\mathrm{lf}} = \mu_1(z^i),$$

where we use the fact that $\chi^j_2(z^i) = 0$ and $\chi^j_1(z^i) = \chi_1(z^i)$.

From Proposition (4.1), we infer that $\chi_1(z^i) = 0$ and:

$$\chi_2(z^i) = -\pi_{\mathrm{H}}\beta u'(c^{\mathrm{lf}}_{\mathrm{H}})x^{\mathrm{lf}}\xi^i_1(z^i).$$

Setting $\xi_1 := (\xi^a_1(z^a) + \xi^b_1(z^b))/2$ produces:

$$\beta\pi_{\mathrm{H}} u'(c^{\mathrm{lf}}_{\mathrm{H}})[\lambda + \xi_1] + \beta\pi_{\mathrm{H}} u'(c^{\mathrm{lf}}_{\mathrm{H}})A_{\mathrm{H}}x^{\mathrm{lf}}\xi_1 q^{\mathrm{lf}} = \beta\pi_{\mathrm{L}} u'(c^{\mathrm{lf}}_{\mathrm{L}})\lambda.$$

Rearranging terms and using the fact that $\beta u'(c^{\mathrm{lf}}_{\mathrm{L}}) = u'(c^{\mathrm{lf}}_{\mathrm{H}})$, we get:

$$\pi_{\mathrm{H}}\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)\xi_1 = (\pi_{\mathrm{L}} - \beta\pi_{\mathrm{H}})\lambda. \tag{4.3}$$

At $t \in \{2, 3\}$. Following similar arguments, we can show that the analogous equalities for periods $t = 2$ and $t = 3$ are, respectively,

$$\pi_{\mathrm{H}}(A_{\mathrm{L}}x^{\mathrm{lf}} - 1)\xi_1 + \pi_{\mathrm{L}}\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)\xi_2 = (\pi_{\mathrm{H}} - \beta\pi_{\mathrm{L}})\lambda, \tag{4.4}$$

$$\pi_{\mathrm{H}}\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)(\xi_1 + \xi_3) + \pi_{\mathrm{L}}(A_{\mathrm{L}}x^{\mathrm{lf}} - 1)\xi_2 = (\pi_{\mathrm{L}} - \beta\pi_{\mathrm{H}})\lambda. \tag{4.5}$$

If $A_{\mathrm{L}}x^{\mathrm{lf}} - 1 > 0$, then direct inspection of (4.3)–(4.5) implies that $\xi_2 = \xi_3 = 0$. So taking into account condition (4.4), the optimality conditions collapse to the following equations:

$$\pi_{\mathrm{H}}\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)\xi_1 = (\pi_{\mathrm{L}} - \beta\pi_{\mathrm{H}})\lambda \quad \text{and} \quad \pi_{\mathrm{H}}(A_{\mathrm{L}}x^{\mathrm{lf}} - 1)\xi_1 = (\pi_{\mathrm{H}} - \beta\pi_{\mathrm{L}})\lambda.$$

We cannot have $\lambda = 0$; otherwise, all Lagrange multipliers are equal to zero. Therefore, we must have $\lambda = 1$. As we assume $\pi_{\mathrm{L}} = \pi_{\mathrm{H}}$ and $\beta \in (0, 1)$, the above system has no solution when $A_{\mathrm{L}}x^{\mathrm{lf}} - 1 \neq \beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)$. Applying Corollary 3.1 and Theorem 3.1, we get the following result.

Theorem 4.1. *Assume that*

$$0 < A_{\mathrm{L}}x^{\mathrm{lf}} - 1 \quad \textit{and} \quad A_{\mathrm{L}}x^{\mathrm{lf}} - 1 \neq \beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1), \tag{4.6}$$

then the laissez-faire equilibrium outcome is finite-time inefficient and, therefore, can be Pareto dominated by another equilibrium with self-enforcing and nonnegative debt limits.

In particular, for the CRRA utility function characterized by a risk-aversion coefficient $\gamma > 0$, the sufficient condition (4.6) can be rewritten as:

$$g(\beta|\zeta) < \gamma \quad \text{and} \quad \gamma \neq h(\beta|\zeta), \tag{4.7}$$

where $g(\beta|\zeta)$ is the value of γ that solves $\zeta + 1 = \gamma(\zeta - \beta^{-1/\gamma})$ and $h(\beta|\zeta)$ the value of γ that solves $\gamma(\zeta - \beta^{-1/\gamma})(1 - \beta^{1+1/\gamma}) = (\zeta + 1)(1 + \beta)$.

In Figure 4.2, we plot the functions $f(\cdot|\underline{\zeta})$, $g(\cdot|\underline{\zeta})$, and $h(\cdot|\underline{\zeta})$ when income dispersion is $\underline{\zeta} = 2$.[24] The green region represents the primitive values (β, γ) satisfying (4.7). The red region represents the set of primitive values (β, γ) where a laissez-faire equilibrium with trade does not exist.[25]

[24] When the discount factor is $\overline{\beta} = 0.9$ and the risk-aversion coefficient satisfies $\gamma \geqslant 4$, then the laissez-faire equilibrium is not a solution to any social planner problem.

[25] Recall that a necessary (and sufficient) condition for the existence of a laissez-faire equilibrium is that interest at autarky is negative. This occurs when $\gamma \leqslant f(\beta|\zeta)$.

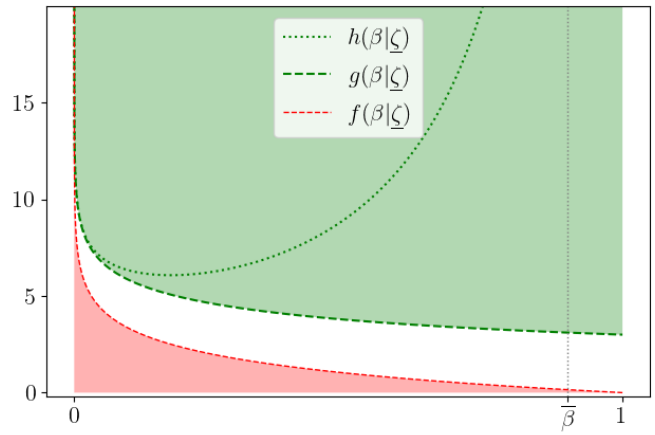


Figure 4.2: Laissez-faire equilibrium is finite-time inefficient.

4.4 Pareto-Improving Policy Intervention

In the previous section, we demonstrated that alternative equilibria, characterized by nonnegative, self-enforcing debt limits, can Pareto dominate laissez-faire equilibria. Our approach was based on a non-constructive proof method, employing a contradiction to illustrate that the optimality conditions necessary for finite-time efficiency are not met. This section provides fresh analytical insights on how superior economic efficiency is obtained in economies with unsecured debt markets by constructing a specific equilibrium with too-tight debt limits that Pareto dominates the laissez-faire outcome.

For clarity, we focus on an equilibrium where agents borrow up to their debt limits contingent on high income, denoted by d_t, and save contingent on low income. Such equilibria are uniquely determined by the sequence $(q_{t+1}, d_t)_{t\geqslant 1}$, where q_{t+1} is the asset price at date t.[26] Here, net trade is given by $x_t := d_t + q_{t+1}d_{t+1}$. Consumption for the high-income agent is given by $c_{\mathrm{H},t} = y_{\mathrm{H}} - x_t$, while for the low-income agent, it is $c_{\mathrm{L},t} = y_{\mathrm{L}} + x_t$. At $t = 0$, both agents consume their endowment y_0, and the two contingent bonds share the same price q_1.

[26] The debt limit contingent on low income need not be specified since our focus is on equilibria where agents save to hedge against the low-income shock.

The conditions ensuring the optimality of individual choices are, at $t = 0$,

$$q_1 = \beta\pi_{\mathrm{L}}\frac{u'(y_{\mathrm{L}} + x_1)}{u'(y_0)} \geqslant \beta\pi_{\mathrm{H}}\frac{u'(y_{\mathrm{H}} - x_1)}{u'(y_0)}; \tag{4.8}$$

and, for any $t > 1$,[27]

$$q_{t+1} = \beta\frac{u'(y_{\mathrm{L}} + x_{t+1})}{u'(y_{\mathrm{H}} - x_t)} \geqslant \beta\frac{u'(y_{\mathrm{H}} - x_{t+1})}{u'(y_{\mathrm{L}} + x_t)}. \tag{4.9}$$

We propose a targeted policy intervention that tightens debt constraints for a single period τ, thereby achieving an equilibrium allocation that is Pareto superior to the laissez-faire allocation. Specifically, we set $\tau = 2$ and construct an equilibrium where the low-income agent faces a too-tight debt limit d_3 at date $t = \tau$. Formally, the equilibrium has the following features:

(a) For periods $t \in \{0, 1\}$, there is no intervention, and the debt limit d_{t+1} is not too tight;

(b) At period $t = 2$, a policy intervention imposes a debt limit $d_3 := (1 - \varepsilon)q_4 d^{\mathrm{lf}}$;

(c) At period $t = 3$, the high-income agent starts with the too-tight debt level d_3. Since the intervention is over, the debt limit reverts to its laissez-faire level d^{lf}, which is not too tight given future debt limits;[28]

(d) For $t \geqslant 4$, the laissez-faire equilibrium prevails: $d_t = d^{\mathrm{lf}}$ and $q_{t+1} = q^{\mathrm{lf}}$.

By construction, all debt limits are self-enforcing. In particular, Proposition 2.2 assures that d_3 is too tight when $\varepsilon > 0$.

We refer to the above as an *equilibrium with ε-tight debt constraints*, and we use the notation $(q_{t+1}(\varepsilon), d_t(\varepsilon))$ to emphasize the dependence on the tightening coefficient ε. Our contribution amounts to identifying a set of parameter values in which such equilibria exist and feature a consumption allocation that is Pareto superior to the laissez-faire allocation. We summarize our main result below.

[27] To ensure that consumption is positive, we also need that $-y_{\mathrm{L}} < x_t < y_{\mathrm{H}}$ for every $t \geqslant 1$.

[28] The not-too-tight debt limits d_1 and d_2 though may differ from d^{lf} as d_3 is too-tight.

Theorem 4.2. *If the interest rate at autarky is negative (i.e., $\beta u'(y_{\mathrm{L}}) > u'(y_{\mathrm{H}})$), then, for sufficiently small $\varepsilon > 0$, there exists an equilibrium with ε-tight debt constraints at $\tau = 2$ that Pareto dominates the laissez-faire equilibrium provided that the following condition holds:*

$$0 < A_{\mathrm{L}} x^{\mathrm{lf}} - 1 < \beta(A_{\mathrm{H}} x^{\mathrm{lf}} + 1). \tag{4.10}$$

It is worth noting that the thresholds $A_{\mathrm{L}} x^{\mathrm{lf}} - 1$ and $\beta(A_{\mathrm{H}} x^{\mathrm{lf}} + 1)$ are exactly those that already appear in the statement of Theorem 4.1. In particular, for the CRRA utility function characterized by a risk-aversion coefficient $\gamma > 0$, the sufficient condition (4.10) can be rewritten as:

$$g(\beta|\zeta) < \gamma < h(\beta|\zeta), \tag{4.11}$$

where $g(\beta|\zeta)$ is the value of γ that solves $\zeta + 1 = \gamma(\zeta - \beta^{-1/\gamma})$ and $h(\beta|\zeta)$ the value of γ that solves $\gamma(\zeta - \beta^{-1/\gamma})(1 - \beta^{1+1/\gamma}) = (\zeta + 1)(1 + \beta)$.

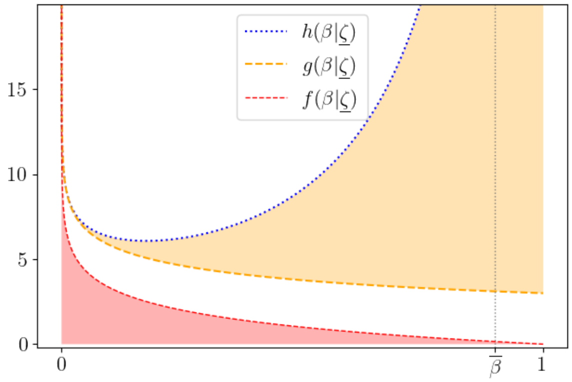
In Figure 4.3(a), we plot the functions $f(\cdot|\underline{\zeta})$, $g(\cdot|\underline{\zeta})$, and $h(\cdot|\underline{\zeta})$ when income dispersion is $\underline{\zeta} = 2$. The orange region represents the set of primitive values (β, γ) where the intervention is Pareto-improving.[29]

Denote by $U_0(\varepsilon)$ the common expected and discounted utility at $t = 0$.[30] We let $\mathrm{ce}(\varepsilon)$ be the certainty equivalent of the expected continuation utility at $t = 1$. This consumption level is determined by the equation $U_0(\varepsilon) = u(y_0) + \beta u(\mathrm{ce}(\varepsilon))$. Figure 4.3(b) offers an alternative illustration when $\bar{\beta} = 0.9$ by plotting the difference of the certainty equivalent consumption, $\varepsilon \mapsto \mathrm{ce}(\varepsilon) - \mathrm{ce}(0)$. Pareto improvement obtains if, and only if, $\mathrm{ce}(\varepsilon) - \mathrm{ce}(0) > 0$.

[29] The red region represents the set of primitive values (β, γ) where a laissez-faire equilibrium with trade does not exist.

[30] It is formally defined as $U_0(\varepsilon) = u(y_0) + \frac{1}{2} \sum_{t \geqslant 1} \beta^t \left[u(y_{\mathrm{L}} + x_t) + u(y_{\mathrm{H}} - x_t) \right]$.

(a) Set of primitive values (β, γ)



(b) $\mathrm{ce}(\varepsilon) - \mathrm{ce}(0)$

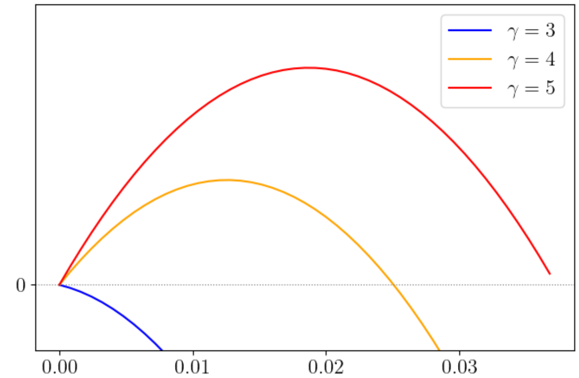


Figure 4.3: Tightening Debt Constraints at $\tau = 2$ is Pareto-improving.

In the following sections, we elucidate the intuition and furnish the analytical underpinnings for Theorem 4.2. In the equilibrium scenario with ε-tight debt constraints, the economy undergoes a transitional phase before reverting to the laissez-faire regime at period $t = 4$. A crucial step in this analysis is determining the not-too-tight debt limits d_1 and d_2 that prevail during the transition. This task is particularly challenging since the powerful characterization result of Hellwig and Lorenzoni (2009) does not apply. Specifically, it is not true that d_1 and d_2, despite being not too tight, permit the exact rollover of debt. This would be the case if all future debt limits were set to be not too tight, a situation that our policy experiment rules out. Therefore, their determination cannot be solely based on the equilibrium price specifications.

To overcome this issue, it becomes necessary to compute the value functions for periods $t \in \{1, 2\}$ that correspond to both equilibrium and out-of-equilibrium paths. The debt limits d_1 and d_2 then emerge as nontrivial solutions to the not-too-tight condition, as expressed in Equation (2.6).

4.5 Equilibrium Construction

To streamline the exposition, we break down the construction of the equilibrium into several steps. We begin by identifying the equilibrium variables for period $t = 3$ and then proceed backward to ascertain the corresponding variables for periods $t \in \{2, 1\}$. At $t = 4$, the economy settles down at the laissez-faire equilibrium described in Proposition 4.2.

The intervention forces the high-income agent to start at period $t = 3$ with a level of debt:

$$d_3(\varepsilon) = (1 - \varepsilon) q_4(\varepsilon) d^{\mathrm{lf}}. \tag{4.12}$$

To support the laissez-faire steady state next period, this agent must save d^{lf} while the low-income agent must find it optimal to borrow the same amount. In this case, the net trade and equilibrium price at period $= 3$ are jointly determined by the following two equations:

$$q_4(\varepsilon) = \beta \frac{u'(y_{\mathrm{L}} + x^{\mathrm{lf}})}{u'(y_{\mathrm{H}} - x_3(\varepsilon))} \quad \text{and} \quad x_3(\varepsilon) = (2 - \varepsilon) q_4(\varepsilon) d^{\mathrm{lf}}. \tag{4.13}$$

The first equation is the FOC of the high-income agent, while the second equation is obtained by substituting $d_3(\varepsilon)$ in $x_3(\varepsilon) = d_3(\varepsilon) + q_4(\varepsilon) d^{\mathrm{lf}}$ that stands for net trade.

It follows from a straightforward application of the Implicit Function Theorem that there exists $\bar{\varepsilon} > 0$ and continuously differentiable functions $q_4 : [0, \bar{\varepsilon}] \to (0, \infty)$ and $x_3 : [0, \bar{\varepsilon}] \to (0, \infty)$ satisfying (4.13) and such that $q_4(0) = q^{\mathrm{lf}} = 1$ and $x_3(0) = x^{\mathrm{lf}}$. To be optimal for the low-income agent to issue the maximum debt level d^{lf}, we must have that:

$$q_4(\varepsilon) \geqslant \beta \frac{u'(c_{\mathrm{H}}^{\mathrm{lf}})}{u'(y_{\mathrm{L}} + x_3(\varepsilon))}. \tag{4.14}$$

As $x_3(0) = x^{\mathrm{lf}}$ and $q_4(0) = 1 > \beta u'(c_{\mathrm{H}}^{\mathrm{lf}})/u'(c_{\mathrm{L}}^{\mathrm{lf}})$, we can reduce $\bar{\varepsilon} > 0$ so that (4.14) is also satisfied for every $\varepsilon \in [0, \bar{\varepsilon}]$. Differentiating the FOC in (4.13) at $\varepsilon = 0$, we get that:

$$q_4'(0) = \frac{A_{\mathrm{H}} x^{\mathrm{lf}}}{2(A_{\mathrm{H}} x^{\mathrm{lf}} + 1)} > 0 \quad \text{and} \quad x_3'(0) = -\frac{1}{A_{\mathrm{H}}} q_4'(0) < 0.$$

Figure 4.4 plots the functions $\varepsilon \mapsto q_4(\varepsilon)$ and $\varepsilon \mapsto x_3(\varepsilon)$ for CRRA utility with different values of γ. The income dispersion coefficient and discount factor values are set to $\underline{\zeta} = 2$ and $\bar{\beta} = 0.9$. Observe that the higher the tightening coefficient ε, the lower the consumption smoothing (i.e., the function $\varepsilon \mapsto x_3(\varepsilon)$ is decreasing), and the higher is the asset price $q_4(\varepsilon)$ (or, equivalently, the lower is the implied interest rate). The contraction of net trade reflects the lower debt ceiling as ε increases. Indeed, $d_3(0) = d^{\text{lf}}$ and $d_3'(0) = (q_4'(0) - 1)d^{\text{lf}} < 0$.

(a) Price $q_4(\varepsilon)$ (b) Net Trade $x_3(\varepsilon)$

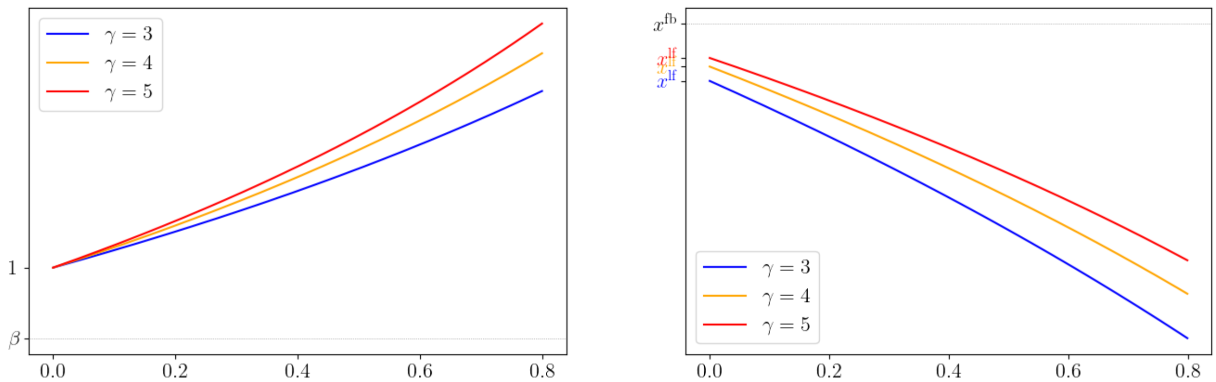


Figure 4.4: Equilibrium variables at $t = 3$ as functions of the tightening coefficient ε.

Tightening debt constraints at period $t = 2$ reduces net trade at $t = 3$. This reduction proves, at first instance, to be disadvantageous when viewed from an ex-ante standpoint. Specifically, the time 0 expected utility is given by:

$$U_0(\varepsilon) := u(y_0) + \frac{1}{2} \sum_{t \geqslant 1} \beta^t \varphi(x_t(\varepsilon)) \quad \text{where} \quad \varphi(x) := u(y_{\text{H}} - x) + u(y_{\text{L}} + x).$$

When $x_t < x^{\text{fb}}$, it holds that $\varphi'(x_t) > 0$. However, these changes produce secondary effects through their feedback on the equilibrium variables of previous periods. In particular, as we illustrate below, these changes alter the default value in periods $t \in \{1, 2\}$, affecting the not-too-tight debt limits d_1 and d_2. If default values reduce, d_1 and d_2 might exceed the laissez-faire limit d^{lf}. This might lift net trade over the laissez-faire value x^{lf}, increasing ex-ante utility. There is scope for Pareto improvement when the reduction in utility due to the contraction of net trade at period $t = 3$ is offset by the increase in utility due to the expansion of net trade in periods $t \in \{1, 2\}$.

4.5.1 Computing the Not-Too-Tight Debt Limit at $t = 2$

We now examine the equilibrium variables for the period $t = 2$. Specifically, we focus on determining the not-too-tight debt limit d_2. This is equivalent to determining the net trade x_2.

The high-income agent starts with debt $d_2(\varepsilon)$ and saves the amount $d_3(\varepsilon)$. This position is accommodated by the low-income agent who borrows the same amount. Net trade at this period therefore equals $x_2(\varepsilon) = d_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon)$. The FOC associated with these saving and borrowing decisions are, respectively,

$$q_3(\varepsilon) = \beta \frac{u'(y_{\mathrm{L}} + x_3(\varepsilon))}{u'(y_{\mathrm{H}} - x_2(\varepsilon))} \quad \text{and} \quad q_3(\varepsilon) \geqslant \beta \frac{u'(y_{\mathrm{H}} - x_3(\varepsilon))}{u'(y_{\mathrm{L}} + x_2(\varepsilon))}. \tag{4.15}$$

Since we require the debt limit $d_2(\varepsilon)$ to be not too tight, condition (2.4) reads as follows:

$$u(y_{\mathrm{H}} - x_2(\varepsilon)) + \beta u(y_{\mathrm{L}} + x_3(\varepsilon)) + \beta^2 U^{\mathrm{lf}}_{\mathrm{H}} = V^{\mathrm{def}}_{\mathrm{H},2}(\varepsilon), \tag{4.16}$$

where $U^{\mathrm{lf}}_{\mathrm{H}}$ is the continuation utility of the high-income agent in the laissez-faire equilibrium, and $V^{\mathrm{def}}_{2,\mathrm{H}}(\varepsilon)$ is the out-of-equilibrium value function of the high-income agent at $t = 2$.

The existence and behavior of functions $\varepsilon \mapsto x_2(\varepsilon)$ and $\varepsilon \mapsto q_3(\varepsilon)$ compatible with (4.15) and (4.16) depend on the behavior of the default value function $\varepsilon \mapsto V^{\mathrm{def}}_{2,\mathrm{H}}(\varepsilon)$. The latter is affected by the equilibrium prices. Therefore, its determination relies on an educated guess about the out-of-equilibrium path. We postulate the following default value function:

$$V^{\mathrm{def}}_{\mathrm{H},2}(\varepsilon) = u(y_{\mathrm{H}} - q_3(\varepsilon)\theta_3(\varepsilon)) + \beta u(y_{\mathrm{L}} + \theta_3(\varepsilon)) + \beta^2 U^{\mathrm{lf}}_{\mathrm{H}}, \tag{4.17}$$

that is derived by assuming that the defaulting agent only saves when income is high. In particular, the agent with high income at period $t = 2$ will not save at period $t = 3$ when income becomes low, and the out-of-equilibrium continuation value at period $t = 4$ will then coincide with the laissez-faire continuation utility $U^{\mathrm{lf}}_{\mathrm{H}}$. This is because d^{lf} is not-too-tight so that $V^{\mathrm{def}}_{\mathrm{H},4}(\varepsilon) = U^{\mathrm{lf}}_{\mathrm{H}}$. The necessary and sufficient FOCs to support the out-of-equilibrium guess are

$$q_3(\varepsilon) = \beta \frac{u'(y_{\mathrm{L}} + \theta_3(\varepsilon))}{u'(y_{\mathrm{H}} - q_3(\varepsilon)\theta_3(\varepsilon))} \quad \text{and} \quad q_4(\varepsilon) \geqslant \beta \frac{u'(c^{\mathrm{lf}}_{\mathrm{H}})}{u'(y_{\mathrm{L}} + \theta_3(\varepsilon))}. \tag{4.18}$$

Replacing the expression of the default option from (4.17) in the not-too-tight condition (4.16) gives:

$$u(y_{\mathrm{H}} - x_2(\varepsilon)) + \beta u(y_{\mathrm{L}} + x_3(\varepsilon)) = \underbrace{u(y_{\mathrm{H}} - q_3(\varepsilon)\theta_3(\varepsilon)) + \beta u(y_{\mathrm{L}} + \theta_3(\varepsilon))}_{=:\, \psi_2(\varepsilon)}. \tag{4.19}$$

Reducing $\bar{\varepsilon} > 0$ if necessary, it follows from the Implicit Function Theorem that there exist continuously differentiable functions $q_3, x_2, \theta_3 : [0, \bar{\varepsilon}] \to (0, \infty)$ satisfying the equations in (4.15) and (4.18) as well as equation (4.19). Moreover, we have $q_3(0) = q_4(0) = 1$, $x_2(0) = x_3(0) = x^{\mathrm{lf}}$ and $\theta_3(0) = x^{\mathrm{lf}}$. Therefore, for $\varepsilon = 0$, the weak inequalities in (4.15) and (4.18) read as $q_3(0) = q_4(0) = 1 > \beta u'(c^{\mathrm{lf}}_{\mathrm{H}})/u'(c^{\mathrm{lf}}_{\mathrm{L}})$, so they are both satisfied for ε small enough. The not-too-tight debt limit $d_2(\varepsilon)$ is then obtained as the difference $x_2(\varepsilon) - q_3(\varepsilon)d_3(\varepsilon)$.

A closer inspection of (4.19) reveals that a necessary condition for net trade $x_2(\varepsilon)$ to be increasing in ε is that the value of the default option $\psi_2(\varepsilon)$ is itself decreasing in ε. Nevertheless, a decreasing default option is still compatible with $x_2(\varepsilon)$ decreasing since net trade $x_3(\varepsilon)$ decreases unambiguously with ε. Differentiating the equation in (4.15) and the not-too-tight condition (4.19) at $\varepsilon = 0$ produces:

$$q_3'(0) = -(x_2'(0)A_{\mathrm{H}} + x_3'(0)A_{\mathrm{L}}) \quad \text{and} \quad x_2'(0) = x_3'(0) + q_3'(0)x^{\mathrm{lf}}. \tag{4.20}$$

Solving the above system, we deduce that

$$q_3'(0) = -x_3'(0)\frac{A_{\mathrm{L}} + A_{\mathrm{H}}}{A_{\mathrm{H}}x^{\mathrm{lf}} + 1} > 0 \quad \text{and} \quad x_2'(0) = -x_3'(0)\frac{A_{\mathrm{L}}x^{\mathrm{lf}} - 1}{A_{\mathrm{H}}x^{\mathrm{lf}} + 1}. \tag{4.21}$$

Given that $u'(c^{\mathrm{lf}}_{\mathrm{H}}) = \beta u'(c^{\mathrm{lf}}_{\mathrm{L}})$, a similar exercise yields $\psi_2'(0) = -q_3'(0)u'(c^{\mathrm{lf}}_{\mathrm{H}})x^{\mathrm{lf}} < 0$.

Our optimality conditions analysis reveals that, for ε small enough, the asset price $q_3(\varepsilon)$ increases. The higher price at $t = 2$ reduces the value of the default option $\psi_2(\varepsilon)$. As $d_2(\varepsilon)$ is required to be not-too-tight, the left-hand side of the not-too-tight condition (4.19) has to adjust accordingly. The adjustment occurs by the unambiguous fall of net trade $x_3(\varepsilon)$ at period $t = 3$. However, when parameters are such that $A_{\mathrm{L}}x^{\mathrm{lf}} > 1$, the decrease of $x_3(\varepsilon)$ is insufficient to compensate for the decrease of the default option.[31] This, in turn,

[31] The restriction corresponds to the first inequality of condition (4.10) in Theorem 4.2.

(a) Price $q_3(\varepsilon)$ (b) Net Trade $x_2(\varepsilon)$

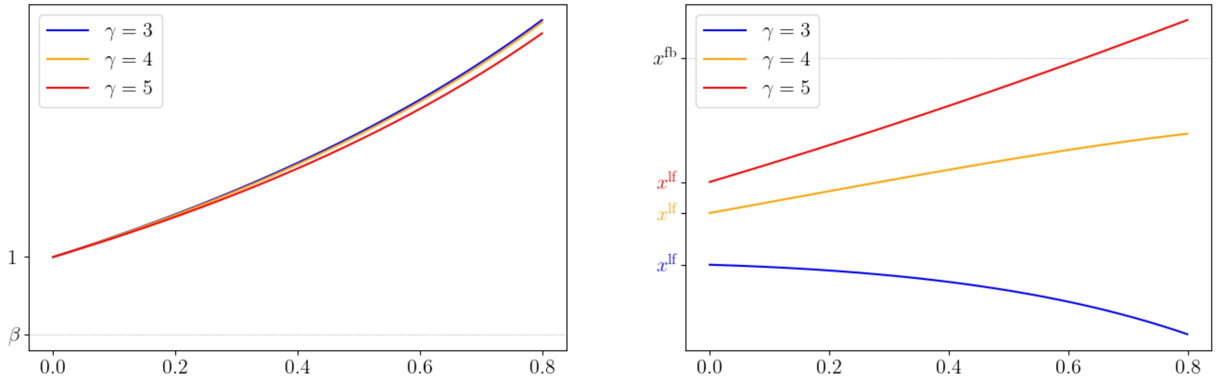


Figure 4.5: Equilibrium variables at $t = 2$ as functions of the tightening coefficient ε.

forces the net trade $x_2(\varepsilon)$ that prevails at period $t = 2$ to increase. The expansion of net trade reflects that $d_2(\varepsilon)$ exceeds the laissez-faire level d^{lf}. Indeed, we have $d_2(0) = d^{\text{lf}}$ and $d_2'(0) = (q_4'(0) + q_3'(0))d^{\text{lf}} > 0$.

Figure 4.5 plots the functions $\varepsilon \mapsto q_3(\varepsilon)$ and $\varepsilon \mapsto x_2(\varepsilon)$ for CRRA utility with different values of γ. The income dispersion and the discount factor equal $\underline{\zeta} = 2$ and $\overline{\beta} = 0.9$. For $\gamma = 3$, we have $A_{\text{L}} x^{\text{lf}} < 1$ and $\varepsilon \to x_2(\varepsilon)$ is decreasing. For $\gamma \in \{4, 5\}$, we have $A_{\text{L}} x^{\text{lf}} > 1$ and $\varepsilon \to x_2(\varepsilon)$ is increasing.

4.5.2 Computing the Not-Too-Tight Debt Limit at $t = 1$

Determining the equilibrium variables at period $t = 1$ follows similar reasoning, so we only discuss the key differences with respect to period $t = 2$.

The high-income agent starts the period with liabilities $d_1(\varepsilon)$ and saves the amount $d_2(\varepsilon)$. The FOC determines the equilibrium price:

$$q_2(\varepsilon) = \beta \frac{u'(y_{\text{L}} + x_2(\varepsilon))}{u'(y_{\text{H}} - x_1(\varepsilon))}. \tag{4.22}$$

As the debt level $d_1(\varepsilon)$ is required to be not-too-tight, the participation constraint reads as follows:

$$
\begin{aligned}
&u(y_{\mathrm{H}} - x_1(\varepsilon)) + \beta u(y_{\mathrm{L}} + x_2(\varepsilon)) + \beta^2 u(y_{\mathrm{H}} - x_3(\varepsilon)) + \beta^3 U^{\mathrm{lf}}_{\mathrm{L}} = \\
&\quad u(y_{\mathrm{H}} - q_2(\varepsilon)\theta_2(\varepsilon)) + \beta u(y_{\mathrm{L}} + \theta_2(\varepsilon)) + \beta^2 u(y_{\mathrm{H}} - q_4(\varepsilon)\theta_4(\varepsilon)) + \beta^3 u(y_{\mathrm{L}} + \theta_4(\varepsilon)) + \beta^4 U^{\mathrm{lf}}_{\mathrm{H}}.
\end{aligned}
\tag{4.23}
$$

The right-hand side stands for the value of the default option $V^{\mathrm{def}}_{\mathrm{H},1}(\varepsilon)$ obtained by guessing that out-of-equilibrium there is saving only when income is high. The variables $\theta_2(\varepsilon)$ and $\theta_4(\varepsilon)$ correspond to the optimal saving decisions at $t = 1$ and $t = 3$ and are determined by the following FOCs:

$$
q_2(\varepsilon) = \beta \frac{u'(y_{\mathrm{L}} + \theta_2(\varepsilon))}{u'(y_{\mathrm{H}} - q_2(\varepsilon)\theta_2(\varepsilon))} \quad \text{and} \quad q_4(\varepsilon) = \beta \frac{u'(y_{\mathrm{L}} + \theta_4(\varepsilon))}{u'(y_{\mathrm{H}} - q_4(\varepsilon)\theta_4(\varepsilon))}. \tag{4.24}
$$

The corresponding marginal price at $\varepsilon = 0$ is $q_2'(0) = -(x_1'(0)A_{\mathrm{H}} + x_2'(0)A_{\mathrm{L}})$. Differentiating the not-too-tight condition (4.23) at $\varepsilon = 0$, we obtain:

$$
-x_1'(0) + x_2'(0) - \beta^2 x_3'(0) = -q_2'(0)x^{\mathrm{lf}} - \beta^2 q_4'(0)x^{\mathrm{lf}}.
$$

Using the expressions for $q_4'(0)$ and $q_2'(0)$, we deduce that:

$$
x_1'(0)(A_{\mathrm{H}}x^{\mathrm{lf}} + 1) = -x_2'(0)(A_{\mathrm{L}}x^{\mathrm{lf}} - 1) - \beta^2 x_3'(0)(A_{\mathrm{H}}x^{\mathrm{lf}} + 1).
$$

As $x_2'(0) = -x_3'(0)(A_{\mathrm{L}}x^{\mathrm{lf}} - 1)/(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)$, we get that $x_1'(0) > 0$ if, and only if, $\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1) > (A_{\mathrm{L}}x^{\mathrm{lf}} - 1)$.[32] To illustrate this possibility, we plot in Figure 4.6 the functions $\varepsilon \mapsto q_2(\varepsilon)$ and $\varepsilon \mapsto x_1(\varepsilon)$ for CRRA utility with different values of γ, setting the values of the income dispersion and discount factor equal to $\underline{\zeta} = 2$ and $\overline{\beta} = 0.9$.

4.6 Pareto Improvement

Since the equilibrium is symmetric, for each agent $i \in I$, the ex-ante (expected and discounted) utility satisfies $U^i(c^{i,\varepsilon}|s^0) = U_0(\varepsilon)$ where:[33]

$$
U_0(\varepsilon) = u(y_0) + \frac{1}{2}\sum_{t \geqslant 1} \beta^t \varphi(x_t(\varepsilon)), \quad \text{with} \quad \varphi(x) = u(y_{\mathrm{H}} - x) + u(y_{\mathrm{L}} + x).
$$

[32] The restriction corresponds to the second inequality of condition (4.10) in Theorem 4.2.

[33] We recall that $\pi_{\mathrm{L}} = \pi_{\mathrm{H}} = 1/2$.

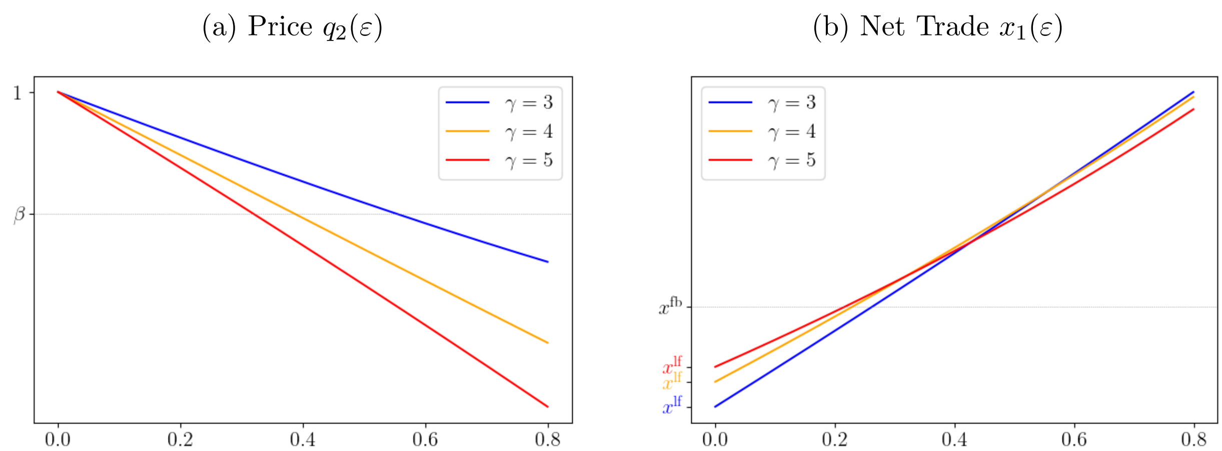


Figure 4.6: Equilibrium variables at $t = 1$ as functions of the tightening coefficient ε.

It is straightforward to verify that if $\varepsilon = 0$, then we recover the laissez-faire equilibrium with not-too-tight debt limits, that is $(q^0, (c^{i,0}, a^{i,0}, D^{i,0})_{i\in I}) = (q, (c^i, a^i, D^i)_{i\in I})$ and we deduce that:

$$U^i(c^i|s^0) = U_0(0) = u(y_0) + \frac{1}{2}\sum_{t\geqslant 1}\beta^t\varphi(x^{\mathrm{lf}}).$$

Therefore, to show that the consumption allocation $(c^{i,\varepsilon})_{i\in I}$ Pareto dominates the laiseez-faire consumption allocation $(c^i)_{i\in I}$, it is sufficient to show that $U_0'(0) > 0$. The variation of the ex-ante utility is given by:

$$U_0(\varepsilon) - U_0(0) = \frac{\beta}{2}[\varphi(x_1) + \beta\varphi(x_2) + \beta^2\varphi(x_3)] - \frac{\beta}{2}[\varphi(x^{\mathrm{lf}}) + \beta\varphi(x^{\mathrm{lf}}) + \beta^2\varphi(x^{\mathrm{lf}})]$$

As $\varphi'(x^{\mathrm{lf}}) = (1-\beta)u'(c_{\mathrm{L}}^{\mathrm{lf}})$, we deduce that:

$$\begin{aligned}
\frac{2U_0'(0)}{\beta(1-\beta)u'(c_{\mathrm{L}}^{\mathrm{lf}})} &= x_1'(0) + \beta x_2'(0) + \beta^2 x_3'(0) \\
&= -x_2'(0)\frac{A_{\mathrm{L}}x^{\mathrm{lf}} - 1}{A_{\mathrm{H}}x^{\mathrm{lf}} + 1} - \beta^2 x_3'(0) + \beta x_2'(0) + \beta^2 x_3'(0) \\
&= \frac{x_2'(0)}{A_{\mathrm{H}}x^{\mathrm{lf}} + 1}\left[\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1) - (A_{\mathrm{L}}x^{\mathrm{lf}} - 1)\right] \\
&= \frac{-x_3'(0)}{A_{\mathrm{H}}x^{\mathrm{lf}} + 1}(A_{\mathrm{L}}x^{\mathrm{lf}} - 1)\left[\beta(A_{\mathrm{H}}x^{\mathrm{lf}} + 1) - (A_{\mathrm{L}}x^{\mathrm{lf}} - 1)\right].
\end{aligned}$$

As a direct consequence of the above arguments, we deduce that there is Pareto improvement for ε sufficiently small if the two inequalities in (4.10) of Theorem 4.2 are satisfied.

Remark 4.1 (Stationary Markovian Intervention). In the simple economy discussed in this section, our baseline scenario is a laissez-faire equilibrium that is symmetric and stationary Markovian. In contrast, our proposed Pareto-improving intervention does not support a stationary Markovian equilibrium. This distinction is crucial since Pareto improvement reflects an intertemporal compromise: the intervention boosts expected utility in periods $t \in \{1, 2\}$ at the expense of reduced utility at $t = 3$. This raises the question: can we obtain Pareto improvement through an intervention supporting a symmetric, stationary Markovian equilibrium? We address this issue in the appendix, Section E.2, and demonstrate that such an approach is infeasible.

The above analysis focused on an intervention tightening the financial constraint at $\tau = 2$. Alternatively, we could analyze whether an intervention at $\tau = 1$ could be Pareto-improving. There is no need to solve for a new equilibrium. Indeed, if we denote by $(\tilde{x}_1, \tilde{x}_2, \tilde{q}_2, \tilde{q}_3, \tilde{d}_1, \tilde{d}_2)$ the variables describing the new equilibrium, then we can set:

$$(\tilde{x}_1, \tilde{x}_2, \tilde{q}_2, \tilde{q}_3, \tilde{d}_1, \tilde{d}_2) = (x_2, x_3, q_3, q_4, d_2, d_3).$$

Denote by $\widetilde{U}_0(\varepsilon)$ the discounted and expected utility at $t = 0$ of the new equilibrium with intervention at $\tau = 1$. There is Pareto improvement for $\varepsilon > 0$ small enough when:

$$0 < \widetilde{U}_0(\varepsilon) = \frac{\beta}{2} \left[\varphi(\tilde{x}_1(\varepsilon)) + \beta\varphi(\tilde{x}_2(\varepsilon))\right] - \frac{\beta}{2} \left[\varphi(x^{\mathrm{lf}}) + \beta\varphi(\tilde{x}_2(\varepsilon))\right].$$

This occurs when $\widetilde{U}_0'(0) > 0$. Observe that:

$$\begin{aligned} \frac{2\widetilde{U}_0'(0)}{\beta(1-\beta)u'(c_{\mathrm{L}}^{\mathrm{lf}})} &= \tilde{x}_1'(0) + \beta\tilde{x}_2'(0) \\ &= \tilde{x}_2'(0) \left[-\frac{A_{\mathrm{L}}x^{\mathrm{lf}} - 1}{A_{\mathrm{H}}x^{\mathrm{lf}} + 1} + \beta\right]. \end{aligned}$$

As

$$\tilde{x}_2(0) = -\frac{1}{A_{\mathrm{H}}}\tilde{q}_2'(0) = -\frac{x^{\mathrm{lf}}}{2(A_{\mathrm{H}}x^{\mathrm{lf}} + 1)} < 0,$$

we deduce the following result.

Theorem 4.3. *If the interest rate at autarky is negative (i.e., $\beta u'(y_{\mathrm{L}}) > u'(y_{\mathrm{H}})$), then, for sufficiently small $\varepsilon > 0$, there exists an equilibrium with ε-tight debt constraints at $\tau = 1$ that Pareto dominates the laissez-faire equilibrium provided that the following condition holds:*

$$A_{\mathrm{L}} x^{\mathrm{lf}} - 1 > \beta(A_{\mathrm{H}} x^{\mathrm{lf}} + 1). \tag{4.25}$$

For the CRRA utility function characterized by a risk-aversion coefficient $\gamma > 0$, the sufficient condition (4.25) is equivalent to:

$$h(\beta|\zeta) < \gamma.$$

The blue region depicted in Figure 4.7(a) illustrates the set of primitive parameters (β, γ) for which an intervention at $\tau = 1$ leads to a Pareto improvement. To facilitate a clearer comparison, we reiterate in Figure 4.7(b) the domain of primitive parameters wherein we have implicitly demonstrated the existence of an intervention that Pareto dominates the laissez-faire allocation. It is noteworthy that the two proposed simple interventions (represented by the blue and yellow regions) encompass all the primitive parameters that are consistent with Pareto improvements, as inferred from the optimality conditions (FOCs) of the social planner's problem.

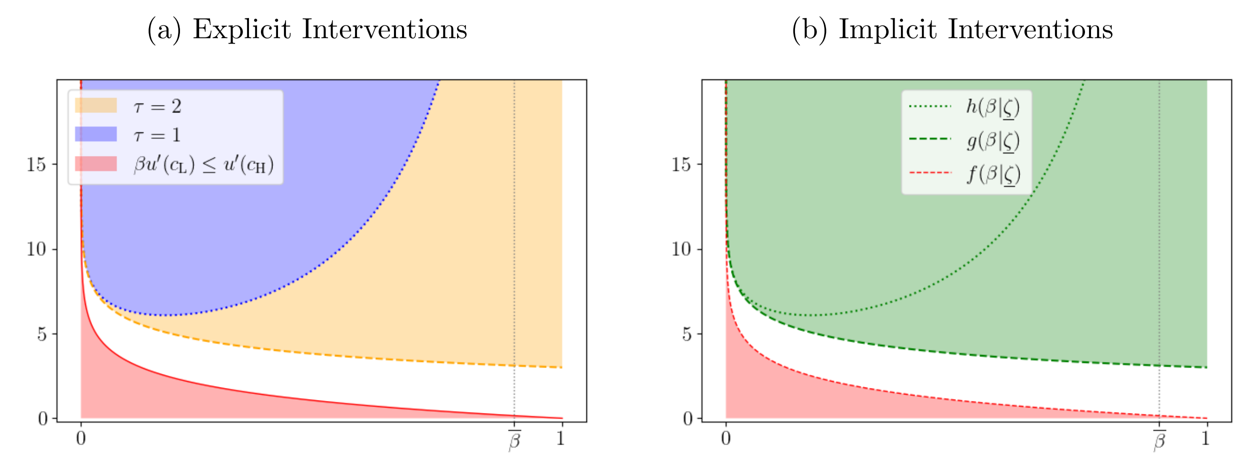


Figure 4.7: Laissez-faire can be Pareto improved.

5 High Interest Rates

In the policy experiment discussed in Section 4, the laissez-faire equilibrium and the equilibrium with ε-tight debt constraints feature low interest rates—specifically, rates that are either zero or negative.[34] This comes as no surprise since we know from Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009) that, in complete markets, unsecured debt is not sustainable at high interest rates.

However, one might wonder whether there is room for Pareto improvement in scenarios where debt remains sustainable despite high interest rates. To address this question, the logical first step is to conduct our policy experiment in a relevant environment that closely resembles the one discussed in previous sections. To this end, we align with the extensive quantitative literature and assume that default results in deadweight endowment losses in addition to credit exclusion.[35] These losses could manifest as output contraction in the case of sovereign default or as seized collateral and legal recourse in the context of consumer and corporate default.

The key modification from the model outlined in Section 2 is the treatment of default. Specifically, if an agent i defaults at s^τ, her endowments for all future events $s^t \succeq s^\tau$ will be reduced to $y^i(s^t) - \ell^i(s^t)$, where $\ell^i(s^t)$ is an exogenously given value within the range $[0, y^i(s^t)]$. Given that this agent is also barred from the credit markets, her outside option is now defined as:

$$V^i_{\text{def}}(s^t) = V^i_{\ell^i}(0, 0|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B^i_{\ell^i}(0, 0|s^t)\}, \tag{5.1}$$

where $B^i_{\ell^i}(0, 0|s^t)$ denotes the budget set for any agent i who has zero liabilities, is unable to borrow, and has resources equal to $y^i - \ell^i$. The condition analogous to the not-too-tight

[34] Formally, $q^{\text{lf}} = 1$ and $q_t(\varepsilon) > 1$ for $t \in \{3, 4, \ldots\}$.

[35] In consumer credit contexts, endowment loss serves as a simplified proxy for the legal consequences of default, as elaborated in works such as Chatterjee et al. 2007, Livshits et al. 2007, Livshits 2015. In the realm of sovereign debt, it encapsulates the negative impact of default on domestic production (e.g., Eaton and Gersovitz 1981, Bulow and Rogoff 1989, Cole and Kehoe 2000, Aguiar and Gopinath 2006, Arellano 2008).

condition (2.4) is then articulated as:

$$V^i(D^i, -D^i(s^t)|s^t) = V^i_{\ell^i}(0, 0|s^t), \quad \text{for all } s^t \succeq s^0.$$

To conduct a policy experiment akin to the one in Section 4, the first step involves extending the characterization of not-too-tight debt limits as presented by Hellwig and Lorenzoni (2009). This extension is not merely a procedural necessity; it significantly simplifies the computation of the laissez-faire equilibrium and the equilibrium with ε-tight debt constraints.[36]

Moreover, this extended characterization has implications for the restrictions on the model's primitives—in our case, the properties of the endowment loss—that can sustain equilibria with high interest rates pre-and post-intervention.

5.1 Characterization of Not-Too-Tight Debt Limits

The following result is the counterpart to the characterization provided by Hellwig and Lorenzoni (2009), adapted for our modified context. It posits that not-too-tight debt limits can be broken down into two components: a fundamental component and a credit bubble component. The latter encapsulates the potential for indefinitely rolling over a portion of the debt. The proof of this result is relegated to the online Appendix.

Theorem 5.1. *A process D^i of debt limits is not too tight (for the default punishment described by (5.1)) if, and only if:*

$$D^i(s^t) = \ell^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \textit{for all } s^t \succ s^0. \tag{5.2}$$

[36] The extension eliminates the typical complexities associated with the fixed-point process of determining not-too-tight debt limits. In this process, the default value is contingent on prices (since defaulting agents can still save), which depend on equilibrium allocations and, consequently, on the debt limits. It's worth noting that this extension is of independent interest. Although it serves as the direct analog of Hellwig and Lorenzoni (2009)'s result in an augmented setting, the proof cannot be derived through a simple adaptation of their arguments. Instead, it relies on novel insights with no counterparts in scenarios without output losses.

It is straightforward that a process of debt limits D^i satisfies property (5.2) if, and only if, it can be decomposed into a *fundamental* and a *bubble* component:

$$D^i(s^t) = \underbrace{\mathrm{PV}(\ell^i|s^t)}_{\text{fundamental}} + \underbrace{M^i(s^t)}_{\text{bubble}}, \quad \text{for all } s^t \succeq s^0. \tag{5.3}$$

Here, the fundamental component is simply the present value of endowment losses:

$$\mathrm{PV}(\ell^i|s^t) := \frac{1}{p(s^t)} \sum_{s^\tau \succeq s^t} p(s^\tau)\ell^i(s^\tau),$$

where $p(s^t)$ is the date-0 price of consumption at event s^t.[37] The bubble component of D^i is a nonnegative process satisfying the following *exact rollover* property:

$$M^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) M^i(s^{t+1}), \quad \text{for all } s^t \succ s^0.$$

Intuitively, the bubble component reflects that credit beyond the fundamental component is sustainable only if agents can roll over their debt.

How can we support equilibria with high interest rates? The following result shows that this is the case when the aggregate losses amount to a *nonnegligible* fraction of aggregate resources. Indeed, since the present value of endowment losses is always finite at equilibrium, under this restriction, we necessarily have that interest rates are high.

Proposition 5.1. *If endowment losses are a nonnegligible fraction of aggregate resources, in the sense that there exists* $\varepsilon > 0$ *such that:*

$$\sum_{i \in I} \ell^i(s^t) \geqslant \varepsilon \sum_{i \in I} y^i(s^t), \quad \textit{for all } s^t \succ s^0, \tag{5.4}$$

then in any equilibrium with not-too-tight debt limits, the bubble component is necessarily zero. As a consequence, $D^i = \mathrm{PV}(\ell^i)$ *for every agent* i.

5.2 Laissez-Faire Equilibrium

Consider the same simplified economy as in Section 4, and assume that the endowment loss upon default is time-invariant and identical for both agents, i.e., $\ell^i(s^t) = \ell$ for all agent i and event s^t.

[37] Formally, $p(s^t)$ is defined recursively by $p(s^0) = 1$ and $p(s^{t+1}) = q(s^{t+1})p(s^t)$ for all $s^{t+1} \succ s^t$.

The (symmetric Markov) laissez-faire equilibrium is characterized by the asset price $q^{\mathrm{lf}}(\ell)$ and the not-too-tight debt level $d^{\mathrm{lf}}(\ell)$. Our characterization result (Theorem 5.1) implies that $d^{\mathrm{lf}}(\ell) = \ell + q^{\mathrm{lf}}(\ell)d^{\mathrm{lf}}(\ell)$, or, equivalently,

$$d^{\mathrm{lf}}(\ell)(1 - q^{\mathrm{lf}}(\ell)) = \ell.$$

When $\ell > 0$, we must have $q^{\mathrm{lf}}(\ell) < 1$, which is consistent with Proposition 5.1. The net trade defined by $x^{\mathrm{lf}}(\ell) = d^{\mathrm{lf}}(\ell) + q^{\mathrm{lf}}(\ell)d^{\mathrm{lf}}(\ell)$, and the asset price $q^{\mathrm{lf}}(\ell)$ are determined by the following equations:[38]

$$x^{\mathrm{lf}}(\ell) = \ell \times \frac{1 + q^{\mathrm{lf}}(\ell)}{1 - q^{\mathrm{lf}}(\ell)} \quad \text{and} \quad q^{\mathrm{lf}}(\ell) = \beta \frac{u'(y_{\mathrm{L}} + x^{\mathrm{lf}}(\ell))}{u'(y_{\mathrm{H}} - x^{\mathrm{lf}}(\ell))}. \tag{5.5}$$

Risk aversion implies that the asset price must decrease with ℓ, and the net trade must increase. Indeed, as the mapping $q \mapsto (1 + q)/(1 - q)$ is increasing, if $\ell \mapsto q^{\mathrm{lf}}(\ell)$ where increasing, then $\ell \mapsto x^{\mathrm{lf}}(\ell)$ would also be increasing by the first equation in (5.5). However, this would contradict the second equation in (5.5) as the mapping $x \mapsto u'(y_{\mathrm{L}} + x)/u'(y_{\mathrm{H}} - x)$ is decreasing by concavity of u. Therefore, the mapping $\ell \mapsto q^{\mathrm{lf}}(\ell)$ must decrease, and the mapping $\ell \mapsto x^{\mathrm{lf}}(\ell)$ must increase. This is confirmed in Figure 5.1 that plots the equilibrium variables as functions of the endowment loss ℓ for CRRA utility for different values of γ when the income dispersion and the discount factor equal to $\underline{\zeta} = 2$ and $\bar{\beta} = 0.9$. We denote by ℓ^{fb} the level of endowment loss that implements perfect risk-sharing, i.e., $x^{\mathrm{lf}}(\ell^{\mathrm{fb}}) = x^{\mathrm{fb}}$. This level is given by $\ell^{\mathrm{fb}} = x^{\mathrm{fb}}(1 - \beta)/(1 + \beta)$.

[38] The first equation corresponds to our characterization of not-too-tight debt limits. The second equation is the FOC of the high-income agent's saving decision.

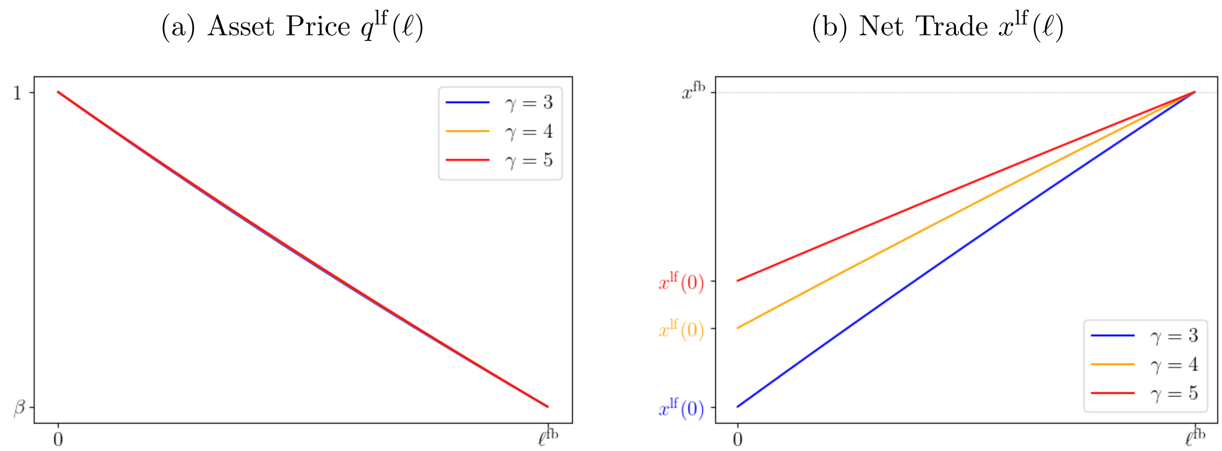


Figure 5.1: Laissez-faire equilibrium variables as functions of the endowment loss ℓ.

5.3 Tightening Debt Constraints

For any arbitrary $\ell \in (0, \ell^{\text{fb}})$, the laissez-faire interest rate is positive. We perform the same policy experiment: we tighten the high-income borrowing constraint at $t = 2$ by imposing the too-tight debt limit:

$$d_3(\varepsilon) := (1 - \varepsilon)[\ell + q_4(\varepsilon) d^{\text{lf}}(\ell)].$$

All the arguments of Section 4 remain qualitatively valid. To illustrate our policy intervention, we set the following arbitrary value $\ell^\star = \ell^{\text{fb}}/2$ for the endowment loss, and plot the equilibrium prices for each period.

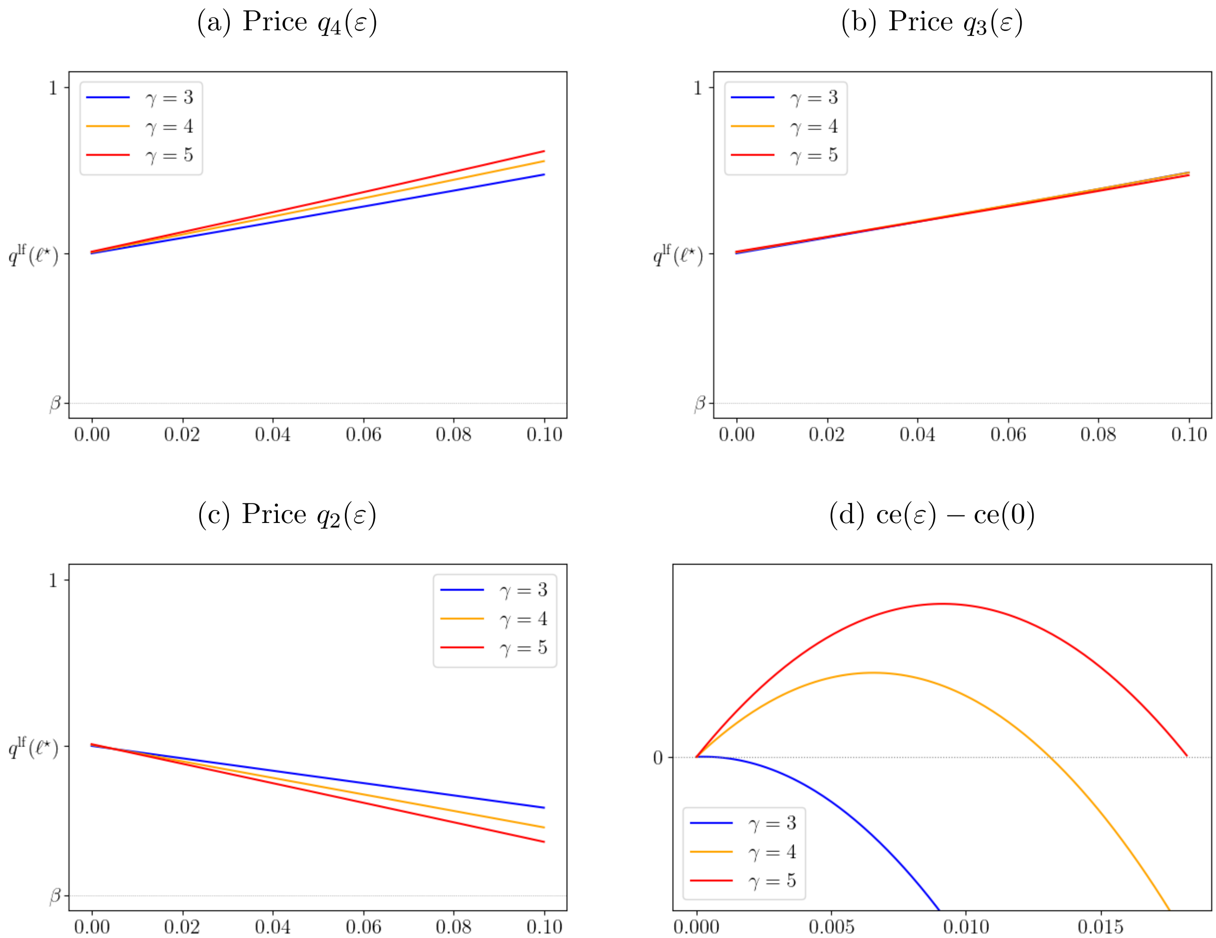


Figure 5.2: Equilibrium variables as functions of the tightening coefficient ε when the endowment loss is $\ell^\star = \ell^{\mathrm{fb}}/2$.

6 Conclusion

This paper revisits the welfare implications of competitive credit markets with endogenously determined debt limits arising from limited commitment. We demonstrate that the nature of default punishments significantly influences market equilibria's efficiency and welfare outcomes. Our analysis reveals that laissez-faire debt limits, although beneficial for risk-sharing for given interest rates, may not always lead to optimal welfare. This suggests that judiciously designed credit rationing could improve welfare by impacting the default option's value and altering consumption and saving decisions.

Building upon the concepts introduced in this paper, several avenues for future exploration could further improve our understanding of credit markets under limited commitment. First, subsequent research could analyze optimal macroprudential policy interventions, for example, by applying the two-period analysis in Dávila and Korinek (2018) to our infinite-horizon environment. Second, it could be valuable to apply our analysis to study the constrained efficiency of an environment with collateralized debts (Gottardi and Kubler 2015). Third, considering our assumption of planner commitment, future investigations could examine scenarios where the planner cannot commit, focusing on time-consistent policies, following Bianchi and Mendoza (2018)'s analysis of optimal time-consistent macroprudential interventions. Finally, while our study provides a qualitative framework, subsequent research could quantify the extent of constrained inefficiency, offering a more detailed understanding of its impacts and how these might vary across different market conditions or consumer behaviors, following the consumer credit quantitative analyses in Livshits et al. (2007, 2010) and Livshits (2015). We believe these explorations could help shed light on the challenges in regulating credit markets.

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Online Appendix to

Credit Rationing in Unsecured Debt Markets

This appendix provides detailed proofs and technical arguments omitted from the main text. Section E provides additional commentary on self-enforcing equilibria in the simple baseline economy presented in Section 4.

A Proof of Proposition 2.2

Fix an event $s^\tau \in \Sigma$ and assume that

$$D^i(s^t) \leqslant \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \text{for all } s^t \succeq s^\tau$$

with a strict inequality in at least one successor event $s^t \succeq s^\tau$. We shall prove that $V^i(D^i, -D^i(s^\tau)|s^\tau) > V^i(0,0|s^\tau)$. Denote by $(\bar{c}^i, \bar{a}^i)$ agent i's optimal choice in the budget set $B^i(0,0|s^\tau)$. Observe that $V^i(0,0|s^\tau) = U(\bar{c}^i|s^\tau)$. Let $a^i := \bar{a}^i - D^i$ and, for every $s^t \succeq s^\tau$, define $\delta^i(s^t) := -D^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1})$. As $\bar{a}^i \geqslant 0$, the pair (c^i, a^i) belongs to the budget set $B^i(D^i, -D^i(s^\tau)|s^\tau)$ where the new consumption plan is defined by $c^i := \bar{c}^i + \delta^i$. This implies that $V^i(D^i, -D^i(s^\tau)|s^\tau) \geqslant U(c^i|s^\tau) > U(\bar{c}^i|s^\tau)$, where the strict inequality follows from the assumption that $\delta^i(s^t) > 0$ for some $s^t \succeq s^\tau$. As $U(\bar{c}^i|s^\tau) = V^i(0,0|s^\tau)$, we get the desired result.

B Proof of Corollary 3.1

Consider a feasible allocation $c = (c^i)_{i \in I}$ that is finite-time efficient. Pose $q := q^\star(c)$ and, for each agent i, $q^i := q^i(c^i)$. Fix some arbitrary finite horizon $T \geqslant 1$ and $\varepsilon > 0$. Recall that $\mathcal{P}(T, \varepsilon)$ is the set of pairs $(\tilde{q}, (\tilde{c}^i)_{i \in I})$ satisfying for every s^t and $s^{t+1} \succ s^t$:

(1) if $t > T$, then $\tilde{c}^i(s^t) = c^i(s^t)$ and $\tilde{q}(s^{t+1}) = q(c|s^{t+1})$;

(2) if $t \leqslant T$, then $|\tilde{c}^i(s^t) - c^i(s^t)| < \varepsilon$ and $|\tilde{q}(s^{t+1}) - q(s^{t+1})| < \varepsilon$;

(3) $\sum_{i \in I}[y^i(s^t) - \tilde{c}^i(s^t)] = 0$ for all s^t with $t \leqslant T$;

(4) $U(\tilde{c}^i|s^t) \geqslant V^i_{\mathrm{def}}(\tilde{q}|s^t)$ for all $i \in I$ and every s^t with $t \leqslant T$;

(5) the price-determination constraint is satisfied, i.e.,

$$\tilde{q}^i(c^i|s^t) = \tilde{q}(s^t), \quad \text{for all } i \in I^\star(s^t), \tag{B.1}$$

where $I^\star(s^t) := \{i \in I\colon q(s^t) = q^i(s^t)\}$ is the set of agents that determine the price $q(s^t)$;

(6) and the Pareto constraint

$$U(\tilde{c}^i|s^0) \geqslant U(c^i|s^0), \quad \text{for all } i \in I. \tag{B.2}$$

By construction, (q, c) belongs to $\mathcal{P}(T, \varepsilon)$. Choosing $\varepsilon > 0$ small enough, we can verify that all pairs $(\tilde{q}, \tilde{c})$ in $\mathcal{P}(T, \varepsilon)$ satisfy the feasible constraints (a), (b), and (c).[1]

Fix an arbitrary family $(\nu^i)_{i \in I}$ of strictly positive welfare weights. We must have

$$\sum_{i \in I} \nu^i U(\tilde{c}^i|s^0) \leqslant \sum_{i \in I} \nu^i U(c^i|s^0), \tag{B.3}$$

for any pair $(\tilde{q}, \tilde{c})$ in $\mathcal{P}(T, \varepsilon)$. Indeed, if (B.3) were not satisfied, then (B.2) would be satisfied with a strict inequality for at least one agent i. This would contradict the assumption that c is weakly constrained efficient.

We have thus proved that (q, c) maximizes the social welfare function $\sum_{i \in I} \nu^i U(\tilde{c}^i|s^0)$ among all pairs $(\tilde{q}, \tilde{c})$ satisfying the constraints (1)–(6). We can write the Fritz John conditions.[2] There exist Lagrange multipliers: $\alpha_0 \geqslant 0$ for the objective function, $\mu(s^t) \in \mathbb{R}$ for the market clearing condition (3); $\xi^i(s^t) \geqslant 0$ for the participation constraint (4); $\chi^i(s^t) \geqslant 0$

[1] We only have to verify the implementability condition (c). Fix an event s^t and an agent i. If $i \in I^\star(s^t)$, then $\tilde{q}(s^t) = \tilde{q}^i(c^i|s^t)$. If $q(s^t) = q^\star(c|s^t) > \tilde{q}^i(c^i|s^t) = \tilde{q}^i(s^t)$, then for $\varepsilon > 0$ small enough, we also have $\tilde{q}(s^t) > \tilde{q}^i(c^i|s^t)$ due to the continuity of u'. This proves that $\tilde{q}(s^t) = q^\star(c|s^t)$.

[2] See Theorem 9.1 in Clarke (2013).

for the price-determination constraint (5), and $\alpha^i \geqslant 0$ for the Pareto constraint (6), not all equal to zero, such that $\nabla\mathcal{L}(q,c)=0$ for the corresponding Lagrangian

$$\mathcal{L}(\tilde{q},\tilde{c}) := \sum_{t=0}^{T} \mathcal{L}(\tilde{q},\tilde{c}|s^t)$$

where

$$\mathcal{L}(\tilde{q},\tilde{c}|s^t) := \sum_{i\in I} \beta^t \pi(s^t) \left\{(\alpha_0\theta^i+\alpha^i)u(\tilde{c}^i(s^t)) + \xi^i(s^t)[U(\tilde{c}^i|s^t) - V^i_{\text{def}}(\tilde{q}|s^t)]\right\}$$
$$+ \sum_{i\in I^\star(s^t)} \chi^i(s^t)[\tilde{q}(s^t) - \tilde{q}^i(s^t)] + \mu(s^t)\sum_{i\in I}\left[y^i(s^t) - \tilde{c}^i(s^t)\right]. \quad \text{(B.4)}$$

For every $i \notin I^\star(s^t)$, we pose $\chi^i(s^t) := 0$. We also let $\lambda^i := \alpha_0\tilde{\lambda}^i + \alpha^i$. Observe that $\lambda^i \geqslant 0$ for every i.

Equation (3.1) reflects the first-order condition associated with the choice $\tilde{c}^i(s^t)$, and Equation (3.2) reflects he first-order condition associated with the choice $\tilde{q}(s^t)$. By construction, if $q(s^t) > q^i(s^t)$, then $\chi^i(s^t) = 0$. We derive from the complementary slackness conditions that $\xi^i(s^t) = 0$ when $U(c^i|s^t) > V^i_{\text{def}}(q|s^t)$.

C Proof of Theorem 3.1

Fix a laissez-faire equilibrium $(q,(c^i,a^i,D^i)_{i\in I})$ in which debt limits are strictly positive. Without any loss of generality, we have $q = q^\star(c)$.[3] Assume that the optimality conditions of Corollary 3.1 are not satisfied. This implies that there exists a finite horizon $T \geqslant 1$ such that for every $\varepsilon > 0$ small enough, there exists a pair $(q_\varepsilon, c_\varepsilon)$ in $\mathcal{P}(T,\varepsilon)$ such that $c_\varepsilon = (c^i_\varepsilon)_{i\in I}$ Pareto dominates $c = (c^i)_{i\in I}$. Recall that $q_\varepsilon = q^\star(c_\varepsilon)$. To simplify the presentation, we let $q^i_\varepsilon := q^i(c^i_\varepsilon)$, for each $i \in I$. Value functions computed with the price q_ε are denoted by V^i_ε, while those computed with the laissez-faire price q are denoted by V^i.

[3] Recall that $I^\star(s^t)$ denotes the set of agents $i \in I$ that determine the asset price at event s^t, i.e., $q^\star(c|s^t) = q^i(c^i|s^t)$.

We shall construct asset positions $(a^i_\varepsilon)_{i\in I}$ and debt limits $(D^i_\varepsilon)_{i\in I}$ such that the family $(q_\varepsilon,(c^i_\varepsilon,a^i_\varepsilon,D^i_\varepsilon)_{i\in I})$ forms a competitive equilibrium with self-enforcing and nonnegative debt limits.

For every $t>T$, we pose $a^i_\varepsilon(s^t):=a^i(s^t)$ and $D^i_\varepsilon(s^t):=D^i(s^t)$.

Fix an event s^T at date T. We set $a^i_\varepsilon(s^T)$ to be the beginning-of-period contingent claim that implements the consumption $c^i_\varepsilon(s^T)$, given asset prices and claims contingent to successor events, i.e.,

$$a^i_\varepsilon(s^T):=-y^i(s^T)+c^i(s^T)+\sum_{s^{T+1}\succ s^T}q_\varepsilon(s^{T+1})a^i_\varepsilon(s^{T+1}).$$

To define the debt limits, we analyze two cases.

1. If $q_\varepsilon(s^T)>q^i_\varepsilon(s^T)$, we pose $D^i_\varepsilon(s^T):=-a^i_\varepsilon(s^T)$. Because of condition (iii) in the definition of $\mathcal{P}(T,\varepsilon)$, we must have $q(s^T)>q^i(s^T)$ at the laissez-faire equilibrium, implying that $a^i(s^T)=-D^i(S^T)$. Therefore, $D^i_\varepsilon(s^t)$ can be made as close to $D^i(s^T)$ as desired, by choosing $\varepsilon>0$ small enough. Since we assume debt limits D^i are positive, we necessarily get $D^i_\varepsilon(s^T)\geqslant 0$. Moreover, as $q_\varepsilon(s^T)\geqslant q^i_\varepsilon(s^T)$, we have $V^i_\varepsilon(D^i_\varepsilon,-D^i(s^T)|s^T)=U(c^i_\varepsilon|s^T)$, and we deduce that $D^i(s^T)$ is self-enforcing (given the successor debt limits).

2. If $q_\varepsilon(s^T)=q^i_\varepsilon(s^T)$, then $V^i_\varepsilon(D^i_\varepsilon,a^i_\varepsilon(s^T)|s^T)=U(c^i_\varepsilon|s^T)\geqslant V^i_{\text{def}}(q_\varepsilon|s^T)$. Applying the Intermediate Value Theorem, we deduce the existence of $D^i_\varepsilon(s^T)$ satisfying:

   $$V^i_\varepsilon(D^i_\varepsilon,-D^i_\varepsilon(s^T)|s^T)=V^i_{\text{def}}(q_\varepsilon|s^T).\tag{C.1}$$

   As $V^i_\varepsilon(D^i_\varepsilon,a^i_\varepsilon(s^T)|s^T)\geqslant V^i_{\text{def}}(q_\varepsilon|s^T)$, we must have $-D^i_\varepsilon(s^T)\leqslant a^i(s^T)$. As $D^i_\varepsilon(s^t)\geqslant 0$ for every successor event $s^t\succ s^T$, we deduce that

   $$V^i_\varepsilon(0,-D^i_\varepsilon(s^T)|s^T)\leqslant V^i_\varepsilon(D^i_\varepsilon,-D^i_\varepsilon(s^T)|s^T)=V^i_{\text{def}}(q_\varepsilon|s^T)=V^i_\varepsilon(0,0|S^T).$$

   This implies $D^i_\varepsilon(s^T)\geqslant 0$.

All the equilibrium variables at each event s^T have been defined by backward induction from the equilibrium variables at all strict successor events. We can continue this backward induction argument to construct recursively all the equilibrium variables $(a^i_\varepsilon(s^t), D^i_\varepsilon(s^t))$ at every event s^t predecessor to s^T. The constructed self-enforcing debt limits $D^i_\varepsilon(s^t)$ are all nonnegative if we choose $\varepsilon > 0$ small enough. This is possible because we have finitely many debt levels to determine.

D Proof of Proposition 4.2

Fix a state $z \in \{z^a, z^b\}$ and two time periods $t \geqslant \tau \geqslant 1$. Denote by $h_\tau \in \{a, b\}$ the agent with high income at event (τ, z). To identify $\chi_t(z)$, we should compute the marginal impact on the value of default at period τ of a marginal change of the asset price in date t,[4]

$$\frac{\partial V^{h_\tau}_{\text{def}}(\cdot|(\tau, z))}{\partial q_t(z)}(q).$$

Our guess for the out-of-equilibrium decision is that the high-income agent saves when income is high and does not save when his income is low. The expected discounted utility of this strategy is

$$V^{h_\tau}_{\text{def}}(q|(\tau, z)) = u(y_{\text{H}} - q_{\tau+1}(z)\theta(q_{\tau+1}(z))) + \beta u(y_{\text{L}} + \theta(q_{\tau+1}(z))) + \beta^2 V^{h_\tau}_{\text{def}}(q|(\tau + 2, z)).$$

The level of saving $\theta(q_{\tau+1}(z))$ at date τ is optimal when

$$q_{\tau+1}(z) = \beta \frac{u'(y_{\text{L}} + \theta(q_{\tau+1}(z)))}{u'(y_{\text{H}} - q_{\tau+1}(z)\theta(q_{\tau+1}(z)))}.$$

As $q_{\tau+1}(z) = q^{\text{lf}} = 1$, we deduce that $\theta(1) = c^{\text{lf}}_{\text{L}} - y_{\text{L}} = y_{\text{H}} - c^{\text{lf}}_{\text{H}} = x^{\text{lf}}$. The decision not to save at date $\tau + 1$ is optimal when

$$q_{\tau+2}(z) \geqslant \beta \frac{u'(y_{\text{H}} - q_{\tau+3}(z)\theta(q_{\tau+3}(z)))}{u'(y_{\text{L}} + \theta(q_{\tau+1}(z)))}.$$

As $q_{\tau+2}(z) = q_{\tau+3}(z) = 1$, the above inequality is satisfied with a strict inequality. Denote by $W_{\text{H}} : q \mapsto W_{\text{H}}(q)$ the function defined on a neighborhood of $q^{\text{lf}} = 1$ by

$$W_{\text{H}}(q) := u(y_{\text{H}} - q\theta(q)) + \beta u(y_{\text{L}} + \theta(q))$$

[4] If h is such that $y^h_\tau(z) = y_{\text{L}}$, then $\xi^h_\tau(z) = 0$ and we do not need to compute the marginal value of default to get an expression of $\chi_t(z)$.

where $\theta(q)$ solves

$$q = \beta \frac{u'(y_{\text{L}} + \theta(q))}{u'(y_{\text{H}} - q\theta(q))}.$$

Given these derivations, we can compute $\chi_t(z)$.[5] The analysis depends on whether t is odd or even.

Assume $t \geqslant 3$ is odd. We have $t = 2n+1$ for some $n \in \{1, \ldots\}$. If τ is also odd, say $\tau = 2k+1$ with $k \leqslant n$, then

$$\begin{aligned} V_{\text{def}}^{h_\tau}(\cdot|s^\tau) &= V_{\text{def}}^{h}(q|(2k+1, z)) \\ &= W_{\text{H}}(q_{2k+2}(z)) + \beta^2 V_{\text{def}}^{h}(q|(2k+3, z)) \\ &= \sum_{\ell=k+1}^{\infty} \beta^{2(\ell-k-1)} W_{\text{H}}(q_{2\ell}(z)). \end{aligned}$$

We deduce that

$$\frac{\partial V_{\text{def}}^{h_\tau}(\cdot|s^\tau)}{\partial q(s^t)}(q) = \frac{\partial V_{\text{def}}^{h_\tau}(\cdot|(2k+1, z))}{\partial q_{2n+1}(z)}(q) = 0$$

as the price $q(s^t) = q_{2n+1}(z)$ does not marginally affect the value of the default option $V_{\text{def}}^{h}(q|s^\tau)$. If τ is even, say $\tau = 2k$ with $k \leqslant n$, then

$$\begin{aligned} V_{\text{def}}^{h_\tau}(\cdot|s^\tau) &= V_{\text{def}}^{h_\tau}(q|(2k, z)) \\ &= W_{\text{H}}(q_{2k+1}(z)) + \beta^2 V_{\text{def}}^{h}(q|(2k+2, z)) \\ &= \sum_{\ell=k}^{\infty} \beta^{2(\ell-k)} W_{\text{H}}(q_{2\ell+1}(z)). \end{aligned}$$

We deduce that

$$\frac{\partial V_{\text{def}}^{h_\tau}(\cdot|s^\tau)}{\partial q(s^t)}(q) = \frac{\partial V_{\text{def}}^{h_\tau}(\cdot|(2k, z))}{\partial q_{2n+1}(z)}(q) = \beta^{2(n-k)} W_{\text{H}}'(1) = -u'(c_{\text{H}}^{\text{lf}}) x^{\text{lf}} \beta^{2(n-k)}.$$

It follows that if $t = 2n+1$ with $n \in \{1, \ldots\}$, then

$$\chi_t(z) = \pi_{\text{L}} \sum_{k=1}^{n} \xi_{2k}^{h_{2k}}(z)(-u'(c_{\text{H}}^{\text{lf}}) x^{\text{lf}}) \beta^{2n} = -\beta^{t-1} \pi_{\text{L}} u'(c_{\text{H}}^{\text{lf}}) x^{\text{lf}} \sum_{k=1}^{n} \xi_{2k}^{h_{2k}}(z) \tag{D.1}$$

where h_{2k} is the agent with high income at event $(2k, z)$.[6]

[5] Observe that $W_{\text{H}}'(q) = -u'(y_{\text{H}} - q\theta(q))\theta(q)$.

[6] Agent h_{2k} has low income at event $(1, z)$ and therefore $\pi^{h_{2k}}(2k, z) = \pi^{h_{2k}}(1, z) = \pi_{\text{L}}$.

Assume $t \geqslant 2$ is even. We have $t = 2n$ for some $n \in \{1, 2, \ldots\}$. If τ is odd, say $\tau = 2k+1$ with $k \leqslant n$, then

$$\frac{\partial V_{\mathrm{def}}^{h_\tau}(\cdot|s^\tau)}{\partial q(s^t)}(q) = \frac{\partial V_{\mathrm{def}}^{h_\tau}(\cdot|(2k+1, z))}{\partial q_{2n}(z)}(q) = \beta^{2(n-k-1)} W_{\mathrm{H}}'(1) = -u'(c_{\mathrm{H}}^{\mathrm{lf}}) x^{\mathrm{lf}} \beta^{2(n-k-1)}.$$

If τ is even, say $\tau = 2k$ with $k \leqslant n$, then

$$\frac{\partial V_{\mathrm{def}}^{h_\tau}(\cdot|s^\tau)}{\partial q(s^t)}(q) = \frac{\partial V_{\mathrm{def}}^{h_\tau}(\cdot|(2k, z))}{\partial q_{2n}(z)}(q) = 0.$$

It follows that if $t = 2n$ with $n \in \{0, 1, \ldots\}$, then

$$\chi_t(z) = \pi_{\mathrm{H}} \sum_{k=0}^{n-1} \xi_{2k+1}^{h_{2k+1}}(z)(-u'(c_{\mathrm{H}}^{\mathrm{lf}}) x^{\mathrm{lf}}) \beta^{2(n-k-1)} = -\beta^{t-1} \pi_{\mathrm{H}} u'(c_{\mathrm{H}}^{\mathrm{lf}}) x^{\mathrm{lf}} \sum_{k=0}^{n-1} \tilde{\xi}_{2k+1}^{h_{2k+1}}(z) \tag{D.2}$$

where h_{2k+1} is the agent with high income at event $(2k+1, z)$.[7]

E Additional Results on Section 4

This section provides additional interesting results regarding the equilibrium solutions of the baseline economy analyzed in Section 4.

E.1 Self-Enforcing Debt Limits Need Not Satisfy Weak Rollover

In the laissez-faire equilibrium, d^{lf} serves as the debt limit for borrowing against the high and low income regimes. In our policy experiment the low-income agent at period $t = 2$ issues debt $d_3(\varepsilon)$ against the next period's high-income realization. However, we have not addressed how much debt, denoted $\tilde{d}_3(\varepsilon)$, can be issued at $t = 2$ by the high-income agent against the next period's low-income realization. This omission is intentional, as its level is irrelevant in the equilibrium under study—agents save against their low income. Consequently, without loss of generality, we can set $\tilde{d}_3(\varepsilon)$ equal to its not-too-tight level, given the future debt limits—all of which are d^{lf} for $t > 3$. The characterization result of Hellwig and Lorenzoni (2009) then applies, so that $\tilde{d}_3(\varepsilon) = q_4(\varepsilon) d^{\mathrm{lf}}$.

[7]Agent h_{2k+1} has high income at event $(1, z)$ and therefore $\pi^{h_{2k+1}}(2k+1, z) = \pi^{h_{2k+1}}(1, z) = \pi_{\mathrm{H}}$.

The low-income agent at period $t = 1$ borrows up to the not-too-tight limit $d_2(\varepsilon)$ against next period's high-income realization. Since this agent will have high income at $t = 2$, from her perspective, borrowing capacity is constrained by the sequence of not-too-tight debt limits $(d_2(\varepsilon), \tilde{d}_3(\varepsilon), d^{\mathrm{lf}}, \ldots, d^{\mathrm{lf}}, \ldots)$. Again, Hellwig and Lorenzoni (2009) applies, so that

$$d_2(\varepsilon) = q_3(\varepsilon)\tilde{d}_3(\varepsilon) = q_3(\varepsilon)q_4(\varepsilon)d^{\mathrm{lf}}.$$

Figure E.1(a) plots the functions $\varepsilon \mapsto d_2(\varepsilon)$ and $\varepsilon \mapsto q_3(\varepsilon)\tilde{d}_3(\varepsilon)$ to demonstrate their equivalence. We also plot in Figure E.1(b) the function $\varepsilon \mapsto d_3(\varepsilon)$ and $\varepsilon \mapsto q_4(\varepsilon)d^{\mathrm{lf}}$ to illustrate that $d_3(\varepsilon)$ is too tight.

(a) High-income Agent's Debt at $t = 2$ (b) High-income Agent's Debt at $t = 3$

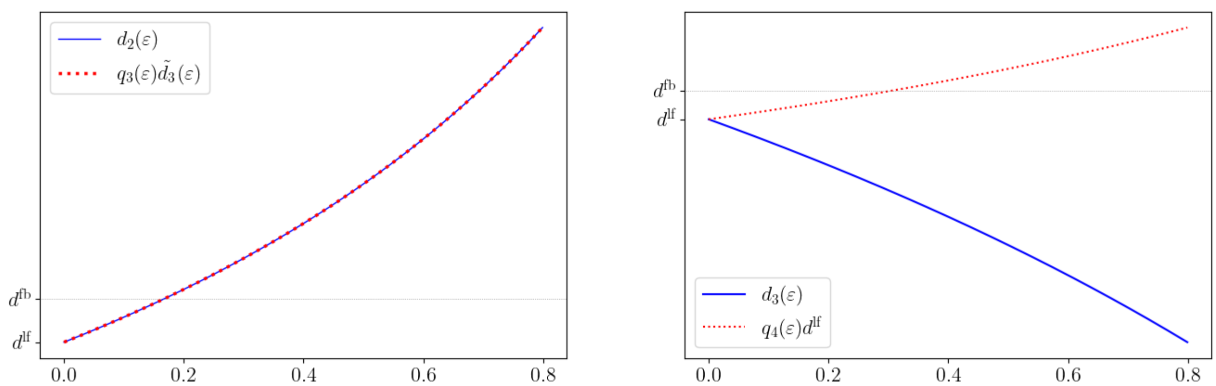

Figure E.1: Debt Limits and Weak Rollover

Let $\tilde{d}_2(\varepsilon)$ denote the debt limit restricting how much debt the high-income agent at period $t = 1$ can issue against next period's low-income realization. Following the same reasoning as before, we can set $\tilde{d}_2(\varepsilon)$ to its not-too-tight level. Notice that we cannot appeal to Hellwig and Lorenzoni (2009) to determine its level as we did with $\tilde{d}_3(\varepsilon)$ since, at period $t = 2$, this agent is constrained by $d_3(\varepsilon)$ that is too-tight. To compute $\tilde{d}_2(\varepsilon)$, we have to solve for the low-income agent's continuation utility when she starts at $t = 2$ with asset holdings equal to $-\tilde{d}_2(\varepsilon)$. Our educated guess is: at $t = 2$ she borrows up to her debt limit $d_3(\varepsilon)$ contingent to next period's high-income realization. Her continuation utility is then

$$u(y_{\mathrm{L}} - \tilde{d}_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon)) + \beta u(y_{\mathrm{H}} - x_3(\varepsilon)) + \beta^2 u(y_{\mathrm{L}} + x^{\mathrm{lf}}) + \beta^3 U_{\mathrm{H}}^{\mathrm{lf}}.$$

This educated guess is correct if, and only if, the first order condition at $t = 2$ is valid, i.e.,

$$q_3(\varepsilon) \geqslant \beta \frac{u'(y_{\mathrm{H}} - x_3(\varepsilon))}{u'(y_{\mathrm{L}} - \tilde{d}_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon))}.$$

The above inequality is satisfied for $\varepsilon > 0$ small enough as it is satisfied with a strict inequality when $\varepsilon = 0$. To compute the default value $V_{2,\mathrm{L}}(\varepsilon)$ of low-income agent at $t = 2$, we make the educated guess that $V_{2,\mathrm{L}}(\varepsilon) = u(y_{\mathrm{L}}) + V_{3,\mathrm{H}}(\varepsilon)$, where $V_{3,\mathrm{H}}(\varepsilon)$ is the default value of the hign-income agent at $t = 3$.[8] This educated guess is correct if, and only if, the following FOC is satisfied

$$q_3(\varepsilon) \geqslant \beta \frac{u'(y_{\mathrm{H}} - q_4(\varepsilon)\theta_4(\varepsilon))}{u'(y_{\mathrm{L}})}.$$

Again, the above inequality is satisfied for $\varepsilon > 0$ small enough as it is satisfied with a strict inequality when $\varepsilon = 0$.

As d^{lf} is not-too-tight, we have $U^{\mathrm{lf}}_{\mathrm{H}} = V^{\mathrm{lf}}_{\mathrm{H}}$. This implies that the not-too-tight debt level $\tilde{d}_2(\varepsilon)$ is determined by the equation

$$\begin{aligned} u(y_{\mathrm{L}} - \tilde{d}_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon)) + \beta u(y_{\mathrm{H}} - x_3(\varepsilon)) + \beta^2 u(y_{\mathrm{L}} + x^{\mathrm{lf}}) = \\ u(y_{\mathrm{L}}) + \beta u(y_{\mathrm{H}} - q_4(\varepsilon)\theta_4(\varepsilon)) + \beta^2 u(y_{\mathrm{L}} + \theta_4(\varepsilon)), \end{aligned} \tag{E.1}$$

where we recall that $\theta_4(\varepsilon) \geqslant 0$ is the optimal saving decision out of equilibrium of the high-income agent at date $t = 4$. It is characterized by the following FOC

$$q_4(\varepsilon) = \beta \frac{u'(y_{\mathrm{L}} + \theta_4(\varepsilon))}{u'(y_{\mathrm{H}} - q_4(\varepsilon)\theta_4(\varepsilon))} \quad \text{and} \quad q^{\mathrm{lf}} \geqslant \beta \frac{u'(y_{\mathrm{H}} - x^{\mathrm{lf}})}{u'(y_{\mathrm{L}} + \theta_4(\varepsilon))}.$$

To illustrate that not-too-tight limits cannot necessarily be rolled over, we plot in Figure E.2(a) the functions $\varepsilon \mapsto \tilde{d}_2(\varepsilon)$ and $\varepsilon \mapsto q_3(\varepsilon)d_3(\varepsilon)$. To reinforce this feature, Figure E.2(b) plots the functions $\varepsilon \mapsto d_1(\varepsilon)$ and $\varepsilon \mapsto q_2(\varepsilon)\tilde{d}_2(\varepsilon)$, showing that they do not coincide. The interesting implication is that the converse of Proposition 2.2 is not necessarily true: self-enforcing debt limits need not satisfy weak rollover.

[8] Recall that $V_{3,\mathrm{H}}(\varepsilon) = u(y_{\mathrm{H}} - q_4(\varepsilon)\theta_4(\varepsilon)) + \beta u(y_{\mathrm{L}} + \theta_4(\varepsilon)) + \beta^2 V^{\mathrm{lf}}_{H}$.

(a) Low-income Agent's Debt at $t = 2$ (b) High-income Agent's Debt at $t = 1$

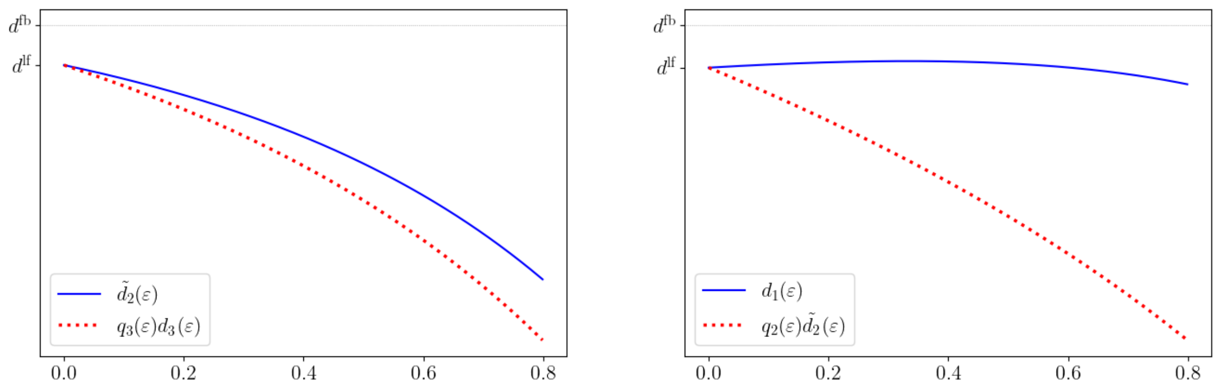

Figure E.2: Debt Limits and Weak Rollover

E.2 Stationary Policy Intervention

This section analyzes possible government interventions–tightening of debt constraints–that implement a symmetric stationary Markovian equilibrium. The associated consumption allocation is parameterized by the stationary net-trade $x \in [0, y_{\text{H}} - y_{\text{L}}]$ as follows: $c_0^i = y_0$ and $c_t^i(z) = c_{\text{H}}(z) := y_{\text{H}} - x$ if $y_t^i(z) = y_{\text{H}}$, and $c_t^i(z) = c_{\text{L}}(z) := y_{\text{L}} + x$ if $y_t^i(z) = y_{\text{L}}$. We should identify the set of net-trade values x that implement a feasible pair. We numerically compute the value of the default option at the initial date $t = 0$ and any $t \geqslant 1$ using the following parametrization: the income dispersion coefficient, discount factor, and risk-aversion coefficient values are set to $\zeta = 1.3$, $\beta = 0.9$, and $\gamma = 4$.[9]

The implied common price of the two contingent assets traded at $t = 0$ is

$$q_1(x) := \beta \max\{\pi_{\text{H}} u'(c_{\text{H}}(x)), \pi_{\text{L}} u'(c_{\text{L}}(x))\}/u'(y_0),$$

and the implied price of the risk-free asset traded at every $t \geqslant 1$ is[10]

$$q(x) := \beta \max\{u'(c_{\text{H}}(x))/u'(c_{\text{L}}(x)), u'(c_{\text{L}}(x))/u'(c_{\text{H}}(x))\}.$$

[9]See Figure E.3(b) for the stationary consumption levels.

[10]See Figure E.3(a).

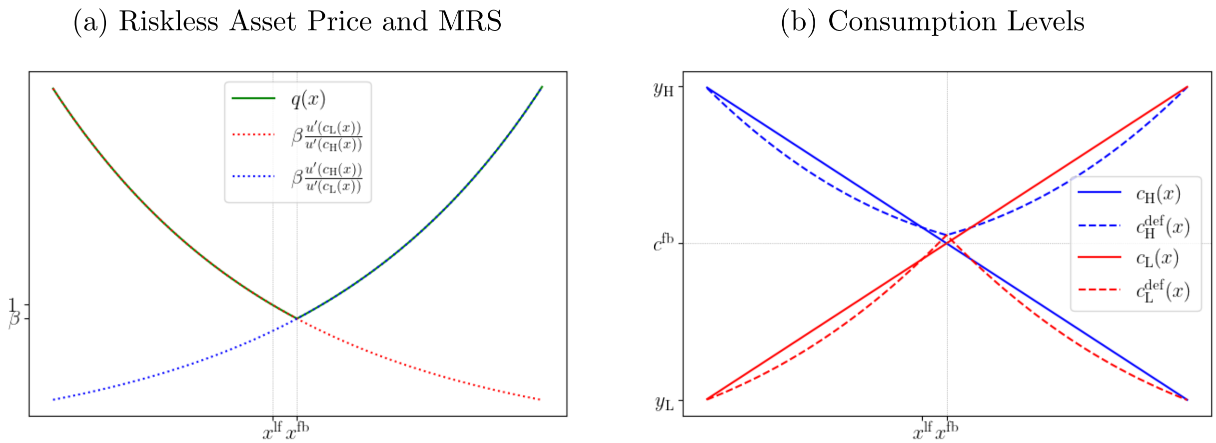

Figure E.3: Stationary equilibrium variables as functions of the net-trade x.

We start by analyzing the self-enforcing constraints at any date $t \geqslant 1$. Denote by $V_{\mathrm{H}}^{\mathrm{def}}(x)$ and $V_{\mathrm{L}}^{\mathrm{def}}(x)$ the default option value of the high and low-income agents, respectively. Their determination relies on an educated guess about the out-of-equilibrium path. We postulate the following default value functions:

$$V_{\mathrm{H}}^{\mathrm{def}}(x) = u(y_{\mathrm{H}} - q(x)\theta(x)) + \beta u(y_{\mathrm{L}} + \theta(x)) + \beta^2 V_{\mathrm{H}}^{\mathrm{def}}(x), \tag{E.2}$$

and

$$V_{\mathrm{L}}^{\mathrm{def}}(x) = u(y_{\mathrm{L}}) + \beta V_{\mathrm{H}}^{\mathrm{def}}(x), \tag{E.3}$$

that are derived by assuming that the defaulting agent only saves when income is high. The level of saving $\theta(x)$ is determined by the FOC of the high-income agent:

$$q(x) = \beta \frac{u'(y_{\mathrm{L}} + \theta(x))}{u'(y_{\mathrm{H}} - q(x)\theta(x))}.$$

To simplify the presentation, we use the following notations: $c_{\mathrm{H}}^{\mathrm{def}}(x) := y_{\mathrm{H}} - q(x)\theta(x)$ and $c_{\mathrm{L}}^{\mathrm{def}}(x) = y_{\mathrm{L}} + \theta(x)$. The low-income agent optimally decides not to save if, and only if, the following FOC are satisfied

$$q(x) \geqslant \beta \frac{u'(c_{\mathrm{H}}^{\mathrm{def}}(x))}{u'(c_{\mathrm{L}}^{\mathrm{def}}(x))} \quad \text{and} \quad q(x) \geqslant \beta \frac{u'(c_{\mathrm{H}}^{\mathrm{def}}(x))}{u'(y_{\mathrm{L}})}.$$

The second inequality is satisfied if the first one also is. We see in Figure E.4(b), that both conditions are satisfied.

(a) Out-of-equilibrium Saving Decision (b) Validity of the Low-Income FOC

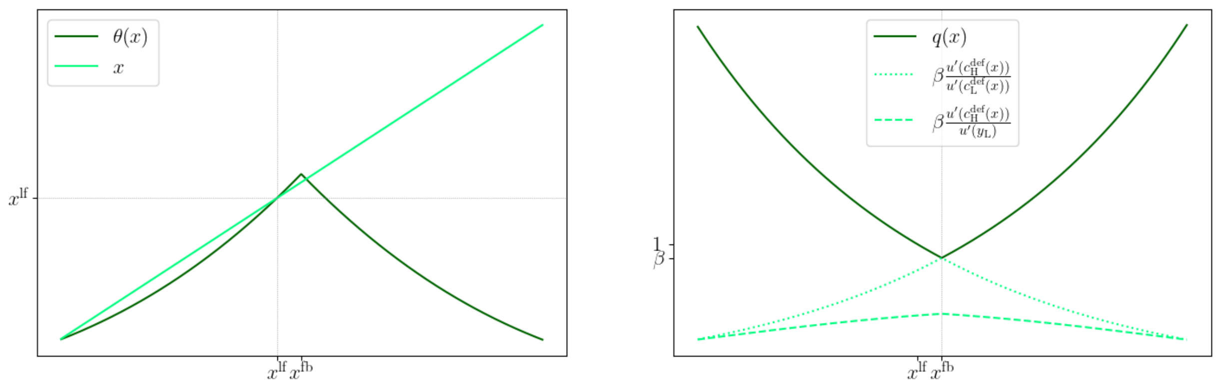

Figure E.4: Optimality of the out-of-equilibrium educated guess.

We see in Figure E.5(a) that the participation constraint, $V_{\mathrm{L}}(x) \geqslant V^{\mathrm{def}}_{\mathrm{L}}(x)$, of the low-income agent is satisfied for any the net-trade value x. However, the participation constraint, $V_{\mathrm{H}}(x) \geqslant V^{\mathrm{def}}_{\mathrm{H}}(x)$, of the high-income agent is satisfied for $x \in [0, x^{\mathrm{lf}}] \cup [\underline{x}, \overline{x}]$ where $x^{\mathrm{fb}} < \underline{x} < \overline{x}$.

We still have to verify the participation constraint at $t = 0$. As both agents are ex-ante identical, there is a single participation constraint

$$u(y_0) + \beta[\pi_{\mathrm{H}} V_{\mathrm{H}}(x) + \pi_{\mathrm{L}} V_{\mathrm{L}}(x)] =: V_0(x) \geqslant V^{\mathrm{def}}_0(x)$$

where $V^{\mathrm{def}}_0(x)$ is the value of the default option at $t = 0$.[11] Our educated guess is that agents optimally decide to save only against high income. This leads to the following value function

$$V^{\mathrm{def}}_0(x) := u(y_0 - q_1(x)\theta_{1,\mathrm{L}}(x)) + \beta\pi_{\mathrm{H}} V^{\mathrm{def}}_H(x) + \beta\pi_{\mathrm{L}}[u(y_{\mathrm{L}} + \theta_{1,\mathrm{L}}(x)) + \beta V^{\mathrm{def}}_{\mathrm{H}}(x)].$$

The level $\theta_{1,\mathrm{L}}(x)$ of savings contingent to low income is determined by the FOC

$$q_1(x) = \beta\pi_{\mathrm{L}} \frac{u'(y_{\mathrm{L}} + \theta_{1,\mathrm{L}}(x))}{u'(y_0 - q_1(x)\theta_{1,\mathrm{L}}(x))}.$$

[11] Agents do not start $t = 0$ with debt. However, for $x > x^{\mathrm{fb}}$, the government intervention imposes mandatory savings above the laissez-faire level. Agents may refuse to follow the government policy and default.

(a) Low-Income Participation Constraint (b) High-Income Participation Constraint

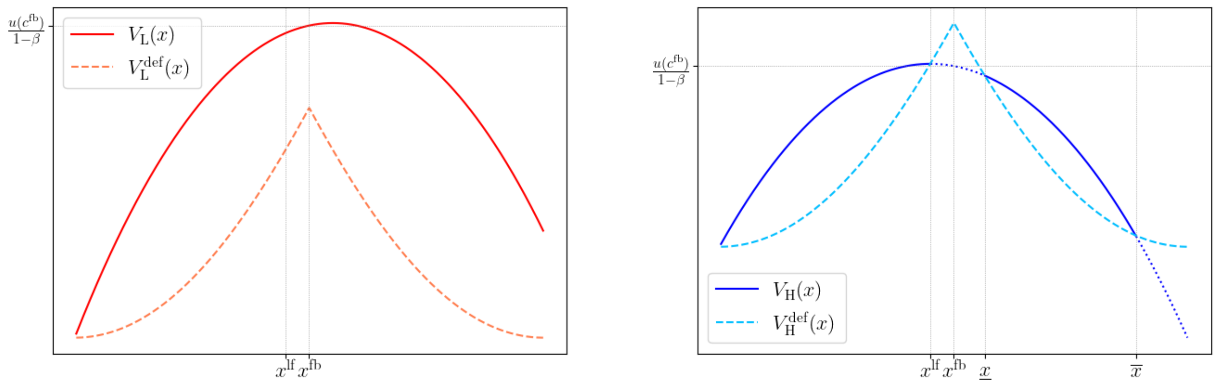

Figure E.5: Participation constraints at $t \geqslant 1$

At $t = 0$, the decision to save only against low income is optimal if, and only if, the following FOC is satisfied

$$q_1(x) \geqslant \beta\pi_{\mathrm{H}}\frac{u'(c_{\mathrm{H}}^{\mathrm{def}}(x))}{u'(c_0^{\mathrm{def}}(x))}$$

where $c_0^{\mathrm{def}}(x) := y_0 - q_1(x)\theta_{1,\mathrm{L}}(x))$ is the consumption level at $t = 0$ along the out-of-equilibrium path. Along this out-of-equilibrium started at $t = 0$, the low-income agent optimally decides not to save at $t = 1$ if, and only if

$$q(x) \geqslant \beta\frac{u'(c_{\mathrm{H}}^{\mathrm{def}}(x))}{u'(c_{1,\mathrm{L}}^{\mathrm{def}}(x))}$$

where $c_{1,\mathrm{L}}^{\mathrm{def}}(x) := y_{\mathrm{L}} + \theta_{1,\mathrm{L}}^{\mathrm{def}}(x)$. Figure E.6 shows numerically that our the two above FOC are satisfied and that our educated guess is indeed optimal.[12]

Figure E.7(a) illustrates that the participation constraint at $t = 0$ is satisfied for all net-trade values $x \in (0, y_{\mathrm{H}} - y_{\mathrm{L}})$. We deduce that the stationary allocation parametrized by the net-trade level x implements a socially feasible pair for any $x \in (0, x^{\mathrm{lf}}) \cup [\underline{x}, \overline{x}]$. For this set of net-trade levels, Figure E.7(b) shows that such a simple intervention does not lead to a Pareto improvement.

[12]We set the time 0 income at the average income level, i.e., $y_0 = (y_{\mathrm{L}} + y_{\mathrm{H}})/2$.

(a) Out-of-equilibrium FOC at $t = 0$

(b) Out-of-equilibrium FOC at $t = 1$

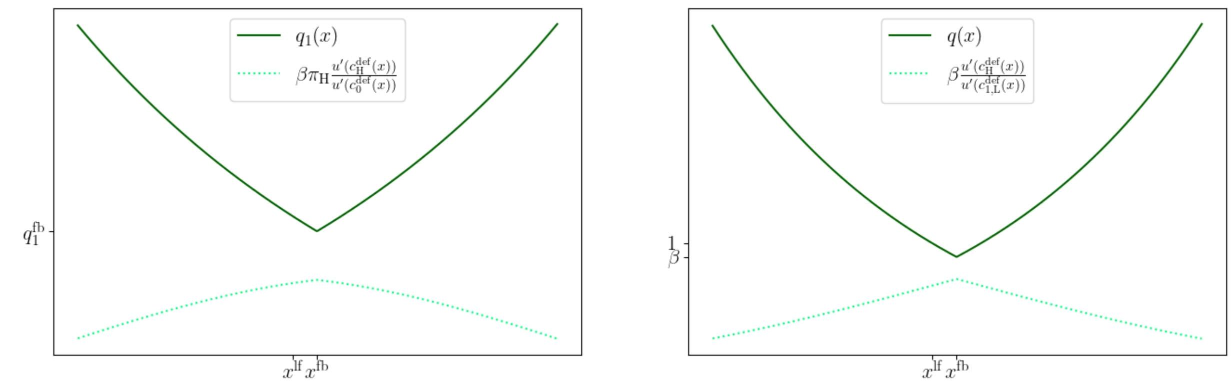

Figure E.6: Optimality of the out-of-equilibrium educated guess for default at $t = 0$.

(a) Participation Constraint at $t = 0$

(b) No Pareto Improvement

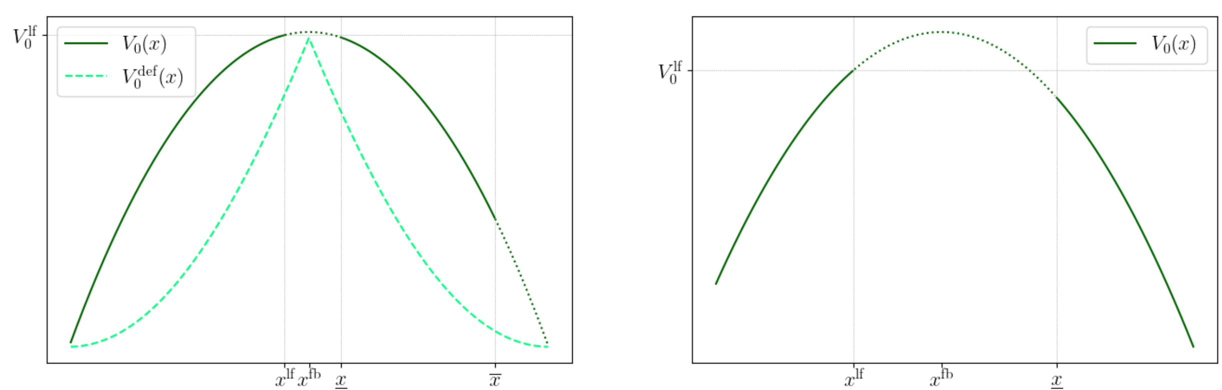

Figure E.7: Equilibrium Values at $t = 0$

F Omitted Proofs of Section 5

To simplify the exposition of the theoretical results, we assume in this section that u is bounded. This restriction ensures that the lifetime utility U is continuous (for the product topology) and the demand set is nonempty. The characterization of debt limits can be extended, and our results in this section continue to hold even when u is unbounded. A general treatment of unbounded utility functions requires additional technical assumptions on endowment processes and a suitable modification of the utility function u outside a specific interval so that the equilibrium outcomes remain unaffected. For a detailed discussion, see Martins-da-Rocha and Santos (2019).

F.1 Proof of Theorem 5.1

The proof of Theorem 5.1 exploits two intermediate results. The first and crucial observation, which has no analog in the absence of output contraction, is to show that the present value of foregone endowment imposes a lower bound on not-too-tight debt limits. A direct implication of this property is that the process $\mathrm{PV}(\ell^i)$ is finite. This is summarized in the following lemma.

Lemma F.1. *Not-too-tight debt limits are at least as large as the present value of endowment losses, i.e., for each agent i, $D^i(s^t) \geqslant \mathrm{PV}(\ell^i|s^t)$ at any event s^t.*

A natural approach to prove this result is to show that $D^i(s^t) \geqslant \ell^i(s^t) + \widetilde{D}^i(s^t)$, where $\widetilde{D}^i(s^t) := \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1})$ is the present value of next period's debt limits, and then use a standard iteration argument. Because, in equilibrium, debt limits are not too tight, this is equivalent to proving that agent i does not have the incentive to default when her net asset position is $\ell^i(s^t) + \widetilde{D}^i(s^t)$, i.e.,

$$V^i(D^i, -\ell^i(s^t) - \widetilde{D}^i(s^t)|s^t) \geqslant V^i_{\ell^i}(0, 0|s^t). \tag{F.1}$$

By definition, the value function $V^i_{\ell^i}$ satisfies:

$$V^i_{\ell^i}(0, 0|s^t) \geqslant u(y^i(s^t) - \ell^i(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V^i_{\ell^i}(0, 0|s^{t+1}). \tag{F.2}$$

If we had an equality in (F.2), then inequality (F.1) would be straightforward. Indeed, consuming $y^i(s^t)-\ell^i(s^t)$ and borrowing up to each debt limit $D^i(s^{t+1})$ at event s^t leads to the right-hand side continuation utility in (F.2) and satisfies the solvency constraint at event s^t in the budget set defining the left-hand side of (F.1). Unfortunately, in our environment where agents can save upon default condition (F.2) may not hold as an equality.[13] Overcoming this problem is the technical challenge in the proof of Lemma F.1. The formal argument is presented below.

The second observation is that the process $\mathrm{PV}(\ell^i)$ of present values of endowment losses, when it is finite, is itself not too tight. The following lemma provides the formal statement. The proof follows from a simple translation invariance of the flow budget constraints.

Lemma F.2. *If* $\mathrm{PV}(\ell^i|s^0)$ *is finite, then the process* $\mathrm{PV}(\ell^i)$ *is not too tight, i.e.,*

$$V^i(\mathrm{PV}(\ell^i), -\mathrm{PV}(\ell^i|s^t)|s^t) = V^i_{\ell^i}(0,0|s^t), \quad \forall s^t \succeq s^0.$$

Equipped with Lemma F.1 and Lemma F.2, we can now provide a simple proof of Theorem 5.1.

Proof of Theorem 5.1. Fix a process D^i of not-too-tight debt limits. Lemma F.1 implies that $\mathrm{PV}(\ell^i|s^0)$ is finite. From Lemma F.2, we also deduce that the process $\underline{D}^i := \mathrm{PV}(\ell^i)$ is not too tight. Martins-da-Rocha and Santos (2019) show that the difference between any two processes of not-too-tight debt limits must be an exact rollover process. Therefore, a process M^i exists satisfying the exact rollover property such that $D^i = \underline{D}^i + M^i$. By Lemma F.1, $D^i \geqslant \underline{D}^i$, in which case the process M^i must be nonnegative. □

F.1.1 Proof of Lemma F.1

Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i. Let D be a process of not-too-tight limits. We first show that

[13] In the simpler environment where, upon default, saving is not possible (as it is the case in Alvarez and Jermann 2000) condition (F.2) always hold as an equality.

there exists a nonnegative process $\underline{D}$ satisfying

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0. \tag{F.3}$$

Indeed, let Φ be the mapping $B \in \mathbb{R}^\Sigma \longmapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leqslant B \leqslant \bar{D}$ where

$$\bar{D}(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \succeq s^0.$$

The mapping Φ is continuous (for the product topology), and we have $\Phi[0, \bar{D}] \subseteq [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that Φ admits a fixed point $\underline{D}$ in $[0, \bar{D}]$.

Claim F.1. *The process $\underline{D}$ is tighter than the process D, i.e., $\underline{D} \leqslant D$.*

Proof of Claim F.1. Fix a node s^t. Since $V_\ell(0, 0|s^t) = V(D, -D(s^t)|s^t)$ and $V(D, \cdot|s^t)$ is strictly increasing, it is sufficient to show that $V(D, -\underline{D}(s^t)|s^t) \geqslant V_\ell(0, 0|s^t)$. Denote by $(c, \tilde{a})$ the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t.[14] We let $\widehat{D}$ be the process defined by $\widehat{D}(s^t) := \min\{D(s^t), \underline{D}(s^t)\}$ for all s^t. Observe that

$$\begin{aligned} y(s^t) - \underline{D}(s^t) &= y(s^t) - \ell(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \widehat{D}(s^{t+1}) \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) [\tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})] \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) \end{aligned}$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})$. Since $\widehat{D} \leqslant D$, we have $a(s^{t+1}) \geqslant -D(s^{t+1})$. At any

[14] That is, the process $\tilde{a}$ supports consumption c such that $U(c|s^t) := V_\ell(0, 0|s^t)$.

successor event $s^{t+1} \succ s^t$, we have

$$\begin{aligned} y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\ &\geqslant y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\ &\geqslant y(s^{t+1}) - \ell(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) \widehat{D}(s^{t+2}) \\ &\geqslant c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) [\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\ &\geqslant c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) a(s^{t+2}) \end{aligned}$$

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.[15] Observe that $a(s^{t+2}) \geqslant -D(s^{t+2})$ as $\widehat{D} \leqslant D$).

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that (c, a) belongs to the budget set $B(D, -\underline{D}(s^t)|s^t)$. It follows that

$$V(D, -\underline{D}(s^t)|s^t) \geqslant U(c|s^t) = V_\ell(0, 0|s^t)$$

implying the desired result: $\underline{D}(s^t) \leqslant D(s^t)$. $\square$

It follows from Claim F.1 that $\underline{D}$ satisfies

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \underline{D}(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \tag{F.4}$$

Applying equation (F.4) recursively, we get

$$p(s^t)\underline{D}(s^t) = p(s^t)\ell(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} p(s^{t+1})\ell(s^{t+1}) + \dots$$
$$\dots + \sum_{s^T \in S^T(s^t)} p(s^T)\ell(s^T) + \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})\underline{D}(s^{T+1})$$

for any $T > t$. Since $\underline{D}$ is nonnegative, it follows that

$$p(s^t)\underline{D}(s^t) \geqslant \sum_{\tau=t}^{T} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)\ell(s^\tau).$$

Passing to the limit when T goes to infinity, we get that $\mathrm{PV}(\ell|s^t)$ is finite for any event s^t (in particular for s^0). Recalling that $D \geqslant \underline{D}$, we also get that $D(s^t) \geqslant \mathrm{PV}(\ell|s^t)$.

[15] To get the second weak inequality, we use equation (F.3).

F.1.2 Proof of Lemma F.2

Denote by $(c, \tilde{a})$ the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t. We pose $\underline{D} := \mathrm{PV}(\ell)$ and observe that

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}).$$

It is easy to show that (c, a) is optimal in the budget set $B(\underline{D}, -\underline{D}(s^0)|s^t)$ where $a := \tilde{a} - \underline{D}$. We then deduce that $V^i(\underline{D}, -\underline{D}(s^t)|s^t) = V_\ell(0, 0|s^t)$, so proving the claim.

F.2 Proof of Proposition 5.1

Let $(q, (c^i, a^i, D^i)_{i \in I})$ be an equilibrium with limited pledgeability. Since pledgeable income is nonnegligible, we must have

$$\sum_{i \in I} \mathrm{PV}(y^i|s^0) \leqslant \frac{1}{\varepsilon} \sum_{i \in I} \mathrm{PV}(\ell^i|s^0).$$

By the decomposition property property (5.3) , we have that $\mathrm{PV}(\ell^i|s^0) < \infty$ for each agent i, so we deduce that the aggregate wealth of the economy $\sum_{i \in I} \mathrm{PV}(y^i|s^0)$ must be finite. Since consumption markets clear, we obtain that the present value of optimal consumption is finite for all agents. In addition, due to the Inada's condition, the optimal consumption is strictly positive.[16] Lemma A.1 in Martins-da-Rocha and Vailakis (2017) then implies that the market transversality condition holds true:[17]

$$\lim_{t \to \infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0. \tag{F.5}$$

The decomposition property property (5.3) implies that, for each i, there exists a nonnegative discounted martingale process M^i such that $D^i = \mathrm{PV}(\ell^i) + M^i$. Condition F.5 can then be

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[16]See the supplemental material of Martins-da-Rocha and Santos (2019) for detailed proof.

[17]The market transversality condition differs from the individual transversality condition. Indeed, due to the lack of commitment, agent i's debt limits may bind, in which case we do not necessarily have that $p(s^t) = \beta^t \pi(s^t) u'(c^i(s^t))/u'(c^i(s^0))$.

rewritten as follows:

$$\lim_{t\to\infty}\sum_{s^t\in S^t} p(s^t)a^i(s^t) = -p(s^0)M^i(s^0).$$

Since bond markets clear, we deduce that $\sum_{i\in I} M^i(s^0) = 0$, proving the desired result: $M^i = 0$ for each i.

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