Endogenous Social Minimum*

Yufeng Shi†

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Abstract

This paper characterizes a class of individual preferences in which heterogeneous social members care not only about their own consumption, but also about the minimum consumption in society. The key axiom triggering such concern is an indifference preference on the consumption distribution of others whenever a social member is "miserable", defined as possessing the lowest disposable endowment after transfer. The characterized individual preferences, represented by a linear combination of a social utility function increasing in the minimum consumption in society and an egoistic utility function increasing in one's own consumption, can support an endogenous social minimum by a benevolent utilitarian social planner. This paper evaluates the dynamics of the endogenous social minimum under various development scenarios, and reveals corresponding sufficient conditions for the social minimum to converge to (or never reach): i. the average endowment (equality of outcome), and ii. the lowest individual endowment (laissez-faire). Even in eventual laissez-faire cases, the social planner can facilitate higher transfers from rich social members to poor social members and minimize the consumption inequality. The endogenous social minimum can also be supported through voluntary contributions from social members following the common approach in public goods literature. Nevertheless, total voluntary contribution will converge to zero if the society expands, keeping the distribution of initial resource endowments unchanged.

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†Economics Department, Boston College. Contact: shibh@bc.edu.

1 Introduction

At least since Charles Fourier and John Stuart Mill, the idea of establishing a social minimum to provide every social member with adequate resources for the subsistence in society has emerged (Cunliffe and Erreygers, 2001). From its origination, the ideal of social minimum has an innate advocacy of social justice, or, in a more liberal sense, a legitimate wealth everyone shall be endowed under certain natural or equal status. Standard literature in economics often assumes an exogenously given social minimum, leaving the methodology of directly obtaining a social minimum from individual preferences an intriguing subject. Intuitively, an endogenous social minimum should be determined by the values of each social member, and social members may hold different opinions on the appropriate level of social minimum that should be established in a society. Still, it shall not be too demanding to assume a social consensus that, without considering its cost, a higher social minimum is better than a lower one. Hence the heterogeneity should matter when establishing certain level of social minimum comes with a corresponding cost, and to the extent that some social members might be particularly happy about paying more for a higher social minimum, while others disagree. Such heterogeneity in preferences on social minimum can be regarded as various social components of individual preferences held by different social members, apart from egoistic components that only care about their own consumption.

This paper characterizes a class of individual preferences represented by a linear combination of a social utility function and a egoistic utility function, under which social members care not only about their own consumption, but also about the social minimum defined as what they think should be the minimum level of individual consumption in society. This paper then advances the approaches of establishing social minimum, which previously often consisting of assuming exogenous or indeterminate existence, by deriving an endogenous social minimum from the characterized individual preferences of social members and the total resources endowed to the social members. We discuss two scenarios of establishing endogenous social minimum: by the ruling of a benevolent utilitarian social planner and through voluntary contributions. Both cases discuss budget-balanced and non-wasteful transfer plans in which the total resources needed for lifting up people who are below the social minimum exactly equal the total resources taken from the people who are above it. The main motivation for social members above the social minimum to contribute to the social support system lies in the social components of their individual preferences, represented by social utility functions strictly increasing in the social minimum.

By the ruling of a benevolent utilitarian social planner, comparative static analysis indicates that the relative social minimum as a fraction of the average endowment in society is not always increasing in individual endowments. Specifically, this is true when the rate of diminishing in marginal egoistic utility *at the* *social minimum* for any one of the poor social members is higher than the rate of diminishing in marginal egoistic utility *at current consumption level* for any one of the rich social members. Under different economic development patterns, this paper contrasts the sufficient conditions for the social minimum to converge to the average endowment with the sufficient conditions for the social minimum to converge to the lowest individual endowment. These two sets of sufficient conditions mainly differ in the limit of marginal rate of contribution of each social member, as the ratio between the marginal utility gain across society from increasing the social minimum and the egoistic utility cost of that particular social member. Notably, the redistribution outcomes under a social planner are path-dependent. Precisely, a single redistribution following a one-time increase in individual endowments results in lower total transfers from rich social members to poor social members compared to the total transfers generated by multiple redistributions occurring after each incremental increase in individual endowments, which collectively equal the same amount as the one-time increase. In addition, even in the worst case where the social minimum will eventually converge to the lowest individual endowment and administrative resources for redistribution are limited, the social planner can still choose specific development stage to redistribute the resources so as to enable the highest possible transfers from rich social members to poor social members and effectively minimize the consumption inequality.

Under voluntary contribution, such motivation quickly diminishes in an expanding society, as additional contributions will always be shared between all social members with initial endowments below the social minimum (henceforth the poor social members).[1] This equally-shared redistribution feature of additional contributions can lead to high marginal rate of substitution between the increase in social minimum (raising contributor's social utility) and the increase in contribution amount (lowering contributor's egoistic utility). Similar to common predictions in the theory of public goods, total voluntary contribution will converge to zero if each social member is self-replicated for a sufficiently large number of times (and the distribution of initial resource endowments is necessarily unchanged). The inadequate redistribution under voluntary contribution naturally calls for an approach in which the social minimum is established under a benevolent utilitarian social planner.

Related Literature

Despite of abundant discussions in the reason of implementing a social minimum, as well as the normative level that a social minimum should be (see White 2021 for a detailed introduction), to the best of our knowledge there does not exist similar modeling work in the current economic literature building an endogenous

[1] Since the social utility function is increasing in the social minimum, and by contrast the social utility function is increasing in one's own consumption, the social members will agree in unanimous that any additional contribution to the social support system should be equally shared between all social members consuming at the current social minimum so that the increase in social minimum (hence the social utility of everyone) is maximized.

social minimum on individual preferences, characterized as a linear combination of social utilities and egoistic utilities, of heterogeneous social members without uncertainty in their individual endowments. Tserenjigmid (2019) characterizes a Min-Min Reference Dependent Choice (MRDC) correspondence which (its reference point) resembles our social preference component that is determined by the minimum consumption in society. However, when interpreting different attributes in the alternatives as allocations on different social members, the representative utility function of such choice correspondence, in its form, depicts the aggregation of individual preferences, rather than the individual preferences per se. Similar issue also occurs in a more generalized version of such choice correspondence characterized by Poterack (2016). Other axiomatization approaches of social preferences fully determined by minimum consumption in society are not unusual in work about Rawlsian social welfare functions (see, for example, Ou-Yang 2016). Sarin (2021) analyses the optimal social minimum as maximizing expected utility for a representative rational (not necessarily altruistic) social member behind the veil of ignorance (Harsanyi, 1953; Rawls, 1999). Nevertheless, establishing social minimum in reality often faces a group of social members knowing their identity and wealth outcome, and hence redistribution relying on maximizing ex ante outcomes is unavailable. Segal and Sobel (2002) axiomatizes a symmetric social preference that may only care the minimum consumption in society, without guaranteeing an exact level of social minimum to be the "turning point" from which the social members feel indifferent about all possible variation in consumption distributions as long as the minimum consumption in society remains unchanged.

Another important line of literature studies a class of individual preferences similar to the one proposed in this paper, namely quasi-maximin preferences (Charness and Rabin, 2002). The individual preferences characterized here differ from quasi-maximin preferences in two notable ways. Firstly, the social components of the individual preferences characterized in this paper allow heterogeneity among social members with regard to (the magnitude of) their positive feelings on a higher minimum consumption in society, while in quasi-maximin preferences the social components are (weighted) social welfare functions invariant across all social members. Secondly, we shut down the channel of sum of payoffs (or utilities) of other social members that also affect one's social utility in quasi-maximin preferences. The exclusion of distributive patterns beyond minimum consumption in social components is motivated by the observation that a person may not fully comprehend the overall distribution in society, but tends to have a better understanding of the disadvantaged. Given these differences, the characterization approach in this paper also deviates from those in current literature for quasi-maximin preferences, including Segal and Sobel (2007) for the Segal-Sobel utility function as a superset that includes quasi-maximin preferences as special cases, as well as Sandbu (2008), Bossert and Kamaga (2020), and Schneider and Kim (2020) for the social welfare functions incorporated in quasi-maximin preferences.

From a broader perspective, our discussion about social minimum should also include those works dealing with sufficientarianism, a similar concept advocating that social members must have as many resources as necessary (Casal, 2007). In an extended set of literature discussing social minimum in a sufficientarian sense, the (minimum) level for the resources to be considered as "sufficient" is often taken as exogenous. Alcantud et al. (2022) characterize a "head-counting" sufficientarian criterion, in which social members will prefer a society with more of its members having enough resources among the two societies of the same size, assuming an existent threshold of enough resources in the axioms. Bossert et al. (2023) generalize such approach, also assuming that such threshold is exogenously given. Chambers and Ye (2024) move one step further by contributing an axiomatization of the "head-counting" approach based on an order filter feature such that all sufficientarian criterions share, without explicitly assuming the existence of a threshold. Nonetheless, the social minimum (including in sufficientarian sense) should have some endogeneity, and hence depends not only on total society endowments, as richer society tends to have higher living cost (Sen, 1987), but also on perspectives of social members from various socioeconomic backgrounds (Davis et al., 2018).[2] This paper intends to incorporate both features as inherent characteristics in individual preferences of all social members, rather than in a social preference shared across social members or aggregated from the individual preferences.

Besides, the potential conflict (from a utilitarian view) between the social welfare objective and the type of "currencies" available for establishing a social minimum (e.g. a benevolent utilitarian social planner cares about the sum of individual utilities but can only reallocate resources) is not discussed extensively in the models aforementioned, for they often end up with an exogenous or indefinite resource (or capability) level as the social minimum. In addition, some of these characterizations aim at constructing social preferences, leaving the budget constraint of the social support system, a general issue that the social planner has to consider in many real applications of establishing a social minimum (e.g., providing basic medicare to low-income personnels, or giving basic subsistence allowance to qualified people) out of the scope.

This paper contributes to the literature by providing a novel characterization for a class of individual preferences that is embedded with an innate concern about the social minimum, exploring possible behavioral incentives to contribute towards the establishment of a social minimum, and evaluating possible evolution of the endogenous social minimum in a developing society. The social minimum established here should be regarded as a benchmark of setting the level of basic allowance or other essential benefits under different scenarios that the social planner might have to deal with.

The rest of the paper is organized as follows. The next section will configure the society being considered

[2] In Sen (1987), it is favored to use capabilities and functionings as a metric for the living standard, while in this paper the social minimum is measured pecuniarily (or by resources). The results in this paper will hold in the capabilities and functionings approach if the cost function of certain levels of capability is weakly convex.

in the model and propose a class of individual preferences for the social members. Section 3 then characterizes the proposed individual preferences. Section 4 presents the endogenous social minimum that can be supported by the ruling of a benevolent utilitarian social planner. Section 5 considers the dynamics of the social minimum that can be supported under a utilitarian social planner in a developing society with various patterns of increase (development) in initial endowments. Section 6 probes into the path dependency in the reallocation results when there are multiple redistributions led by a utilitarian social planner, and points out the conditions when choosing the optimal development stage for another round of redistribution to minimize the consumption inequality. Section 7 briefly discusses the endogenous social minimum that can be supported through voluntary contributions. Section 8 concludes.

2 Model

Consider a society consisting of n social members denoted by the set $N = \{1, 2, \ldots, n\}$. Given an initial endowment vector $x = (x_1, x_2, \ldots, x_n)$, let x_i be the endowment (single consumption good for simplicity) of social member i, and $\bar{x} = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$ be the average endowment in society. The individual preferences to be characterized will ensure that everyone cares about the living standard of the poorest people in society. Hence the social members seek to establish a social minimum that should be enjoyed by each social member below it (henceforth the poor people in society). Namely, each poor people will be lifted up to the social minimum. However, people also care about their own consumption. Particularly, their individual preferences on the consumption allocation in this society can be represented as a linear combination of their social and egoistic utility functions:

$$U_i = f_i(\cdot) + u_i(\cdot)$$

Morally speaking, it is possible to add explicit weights $\alpha_i \in (0, 1)$ that social members put on the living of the poor people in society and its complement weight to their own consumption, representing the trade-offs between the social utility $f_i(\cdot)$ and egoistic utility $u_i(\cdot)$. Nevertheless, such configuration will result in utility functions of the form $U_i = \alpha_i f_i(\cdot) + (1 - \alpha_i) u_i(\cdot)$, where such weights can be absorbed into corresponding components (social and egoistic utilities) without loss of generality, and therefore we omit such explicit weights henceforth.

In order to lift the poor people who live below the social minimum exactly to the social minimum, there will be a transfer collected from the social members with endowments above the social minimum (henceforth the rich people in society) to have sufficient funding for the social support system. Let t_i be the transfer either taken from (if positive) or given to (if negative) person i, then naturally the following balanced budget condition has to be satisfied

$$\sum_{i=1}^{n} t_i = 0$$

Define the individual consumption equals disposable endowment after transfer as $c_i = x_i - t_i$. The egoistic utility of any social member $i \in N$ is determined by one's own consumption $c_i \in \mathbb{R}_+$.

$$u_i(c_i)$$

such that the egoistic utility function of social member i, u_i is assumed to be twice differentiable, increasing in its argument, and concave. Hence $u_i' > 0$ and $u_i'' < 0$.

Let $c = (c_1, c_2, \ldots, c_n)$ be the consumption vector of all social members. The social utility of any social member $i \in N$ is determined by the social minimum, defined as the minimum individual consumption in society, $\min\{c\}$. Later in this paper, the social minimum will often be intentionally denoted by a multiplier times the average endowment in society, $m\bar{x} \equiv \min\{c\}$, so that the multiplier $m = \frac{\min\{c\}}{\bar{x}} \in [0,1]$ has a natural interpretation as the social minimum coefficient indicating the relative abundance of the (absolute) social minimum comparing with the average endowment in society.[3]

$$f_i(\min\{c\}) = f_i(m\bar{x})$$

such that the social utility function of social member i, f_i is assumed to be twice differentiable, increasing in its argument, and concave. Hence $f_i' > 0$ and $f_i'' < 0$. Note that both social and egoistic utility functions variate between social members. Social members are considered heterogeneous in their preferences on the consumption allocation in this society, given the non-identical social and egoistic utility functions, as well as their different initial allocation of disposable endowments, $x = (x_1, x_2, \ldots, x_n)$.

Since the social minimum coefficient m and individual endowments $x_1, x_2, \ldots, x_n$ are the only parameters that enter one's egoistic and social utility functions, the transfer t_i may be considered as a function of the aforementioned parameters, $t_i(m, x)$, with a slight abuse of notation. It is trivial that if the social minimum is set below the initial endowment of the poorest people in society, then no transfer is necessary to support such social minimum, viz., if $m\bar{x} \le \min\{x\}$, then $t_i(m, x) = 0$ for all $i \in N$. For notational convenience, the proportion of social members who are initially below the social minimum before any transfer, i.e., when

[3]Some readers may argue that it would be better to relate the social utility function to the relative social minimum (as a percentage of the average endowment in society) rather than its absolute level. For that possibility, a detailed discussion on possible motivations for the social members to consider relative social minimum as the sole component in the social utility function can be found in the section A of the appendix.

$t_i = 0$ for all $i \in N$, is denoted by $p(m) \equiv \frac{|\{i:x_i \leq m\bar{x}\}|}{n} \in [0,1]$. Besides, since the poor people who live below the social minimum will be lifted up exactly to the social minimum, and the rich people who live above the social minimum shall not go below it. That is, $t_i(m,x) = x_i - m\bar{x}$ if $x_i \leq m\bar{x}$, and $t_i(m,x) \leq x_i - m\bar{x}$ otherwise. In addition, under dynamic settings where the initial allocation of disposable endowments may change, the corresponding comparative statics conditions for the transfer amount taken from someone to be increasing in one's own individual endowment and decreasing in others' endowments can be represented as $\frac{dt_i}{dx_i} \geq 0$ and $\frac{dt_i}{dx_{j\neq i}} \leq 0$.

3 Axioms

In this section, we would like to characterize a class of individual utility functions of functional form $U_i(c) = f_i(\min\{c\}) + u_i(c_i)$. Again, given a society of $N = \{1, 2, \ldots, n\}$ members, $c = (c_1, c_2, \ldots, c_n)$ is the consumption vector and c_i is the consumption of social member i. We will start from a basic form assuming that individual utility is determined by the consumption vector, hence $U_i(c)$ as a representation of the complete, continuous, and transitive individual preference of social member i, $\succeq_i$. The following two axioms trigger a focus on the social minimum for every social member from a rather egoistic ideological foundation that a social member who is already the poorest people in the society (having the lowest consumption) would no longer care about the distributive justice for others, perhaps due to the miserable life that this poorest member must deal with. For the preference on distributive justice, or on resource allocation regarding to consumptions of others, the axioms do not impose much limitation on it except ruling out a cynical idea by regulating that people should in general be happy about all other social members having a strictly better life when they are not the poorest member in society.

Complete Separability of Self-consumption until Miserable (**CSSM**): for all $c_i > \min\{c_{-i}, c'_{-i}\}$ and $c'_i > \min\{c_{-i}, c'_{-i}\}$, $(c_{-i}, c_i) \succeq_i (c'_{-i}, c_i)$ if and only if $(c_{-i}, c'_i) \succeq_i (c'_{-i}, c'_i)$. Moreover, for all $c''_i \leq \min\{c_{-i}, c'_{-i}\}$, $(c_{-i}, c''_i) \sim_i (c'_{-i}, c''_i)$.

Weak Monotonicity in Others' Consumption until Miserable (**WMOM**): for any (c_{-i}, c_i), (c'_{-i}, c_i) such that $c_{-i} \gg c'_{-i}$ and $c_i > \min\{c_{-i}, c'_{-i}\}$, we have $(c_{-i}, c_i) \succ_i (c'_{-i}, c_i)$.

Before the addition of miserable judgement on whether the social member is the poorest people in the society, axioms CSSM and WMOM indicate that every social member is evaluating the overall consumption distribution of other social members from the perspective of an impartial observer following the weak Pareto principle. Adding the miserable judgement allows the social member to deviate from such impartial role whenever the social planner is beset with the most miserable situation (lowest consumption) and no longer make positive or negative judgement on the overall consumption distribution of other social members. Surprisingly, these two axioms, although endow every social member with pure egoistic individual preference under the most miserable situation, are sufficient to develop an unbiased attentiveness to the lowest consumption in society regardless of whether that particular social member is the person with the lowest consumption in society or not. The following lemma demonstrates such unbiased attentiveness.

Lemma 1. *If a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies the axioms CSSM and WMOM, then for all (c_{-i}, c_i) and $\left(c'_{-i}, c_i\right)$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if* $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}, c_i\right\}$.

Proof. See section B.1 of the appendix.

So far, all the comparisons in Lemma 1 are between two consumption allocations where one's own consumption remains unchanged. In order to let the social member has a normal sense that a higher own consumption is always better given that everybody else remains the same, we add a standard axiom to let the individual preference be strictly increasing in one's own consumption.

Monotonicity in Self-consumption (**MS**): for any (c_{-i}, c_i), (c_{-i}, c'_i) such that $c_i > c'_i$, we have $(c_{-i}, c_i) \succ_i (c_{-i}, c'_i)$.

Corollary 1. *If a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies the axioms CSSM, WMOM, and MS, then $(c_{-i}, c_i) \succeq_i (c_{-i}, c'_i)$ if and only if $\left(c'_{-i}, c_i\right) \succeq_i \left(c'_{-i}, c'_i\right)$.*

Up to this point, we can conclude that a complete, continuous, and transitive individual preference following axioms CSSM, WMOM, and MS can be represented by a utility function of functional form $U_i(c) = u_i(\min\{c\}, c_i)$ that is strictly increasing in both its first and second arguments up to a positive affine transformation. In order to have additive separability between the minimal consumption among social members and one's own consumption, we add the triple cancellation condition proposed by Wakker (1988), who extends the work of Debreu (1960) and Krantz et al. (1971).

Triple Cancellation (**TC**): If $(c_{-i}, c_i) \preceq_i \left(c'_{-i}, c'_i\right)$, $c_i \geq \min\{c_{-i}\}$, $\left(c''_{-i}, c_i\right) \succeq_i \left(c'''_{-i}, c'_i\right)$, $c'_i \geq \min\left\{c'''_{-i}\right\}$, and $(c_{-i}, c''_i) \succeq_i \left(c'_{-i}, c'''_i\right)$, $c'''_i \geq \min\left\{c'_{-i}\right\}$, then $\left(c''_{-i}, c''_i\right) \succeq_i \left(c'''_{-i}, c'''_i\right)$ for all $c''_i \geq \min\left\{c''_{-i}\right\}$.

Axiom TC enters our axiomatic system in a slightly weaker form. Specifically, we tighten up the antecedents to ensure that the inferior society in comparison does not trigger the miserable judgement in axiom CSSM and loses information on other's resource allocation given such comparison result. On its consequent, we limit the applicable cases to rule out the cases when the superior society in comparison triggers the miserable judgement in axiom CSSM and becomes less desirable regardless of other's resource allocation. The following theorem provides a full characterization of the individual utility representation.

Theorem 1. *For a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c, the following conditions are equivalent:*

1. $\succeq_i$ satisfies axioms CSSM, WMOM, MS, and TC.

2. There exists, up to a positive affine transformation, a continuous utility function $U_i(c) = f_i(\min\{c\}) + u_i(c_i)$ representing $\succeq_i$, in which both f_i and u_i are strictly increasing in their arguments.

Proof. See section B.2 of the appendix.

So far the characterization does not guarantee the social utility function f_i and the egoistic utility function u_i to be strictly concave for all social member $i \in N$. Since social members are assumed to know their initial endowments without uncertainty, adding risk aversion axioms to induce concave utility functions can be unnatural. Instead, we choose to interpret strict concavity in the egoistic utility function u_i as assuming diminishing marginal utility in one's own consumption, and similarly strict concavity in the social utility function f_i as assuming diminishing marginal utility in improvement of consumption inequality.

4 Endogenous Social Minimum

Given the representative individual utility functions characterized above, the marginal utility for social member i from increasing the social minimum **coefficient** m is

$$\frac{dU_i(m,x)}{dm} = f_i'(m\bar{x})\,\bar{x} - u_i'(x_i - t_i(m,x))\,\frac{dt_i}{dm}$$

Let $j(m) = [1 - p(m)]\,n$ stands for the number of rich people given a social minimum coefficient m. Then in aggregate, the transfer $t_i(m,x)$ from the rich people in society should satisfy $\sum_{i=1}^{j(m)} \frac{dt_i}{dm} = p(m)\,n\bar{x}$. Also, let $J(m) = \{1, 2, \ldots, j(m)\}$ indicates the set of rich people who live above the social minimum in society. The social member i will agree on increasing the social minimum at some status quo if the marginal utility from doing so is strictly positive, i.e., if $\frac{dU_i(m,x)}{dm} > 0$. W.l.o.g. assume $x_1 \geq x_2 \geq \ldots \geq x_n$. The optimal social minimum $m^* \in \left(\frac{x_j}{\bar{x}}, \frac{x_{j+1}}{\bar{x}}\right)$ should satisfy the first order condition $\frac{dU_i(m,x)}{dm} = 0$ for all i.[4] For simplicity we only consider the case when $m^* \in \left(\frac{x_j}{\bar{x}}, \frac{x_{j+1}}{\bar{x}}\right)$ below, so that $p(m^*)$ and $j(m^*)$ are constants, and $\frac{dU_i(m^*,x)}{dm}$ is continuous in m. The second order condition in this case is $\frac{d^2U_i}{dm^2} = f_i''(m\bar{x})\bar{x}^2 - \left[u_i'(x_i - t_i(m,x))\frac{d^2t_i}{dm^2} - u_i''(x_i - t_i(m,x))\left(\frac{dt_i}{dm}\right)^2\right] < 0$ if $\frac{d^2t_i}{dm^2} \geq 0$, as $f_i''(m\bar{x}) < 0$, $u_i'(x_i - t_i(m,x)) > 0$, and $u_i''(x_i - t_i(m,x)) < 0$.

By the ruling of a benevolent utilitarian social planner, we assume that the goal of the social planner is to choose a transfer scheme $t = (t_1, t_2, \ldots, t_n)$, such that t_i is the transfer for social member i, to maximize the sum of utilities among all social members given the initial endowments $x = (x_1, x_2, \ldots, x_n)$.[5] In other words, the objective social welfare function that the social planner would like to maximize is:

$$SWF(t \mid x) \equiv \sum_{i=1}^{n} f_i(m(t)\bar{x}) + u_i(x_i - t_i)$$

in which the social minimum $m(t)\bar{x}$ is implicitly determined by the balanced budget condition $\sum_{i=1}^{j} t_i = \sum_{i=j+1}^{n} (m\bar{x} - x_i)$.[6] Let the optimal social minimum under such maximization problem be $m_{CU}^*\bar{x}$, and for notation convenience assume w.l.o.g. that $j = [1 - p(m_{CU}^*)]\,n$ is the number of rich people in society, and $1 \leq j \leq n-1$. In accordance with the definition of social minimum, the social planner will comply with the social norm that rich people shall not go below the social minimum, i.e., $t_i \leq x_i - m\bar{x}$ for all $i \in \{1, 2, \ldots, j\}$. Also, the poor people who live below the social minimum shall not have the duty to contribute to the social support system and will be lifted up to the social minimum, i.e., $t_i = x_i - m\bar{x}$ for all $i \in \{j+1, j+2, \ldots, n\}$. Formally, the social planner is facing the following maximization problem:

[4] If $\lim_{m \downarrow \frac{x_j}{\bar{x}}} \frac{dU_i(m,x)}{dm} < 0$, $\lim_{m \uparrow \frac{x_j}{\bar{x}}} \frac{dU_i(m,x)}{dm} > 0$, and $\frac{dU_i(m,x)}{dm} > 0$ for all i, then $m^* = \frac{x_j}{\bar{x}}$.

[5] To establish an upper bound of the social minimum that can be supported based on the individual preferences characterized earlier, we deliberately introduce a benevolent utilitarian social planner who is responsible for collecting the funding for the social support system, particularly in the context of a large society. We discuss the results under voluntary contributions in section 7.

[6] For an axiomatization of the preference aggregation introduced here, see a recent novel approach in Li et al. (2024). Note that the axioms proposed earlier in this paper are for characterizing the individual preferences, and hence they do not conflict with those aiming at characterizing the aggregation of individual preferences. For further discussion regarding the gap between representations of social preferences and the axiomatization of utilitarianism, as well as interpersonal utility comparisons, see arguments in Harsanyi (1955, 1979), analysis in Weymark (1991), and a more recent discussion in Greaves (2017).

$$
\begin{aligned}
&\max_{m,t} \sum_{i=1}^{n} f_i\left(m\bar{x}\right) + u_i\left(x_i - t_i\right) \\
&s.t. \sum_{i=1}^{j} t_i = \sum_{i=j+1}^{n} \left(m\bar{x} - x_i\right) \\
&\quad t_i \leq x_i - m\bar{x} \; \forall i \in \{1, 2, \ldots, j\} \\
&\quad t_i = x_i - m\bar{x} \; \forall i \in \{j+1, j+2, \ldots, n\}
\end{aligned}
$$

Note that a benevolent utilitarian social planner giving equal weights to individual utility functions can be too progressive in some cases of redistributing resources. For example, for all societies consist of social members with homogenous social and egoistic utility functions, the social planner will always choose an equality-of-outcome redistribution scheme.

Example 1. Consider a society consists of $2n$ social members, n rich people with 1 unit of resources, and n poor people with 0 unit of resources. All the social members have identical individual utility functions such that $U_i = \frac{1}{2}\sqrt{m\bar{x}} + \frac{1}{2}\sqrt{c_i}$ for all $i \in N$. Hence $f_i\left(m\bar{x}\right) = \frac{1}{2}\sqrt{m\bar{x}}$ is the social utility function and $u\left(c_i\right) = \frac{1}{2}\sqrt{c_i}$ is the egoistic utility function. By symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t. Then every rich social member will consume $1 - t$ unit of resources after the transfer, while every poor social member will consume t unit of resources after the transfer. The endogenous social minimum under this circumstance will be t and hence $m\bar{x} = t$. The social planner is then maximizing the sum of identical utilities by choosing the optimal transfer t.

$$
\begin{aligned}
&\max_{t} \frac{1}{2}\left(2n\right)\sqrt{t} + \frac{1}{2}n\sqrt{1-t} + \frac{1}{2}n\sqrt{t} \\
&F.O.C. 3t^{-\frac{1}{2}} - \left(1-t\right)^{-\frac{1}{2}} = 0
\end{aligned}
$$

The first order condition suggests that $t^* = 0.9 > 0.5 = \bar{x}$, and hence the social planner will redistribute the initial endowments evenly among all social members. This equality-of-outcome result is true in general for social members with homogeneous individual utility functions, ensured by the monotonically increasing social utility function and concave egoistic utility function. Nevertheless, in other cases when social members have heterogeneous individual utility functions, such interpersonal comparison of utility may result in extremely low social minimum, especially when rich people have much higher marginal utility of consumption from their egoistic utility functions.

5 Dynamics

Since the endogenous social minimum will be determined by both the individual preferences of social members and the initial allocation of disposable endowments in society, different societies with different set of social members (and their individual preferences) and different initial allocation of disposable endowments will have different social minima. The first part of this section analyzes the comparative statics of the endogenous social minimum by the ruling of a benevolent utilitarian social planner. Specifically, it compares two societies with the same set of social members but slightly different initial allocation of disposable endowments, and explores how these differences affect their resulting social minima after redistribution by the social planner. The comparative statics can be directly calculated from the balanced budget condition and first order conditions. Likewise, assume w.l.o.g. that $j = [1 - p(m^*_{CU})]\, n$ is the number of rich people in society, and $1 \leq j \leq n-1$. The comparative statics for the absolute level of social minimum can be derived from $\frac{d(m\bar{x})}{dx_k}\, |_{m=m^*_{CU}} = \frac{m^*_{CU}}{n} + \bar{x}\frac{dm}{dx_k}\, |_{m=m^*_{CU}}$.[7]

$$\frac{d(m\bar{x})}{dx_k}\, |_{m=m^*_{CU}} = \frac{1}{p(m^*_{CU})\, n + \sum\limits_{i=1}^{j} (\Gamma_i + \Phi_i)}$$

in which $\Gamma_i = \frac{\sum_{l=1}^{n} f_l''(m^*_{CU}\bar{x})}{u_i''(x_i - t_i)\, p(m^*_{CU})\, n} > 0$ and $\Phi_i = \frac{\sum_{l=j+1}^{n} u_l''(m^*_{CU}\bar{x})}{u_i''(x_i - t_i)\, p(m^*_{CU})\, n} > 0$. Therefore $0 < \frac{d(m\bar{x})}{dx_k}\, |_{m=m^*_{CU}} < \frac{1}{p(m^*_{CU})\, n}$, and hence $\frac{dm}{dx_k}\, |_{m=m^*_{CU}} < \frac{p(m^*_{CU})(1-m^*_{CU})}{p(m^*_{CU})\, n\bar{x}}$. For now, it seems that such upper bound is a trivial result from concavity of the social utility function (in each of its arguments) and the egoistic utility function, as it translates to $\sum\limits_{i=1}^{j} \frac{dt_i}{dx_k} < 1$ for $k \in \{1, 2, \ldots, j\}$, i.e., the marginal propensity of contribution under a benevolent social planner is smaller than 1. Nevertheless, if it is assumed that for every poor social member their egoistic utility functions are less concave at the point of social minimum than the egoistic utility functions of all rich social members at the point of their corresponding consumption levels, then the upper bound of marginal propensity of contribution will be lower than the proportion of poor people in that society.

Theorem 2. *If for all $i \in N \setminus J$ and all $j \in J$, $|u_i''(m^*_{CU}\bar{x})| \geq |u_j''(x_j - t_j)|$, then $\frac{d(m\bar{x})}{dx_k}\, |_{m=m^*_{CU}} < \frac{1}{n}$ and hence $\sum\limits_{i=1}^{j} \frac{dt_i}{dx_k} < p(m^*_{CU})$ for all $k \in \{1, 2, \ldots, j\}$.*

[7] Full deviation of the comparative statics can be found in section C of the appendix.

Proof. See section B.3 of the appendix.

In some cases the marginal propensity of contribution can be far lower than this upper bound, as the following example shows.

Example 2. Consider a society consists of $2n$ social members, n rich people with 1 unit of resources, and n poor people with 0 unit of resources. All rich social members j have identical individual utility functions such that $U_j = \frac{1}{10}(m\bar{x})^{\frac{1}{4}} + \frac{9}{10}\sqrt{c_j}$ for all $j \in J = \{1, 2, \ldots, n\}$. Hence $f_j(m, \bar{x}) = \frac{1}{10}(m\bar{x})^{\frac{1}{4}}$ is the social utility function and $u_j(c_j) = \frac{9}{10}\sqrt{c_j}$ is the egoistic utility function. All poor social members i have identical individual utility functions such that $U_i = \frac{1}{10}(m\bar{x})^{\frac{1}{4}} + \frac{9}{10}(c_i)^{\frac{1}{4}}$ for all $i \in N \setminus J = \{n+1, n+2, \ldots, 2n\}$. Hence $f_i(m, \bar{x}) = \frac{1}{10}(m\bar{x})^{\frac{1}{4}}$ is the social utility function and $u_i(c_i) = \frac{9}{10}(c_i)^{\frac{1}{4}}$ is the egoistic utility function. By symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t. Then every rich social member will consume $1-t$ unit of resources after the transfer, while every poor social member will consume t unit of resources after the transfer. The endogenous social minimum under this circumstance will be t and hence $m\bar{x} = t$. The social planner is then maximizing the sum of individual utilities by choosing the optimal transfer t.

$$\max_t \frac{1}{10}(2n)\, t^{\frac{1}{4}} + \frac{9}{10} n\sqrt{1-t} + \frac{9}{10} n t^{\frac{1}{4}}$$
$$F.O.C. \frac{11}{40} t^{-\frac{3}{4}} - \frac{9}{20}(1-t)^{-\frac{1}{2}} = 0$$

The first order condition suggests that $t^* \approx 0.3779 < 0.5 = \bar{x}$, and accurate calibrations for A_j, Γ_j, Φ_j, and hence $\frac{d(m\bar{x})}{dx_k}\mid_{m=m^*_{CU}}$ are possible.[8] The calibration results suggest that $\frac{d(m\bar{x})}{dx_k}\mid_{m=m^*_{CU}} \approx 0.2883n^{-1}$, which indicates that even in a society of $2n = 2$ people $\frac{dm}{dx_k}\mid_{m=m^*_{CU}} = \frac{1}{\bar{x}}\left(\frac{d(m\bar{x})}{dx_k}\mid_{m=m^*_{CU}} - \frac{m^*_{CU}}{n}\right) \approx -0.1793 < 0$.

The social minimum coefficient is decreasing in the individual endowment of every social member in Example 2, and hence the endogenous social minimum is decreasing in a relative sense as its ratio to the average social endowment. Example 2 suggests that in some cases a benevolent utilitarian social planner is far less progressive when examining in a relative sense, and the aforementioned upper bound of marginal propensity of contribution can be smaller under some circumstances. Specifically, it is possible for the social minimum coefficient to be decreasing in the endowment of every social member. Note that $\frac{dm}{dx_k}\mid_{m=m^*_{CU}} < 0$ if and only if $\frac{d(m\bar{x})}{dx_k}\mid_{m=m^*_{CU}} < \frac{m^*_{CU}}{n}$. That is,

[8] Exact calibration results are available in section E of the appendix.

$$\frac{1}{p\left(m_{CU}^{*}\right)n+\sum_{i=1}^{j}\left(\Gamma_i+\Phi_i\right)}<\frac{m_{CU}^{*}}{n}$$

If we take the required conditions in Proposition 1 as granted, then $\Phi_i > 1$ for all $i \in \{1, 2, \ldots, j\}$ and hence a sufficient condition for $\frac{dm}{dx_k} \mid_{m=m_{CU}^{*}} < 0$ is

$$\sum_{i=1}^{j}\Gamma_i > \frac{1-m_{CU}^{*}}{m_{CU}^{*}}n$$

By expanding the term Γ_i and multiplying $\frac{1}{\left[1-p\left(m_{CU}^{*}\right)\right]n}$ on both sides, we may rewrite the sufficient condition as

$$\frac{1}{\left[1-p\left(m_{CU}^{*}\right)\right]n}\sum_{i=1}^{j}\frac{f_i''\left(m_{CU}^{*}\bar{x}\right)}{u_i''\left(x_i-t_i\right)} > \frac{\frac{1-m_{CU}^{*}}{m_{CU}^{*}}}{\frac{1-p\left(m_{CU}^{*}\right)}{p\left(m_{CU}^{*}\right)}}$$

The right hand side of the inequality condition above can be regarded as a measure of consumption inequality that will be smaller if the poor social members living below the average social endowment obtain increment in their consumption, which indicates transformation towards a more equality-of-outcome society. Therefore, the redistribution led by a benevolent utilitarian social planner has an intention to fade out when the consumption inequality in society get sufficiently improved.

To probe into the exact conditions that determine such gradual disappearance of redistribution, we consider a particular scenario of economic development in which the inequality in initial endowments will gradually fade out, namely a uniform amount of increase in individual endowments. Given any benchmark society with an arbitrary initial endowment vector $x = (x_1, x_2, \ldots, x_n)$, a uniform amount of increase in endowments will impose a constant increase of $\kappa > 0$ units of resources on the endowment of every social member, and hence add an n-dimensional vector of ones $e = (1, 1, \ldots, 1)$ to the initial endowment vector, resulting in $x + \kappa e = (x_1 + \kappa, x_2 + \kappa, \ldots, x_n + \kappa)$ after the increase. On the top of the conditions required in Theorem 2, if in addition for every rich social member their social utility functions are less concave at the point of social minimum than their egoistic utility functions at the point of their corresponding consumption levels, then the social minimum that can be supported by a benevolent utilitarian social planner will eventually equal the endowment of the poorest social member given sufficiently high uniform amount of increase in the initial endowments.[9] The exact value of $\bar{\kappa}$ can be then interpreted as a threshold beyond which the utilitarian social planner will be satisfied with the inclusive growth brought by the uniform amount of increase in endowments given the egoistic and social utility functions of all the social members and believe no additional redistribution is beneficial for the entire society.

[9]For the formal statement of this claim, see section A.4 of the appendix for Proposition 2.

Example 3. Consider a society consists of $2n$ social members, n rich people with 0.4 unit of resources, and n poor people with 0 unit of resources. All rich social members j have identical individual utility functions such that $U_j = \frac{1}{10}(m\bar{x})^{\frac{1}{4}} + \frac{9}{10}\sqrt{c_j}$ for all $j \in J = \{1, 2, \ldots, n\}$. Hence $f_j(m, \bar{x}) = \frac{1}{10}(m\bar{x})^{\frac{1}{4}}$ is the social utility function and $u_j(c_j) = \frac{9}{10}\sqrt{c_j}$ is the egoistic utility function. All poor social members i have identical individual utility functions such that $U_i = \frac{1}{10}(m\bar{x})^{\frac{1}{4}} + \frac{9}{10}(c_i)^{\frac{1}{4}}$ for all $i \in N \setminus J = \{n+1, n+2, \ldots, 2n\}$. Hence $f_i(m, \bar{x}) = \frac{1}{10}(m\bar{x})^{\frac{1}{4}}$ is the social utility function and $u_i(c_i) = \frac{9}{10}(c_i)^{\frac{1}{4}}$ is the egoistic utility function. By symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t. Then every rich social member will consume $0.4 - t$ unit of resources after the transfer, while every poor social member will consume t unit of resources after the transfer. The endogenous social minimum under this circumstance will be t and hence $m\bar{x} = t$. The social planner is then maximizing the sum of individual utilities by choosing the optimal transfer t.

$$\max_t \frac{1}{10}(2n)\, t^{\frac{1}{4}} + \frac{9}{10} n\sqrt{0.4 - t} + \frac{9}{10} n t^{\frac{1}{4}}$$
$$F.O.C. \frac{11}{40} t^{-\frac{3}{4}} - \frac{9}{20}(0.4 - t)^{-\frac{1}{2}} = 0$$

The first order condition suggests that $t^* \approx 0.1857 < 0.2 = \bar{x}$. Now consider a uniform endowment increase κ on every social member, so now each rich people has $0.4 + \kappa$ unit of resources, and each poor people has κ unit of resources. Again by symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by $t(\kappa)$ as a function of κ. The endogenous social minimum under this circumstance will be $\kappa + t(\kappa)$ and hence $m\bar{x} = \kappa + t(\kappa)$. The social planner is then maximizing the sum of individual utilities by choosing the optimal transfer $t(\kappa)$.

$$\max_t \frac{1}{10}(2n)\left[\kappa + t(\kappa)\right]^{\frac{1}{4}} + \frac{9}{10} n\sqrt{0.4 + \kappa - t(\kappa)} + \frac{9}{10} n\left[\kappa + t(\kappa)\right]^{\frac{1}{4}}$$
$$F.O.C. \frac{11}{40}\left[\kappa + t(\kappa)\right]^{-\frac{3}{4}} - \frac{9}{20}\left[0.4 + \kappa - t(\kappa)\right]^{-\frac{1}{2}} = 0$$

We can find the $\bar{\kappa}$ that will result in a futile social minimum by setting $t(\kappa) = 0$ and hence $\bar{\kappa} \approx 0.4741$.[10] The positive third derivatives on the social and egoistic utility functions ensure that $t(\kappa) < 0$ for all $\kappa > \bar{\kappa}$.

In Theorem 2 and the omitted Proposition 2, the conditions on the second derivatives of the social utility functions and the egoistic utility functions need to be satisfied along the entire path of redistribution, which can be hard to verify. Therefore, we would like to have the conditions related to the first derivatives

[10] For a graphical illustration see Figure 1. The social minimum equals the lowest individual endowment at the intersection of the blue solid line and the red dashed line.

of the social utility functions and the egoistic utility functions so that the conditions are much easier to verify. Starting from an arbitrary initial endowment vector $x = (x_1, \ldots, x_n)$, let $\underline{c}(x) \equiv m(x)\bar{x}$ be the social minimum that will result from the transfer scheme that a utilitarian social planner will choose to maximize the sum of utilities over all social members. Respectively, let J_0 be the set of rich social members with an initial endowment x_j higher than the social minimum $\underline{c}(x)$, and its cardinality $|J_0|$ be the number of such rich social members. We define the Marginal Rate of Contribution (MRC) for every rich social member $j \in J_0$ given any social minimum $\underline{c} \geq \underline{c}(x)$, any own consumption level $c_j \in (\underline{c}(x), x_j]$, and any number of poor social members n_p such has an endowment lower than $\underline{c}$ before receiving any transfer as

$$MRC_j(\underline{c}, c_j, n_p) = \frac{\overbrace{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{l=j+1}^{n} u_l'(\underline{c})}^{\text{total marginal utility gain at social minimum level } \underline{c}}}{\underbrace{u_i'(c_j)\, n_p}_{\text{marginal egoistic utility cost at consumption level } c_j}}$$

In addition, for notation convenience, let $d_j(x) = x_j - x_n$ be the difference between the endowments of rich person j and the poorest person in the society of any given endowment vector $x = (x_1, x_2, \ldots, x_n)$. Observe that for any initial endowment vector $x = (x_1, \ldots, x_n)$ it must be the case that the progressed social minimum $\underline{c}(x) \geq x_n$. Therefore, under sufficiently large amount of uniform increase in endowments, that is, $\kappa \to \infty$, $\underline{c}(x + \kappa e) \to \infty$ as $x_n + \kappa \to \infty$. It turns out whether the society can advance towards an equality-of-outcome society after sufficiently large amount of uniform increase in endowments depends on whether the limit of MRC will be strictly larger than 1. On the contrary, when the limit of MRC is strictly less than 1, even by the ruling of a benevolent utilitarian social planner the endogenous social minimum will eventually not be able to stand at anywhere above the initial endowment of the poorest social member, and hence become paltry.

Specifically, if for all rich people the limit of the ratio between the total marginal utility gain obtained by one's contribution to the society and one's marginal egoistic utility, considering the highest possible difference between their own consumption and the social minimum (which equals the difference between their own endowment and the poorest social member's endowment), is strictly lower than 1, then under a utilitarian social planner all rich people will eventually stop contributing, and hence the social minimum will be lower than the individual endowment of the poorest social member if society is getting sufficiently rich, representing by a uniform amount of increase in individual endowments. Moreover, if a subgroup of rich people can satisfy a set of similar but more strict conditions, then this subgroup of rich people will eventually stop contributing regardless of the contribution from other rich social members.[11]

[11] For the formal statement of these claims, see section B.5 of the appendix for Proposition 3.

On the contrary, if for all rich people the limit of the ratio between the total marginal utility gain obtained by one's contribution to the society and one's marginal egoistic utility, considering the equality of outcome case in which their own consumption equals the social minimum, is strictly higher than 1, then under a utilitarian social planner all rich people will eventually contribute as much as they can, and hence the social minimum will reach the average endowment in society if the society is getting sufficiently rich, representing by a uniform amount of increase in individual endowments. Moreover, if a subgroup of rich people can satisfy a set of similar but more strict conditions, then this subgroup of rich people will eventually contribute as much as they can regardless of the contribution from other rich social members.[12]

One caveat for interpreting the statements above is that they only reveals the sufficient conditions under which the society will eventually become an equality-of-outcome one after a *sufficiently large* amount of uniform increase in individual endowments and the redistribution conducted by a utilitarian social planner. It says nothing on the amount of uniform increase needed, or what development stage the society shall reach, such that the equality-of-outcome result can be realized. Similarly, it also has no control on the evolution of the social minimum that can be supported by a utilitarian social planner along the development path towards such equality-of-outcome end. The following example shows an unfortunate case in which the amount of uniform increase needed to reach an equality-of-outcome society is so large such that the original inequality among social members becomes absolutely meaningless after such advanced level of development, *and* for most time along the development path, the social minimum that can be supported is below the individual endowment of the poorest social member.

Example 4. Consider a society consists of $2n$ social members, n rich people with 1 unit of resources, and n poor people with 0 unit of resources. All rich social members j have identical individual utility functions such that $U_j = \frac{1}{10}\left(m\bar{x}\right)^{\frac{1}{10}} + \frac{9}{10}\left(c_i\right)^{\frac{2}{3}}$ for all $j \in J = \{1, 2, \ldots, n\}$. Hence $f_j\left(m, \bar{x}\right) = \frac{1}{10}\left(m\bar{x}\right)^{\frac{1}{10}}$ is the social utility function and $u_j\left(c_j\right) = \frac{9}{10}\left(c_i\right)^{\frac{2}{3}}$ is the egoistic utility function. All poor social members i have identical individual utility functions such that $U_i = \frac{1}{2}\left(m\bar{x}\right)^{\frac{1}{10}} + \frac{1}{2}\left(c_j\right)^{\frac{7}{10}}$ for all $i \in N \setminus J = \{n+1, n+2, \ldots, 2n\}$. Hence $f_i\left(m, \bar{x}\right) = \frac{1}{2}\left(m\bar{x}\right)^{\frac{1}{4}}$ is the social utility function and $u_i\left(c_i\right) = \frac{1}{2}\left(c_j\right)^{\frac{7}{10}}$ is the egoistic utility function. By symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t. Then every rich social member will consume $1-t$ unit of resources after the transfer, while every poor social member will consume t unit of resources after the transfer. The endogenous social minimum under this circumstance will be t and hence $m\bar{x} = t$. The social planner is then maximizing the sum of individual utilities by choosing the optimal transfer t.

[12]For the formal statement of these claims, see section B.6 of the appendix for Proposition 4.

$$\max_t \left(\frac{1}{10}+\frac{1}{2}\right) nt^{\frac{1}{10}} + \frac{9}{10}n\left(1-t\right)^{\frac{2}{3}} + \frac{1}{2}nt^{\frac{7}{10}}$$
$$F.O.C. \frac{3}{50}t^{-\frac{9}{10}} - \frac{3}{5}\left(1-t\right)^{-\frac{1}{3}} + \frac{7}{20}t^{-\frac{3}{10}} = 0$$

The first order condition suggests that $t^* \approx 0.4518 < 0.5 = \bar{x}$. Now consider a uniform endowment increase κ on every social member, so now each rich people has $1+\kappa$ unit of resources, and each poor people has κ unit of resources. Again by symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by $t(\kappa)$ as a function of κ. The endogenous social minimum under this circumstance will be $\kappa + t(\kappa)$ and hence $m\bar{x} = \kappa + t(\kappa)$. The social planner is then maximizing the sum of individual utilities by choosing the optimal transfer $t(\kappa)$.

$$\max_t \left(\frac{1}{10}+\frac{1}{2}\right) n\left[\kappa + t(\kappa)\right]^{\frac{1}{10}} + \frac{9}{10}n\left[1+\kappa - t(\kappa)\right]^{\frac{2}{3}} + \frac{1}{2}n\left[\kappa + t(\kappa)\right]^{\frac{7}{10}}$$
$$F.O.C. \frac{3}{50}\left[\kappa + t(\kappa)\right]^{-\frac{9}{10}} - \frac{3}{5}\left[1+\kappa - t(\kappa)\right]^{-\frac{1}{3}} + \frac{7}{20}\left[\kappa + t(\kappa)\right]^{-\frac{3}{10}} = 0$$

We can find the $\bar{\kappa}$ that will result in an equality-of-outcome society by setting $t(\kappa) = \bar{x} = 0.5$ and hence $\bar{\kappa} \approx 10528330.81$.[13] The positive third derivatives on the social and egoistic utility functions ensure that $t(\kappa) > 0.5$ for all $\kappa > \bar{\kappa}$.

From the omitted Proposition 3 and Proposition 4 it is not difficult to also derive the sufficient conditions for the endogenous social minimum to never converge to the two extreme cases mentioned above under arbitrarily large uniform amount of increase in endowments, as the following theorem indicates.

Theorem 3. *If $\exists R \subseteq J_0$ such that for all $j \in R$, $\inf_{\underline{c} \geq x_n} MRC_j\left(\underline{c}, \underline{c} + d_j(x), n - |R|\right) > 1$ and $\sup_{\underline{c} \geq x_n} MRC_j\left(\underline{c}, \underline{c}, n-1\right) < 1$, then the social minimum that can be supported by a benevolent utilitarian social planner will always strictly lie between the endowment of the poorest social member and the average endowment in society, whatever uniform amount of increase has been applied to the initial endowments.*

Proof. See section B.7 of the appendix.

That is to say, if for some subgroup of rich people after arbitrarily large uniform amount of increase in endowments, their MRCs have never gone below 1 given the highest possible difference between their

[13] For a graphical illustration see Figure 2. The social minimum converges to the average endowment as the blue solid line moves away from the red dashed line.

own consumption and the social minimum, *and* never gone above 1 given the equality of outcome case, the endogenous social minimum will always be between the extreme positions representing equality of outcome and laissez-faire.

Now we consider the change in rich people contribution under a utilitarian social planner considering a particular scenario of economic development in which the inequality in initial endowments will not fade out but persist, namely a uniform *percentage* of relative increase in individual endowments. Given any benchmark society with an arbitrary initial endowment vector $x = (x_1, x_2, \ldots, x_n)$, a uniform *percentage* of relative increase in endowments will impose a constant percentage increase of $\kappa \times 100\% > 0$ on the endowment of every social member, which will result in an endowment vector $\kappa x = (\kappa x_1, \kappa x_2, \ldots, \kappa x_n)$ after the increase. It turns out that the conditions for the contribution converging to zero is quite similar with the ones we have in the uniform increase case, except that here the highest possible (relative) *percentage* difference between their own endowment and the poorest social member's endowment is going to replace the absolute difference used in denominator before. In order to define such relative difference, let $r_j(x) = \frac{x_j}{x_n}$ be the *percentage* difference between the endowments of rich person j and the poorest person in the society of any given endowment vector $x = (x_1, x_2, \ldots, x_n)$.

Precisely, if for all rich people the limit of the ratio between the total marginal utility gain obtained by one's contribution to the society and one's marginal egoistic utility, considering the highest possible (relative) *percentage* difference their own consumption and the social minimum (which equals the *percentage* difference between their own endowment and the poorest social member's endowment), is strictly lower than 1, then under a utilitarian social planner all rich people will eventually stop contributing, and hence the social minimum will be lower than the individual endowment of the poorest social member if society is getting sufficiently rich, representing by a uniform *percentage* of relative increase in individual endowments. Moreover, if a subgroup of rich people can satisfy a set of similar but more strict conditions, then this subgroup of rich people will eventually stop contributing regardless of the contribution from other rich social members.[14]

If instead for all rich people, the limit of the ratio between the total marginal utility gain obtained by one's contribution to the society and one's marginal egoistic utility, considering the equality of outcome case in which their own consumption equals the social minimum, is strictly higher than 1, then under a utilitarian social planner all rich people will eventually contribute as much as they can, and hence the social minimum will reach the average endowment in society if the society is getting sufficiently rich, representing by a uniform *percentage* of relative increase in individual endowments. Moreover, if a subgroup of rich people can satisfy a set of similar but more strict conditions, then this subgroup of rich people will eventually contribute as

[14] For the formal statement of these claims, see section B.8 of the appendix for Proposition 5.

much as they can regardless of the contribution from other rich social members.[15]

Similarly, from the Proposition 5 and Proposition 6 we can derive the sufficient conditions for the endogenous social minimum to never converge to the two extreme cases mentioned above under arbitrarily large uniform *percentage* of relative increase in endowments, as the following theorem indicates.

Theorem 4. *If* $\exists R \subseteq J_0$ *such that for all* $j \in R$, $\inf_{\underline{c} \geq x_n} MRC_j\left(\underline{c}, r_j(x)\,\underline{c}, n - |R|\right) > 1$ *and* $\sup_{\underline{c} \geq x_n} MRC_j\left(\underline{c}, \underline{c}, n-1\right) < 1$, *then the social minimum that can be supported by a benevolent utilitarian social planner will always strictly lie between the endowment of the poorest social member and the average endowment in society, whatever uniform percentage of relative increase has been applied to the initial endowments.*

Proof. See section B.10 of the appendix.

By contrast to Theorem 3, now it requires a subgroup of rich people after arbitrarily large uniform *percentage* of relative increase in endowments to have their MRCs never below 1 given the highest possible (relative) *percentage* difference between their own consumption and the social minimum, *and* never above 1 given the equality of outcome case, for the endogenous social minimum will always be between the extreme positions representing equality of outcome and laissez-faire.[16]

6 Path Dependence

In the previous chapter we have discovered that even if social members (and their preferences) remain unchanged in a society, any economic development that can be represented by an increase in the initial allocation of disposable endowments will also change the endogenous social minimum. So far, when considering the comparative statics of the endogenous social minimum under a utilitarian social planner, the comparison is between two societies consisting of same set of social members but with different initial allocation of disposable endowments. Therefore, we deliberately exclude the irreversible feature of redistribution and increase in the social minimum: if at a later development stage the social planner cannot support such high amount of transfer from the rich social members to the poor social members, the previously transferred resources cannot be turned back from the poor social members. In addition, it is implicitly assumed that the social planner always gives immediate response to the economic development (represented by an increase in the disposable

[15] For the formal statement of these claims, see section B.9 of the appendix for Proposition 6.

[16] For a preliminary investigation of the mixed scenarios blending a uniform amount of increase in endowments and a uniform *percentage* of relative increase in individual endowments, see section F of the appendix.

endowments of some social members) by adjusting the transfers collected from all rich social members. However, it might not be always possible to continuously adjust the transfer scheme. For example, the income taxes are usually collected (as tax withholding) on a monthly or weekly basis, depending on the payment roll. One important question is whether the social planner, adjusting the transfer scheme under the same economic development pattern but in different frequencies of adjustment, will reach the same endogenous social minimum. The following example indicates a negative answer to such path independence presumption. In general, the endogenous social minimum will be different under various redistribution timetables, and the exact schedule of reallocation may affect the overall consumption inequality in society.[17]

Example 5. Consider a society consists of $2n$ social members, n rich people with 1 unit of resources, and n poor people with 0 unit of resources. All rich social members j have identical individual utility functions such that $U_j = \begin{cases} \frac{5}{12}(m\bar{x})^{\frac{1}{3}} + \frac{7}{12}[\ln(c_j)+1] & 0 < c_j \leq 1 \\ \frac{5}{12}(m\bar{x})^{\frac{1}{3}} + \frac{7}{12}\sqrt{c_j} & c_j > 1 \end{cases}$ for all $j \in J = \{1, 2, \ldots, n\}$. Hence $f_j(m, \bar{x}) = \frac{5}{12}(m\bar{x})^{\frac{1}{3}}$ is the social utility function and $u_j(c_j) = \begin{cases} \frac{7}{12}[\ln(c_j)+1] & 0 < c_j \leq 1 \\ \frac{7}{12}\sqrt{c_j} & c_j > 1 \end{cases}$ is the egoistic utility function. All poor social members i have identical individual utility functions such that $U_i = \frac{1}{3}(m\bar{x})^{\frac{1}{3}} + \frac{2}{3}(c_i)^{\frac{1}{3}}$ for all $i \in N \setminus J = \{n+1, n+2, \ldots, 2n\}$. Hence $f_i(m, \bar{x}) = \frac{1}{3}(m\bar{x})^{\frac{1}{3}}$ is the social utility function and $u_i(c_i) = \frac{2}{3}(c_i)^{\frac{1}{3}}$ is the egoistic utility function. By symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t. Then every rich social member will consume $1-t$ unit of resources after the transfer, while every poor social member will consume t unit of resources after the transfer. The endogenous social minimum under this circumstance will be t and hence $m\bar{x} = t$. Since the consumption of rich people must be less than 1 in this case, we take $U_j = \frac{5}{12}(m\bar{x})^{\frac{1}{3}} + \frac{7}{12}\ln(c_j)$ to represent their individual preferences. The social planner is then maximizing the sum of personal utilities by choosing the optimal transfer t.

$$\max_t \frac{5}{12}(2n)\,t^{\frac{1}{3}} + \frac{7}{12}n\left[\ln(1-t)+1\right] + \frac{7}{12}nt^{\frac{1}{3}}$$
$$F.O.C. \frac{17}{36}t^{-\frac{2}{3}} - \frac{7}{12(1-t)} = 0$$

[17] Given the path-dependent property of the endogenous social minimum, a representative of the rich segment of the population, somewhat similar to the right-wing party described in Roemer (1994), might not need to explicitly express its objection to the benevolent social planner idea, but can instead deliberately choose a long break between two reallocations to alleviate the tax burden on the rich social members. We will leave the discussion on such possible propaganda strategy to future studies.

The first order condition suggests that $t^* \approx 0.3669 < 0.5 = \bar{x}$. Now consider a 10-unit uniform endowment increase on every social member, so now each rich people has 11 units of resources, and each poor people has 10 unit of resources. Again by symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t_{10}. The endogenous social minimum under this circumstance will be $10 + t_{10}$ and hence $m\bar{x} = 10 + t_{10}$. Since the consumption of rich people must be more than 1 in this case, we take $U_j = \frac{5}{12}(m\bar{x})^{\frac{1}{3}} + \frac{7}{12}\sqrt{c_j}$ to represent their individual preferences. The social planner is then maximizing the sum of personal utilities by choosing the optimal transfer t_{10}.

$$\max_t \frac{5}{12}(2n)(10 + t_{10})^{\frac{1}{3}} + \frac{7}{12}n\sqrt{11 - t} + \frac{7}{12}n(10 + t_{10})^{\frac{1}{3}}$$
$$F.O.C. \frac{17}{36}(10 + t_{10})^{-\frac{2}{3}} - \frac{7}{24}(11 - t_{10})^{-\frac{1}{2}} = 0$$

The first order condition suggests that $t_{10} \approx 1.3124 > 0.5$. Since the social planner cannot transfer more resources from the rich people when they actually reach the social minimum, $t_{10} = 0.5$ and hence all social members consume 10.5 units of resources in this case. An immediate conclusion is that if after the redistribution there is another 10-unit uniform endowment increase on every social member, then each social member will consume 20.5 units of resources after increase and the society remains an equality-of-outcome society. However, if instead it is a 20-unit uniform endowment increase directly being imposed on every social member, then each rich people has 21 units of resources before redistribution, and each poor people has 20 unit of resources. Similarly by symmetry the utilitarian social planner will transfer the same unit of resources from each rich social member to a poor social member, which can be denoted by t_{20}. The endogenous social minimum under this circumstance will be $10 + t_{20}$ and hence $m\bar{x} = 10 + t_{20}$. Since the consumption of rich people must be more than 1 in this case, we take $U_j = \frac{5}{12}(m\bar{x})^{\frac{1}{3}} + \frac{7}{12}\sqrt{c_j}$ to represent their individual preferences. The social planner is then maximizing the sum of personal utilities by choosing the optimal transfer t_{20}.

$$\max_t \frac{5}{12}(2n)(20 + t_{20})^{\frac{1}{3}} + \frac{7}{12}n\sqrt{21 - t} + \frac{7}{12}n(20 + t_{20})^{\frac{1}{3}}$$
$$F.O.C. \frac{17}{36}(20 + t_{20})^{-\frac{2}{3}} - \frac{7}{24}(21 - t_{20})^{-\frac{1}{2}} = 0$$

The first order condition suggests that $t_{20} \approx 0.1216 < 0.5$, so after the redistribution by the social planner the society will not become an equality-of-outcome one if the 20-unit uniform increase comes all at once.

The redistribution result is path-dependent in this example.[18]

Moreover, even if some or all of the rich people will eventually stop contributing, it does not necessarily mean that the social planner can do nothing in minimizing social inequality in individual endowments. In the diminishing consumption inequality scenario that economic development is represented by a uniform amount of increase in individual endowments, the discrepancy between rich people and poor people in their individual endowments after redistribution will never enlarge thereafter, and hence the social planner can choose a particular development stage for an additional round of redistribution such that, after the redistribution, the social inequality in individual endowments is decreased to the minimum possible level. In order to find the optimal development stage in which the rich people have the highest incentive to contribute, we need to define the resources that can be collected from the rich people by a utilitarian social planner, henceforth contribution potential, under various levels of social minimum. For any initial social endowment vector $x = (x_1, x_2, \ldots, x_n)$ such that x_i is the individual endowment of social member i, let $\rho_j(\underline{c}, x)$ be the contribution potential of rich social member $j \in J_0$ under any given social minimum $\underline{c} \geq x_n$, and J_0 be the set of rich social members with initial endowment x_j higher than the given social minimum $\underline{c}$. Define the total contribution potential $T(\underline{c}, x) \equiv \sum_{j \in J} \rho_j(\underline{c}, x)$, then given the objective function of the utilitarian social planner,

$$\rho_j(\underline{c}, x) = \max\left\{\min\left\{\sup\left\{t \colon MRC_j\left(\underline{c}, \underline{c} + d_j(x) - t, n - |R(\underline{c}, x)|\right) \geq 1\right\}, d_j(x)\right\}, 0\right\}$$

in which $MRC_j\left(\underline{c}, \underline{c} + d_j(x) - t, n - |R(\underline{c}, x)|\right) = \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i \in N \setminus R} u_i'(\underline{c})}{u_j'(\underline{c} + d_j(x) - t)(n - |R(\underline{c}, x)|)}$ is the marginal rate of contribution for every rich social member $j \in J_0$ given any social minimum $\underline{c} \geq x_n$, and $R(\underline{c}, x) = \{j \in J_0 \colon \rho_j(\underline{c}, x) < d_j(x)\} \subseteq J$ is the set of rich people given certain social minimum $\underline{c}$. Since the total contribution potential is the maximum resources a utilitarian social planner can collect from the rich people if the social minimum reaches $\underline{c}$, the social planner will choose the highest possible social minimum $\underline{c}^*(x)$ such that the total contribution potential is no less than the resources needed to lift up poor social members to $\underline{c}^*(x)$

$$\underline{c}^*(x) = \max_{\underline{c}} \left\{\underline{c} : T(\underline{c}, x) \geq \sum_{i=1}^{n} \max\{\underline{c} - x_i, 0\}\right\}$$

To pin down the optimal development stage for an additional round of redistribution, the social planner will search for the uniform amount increase κ on everyone's endowment such that the total contribution potential given the highest possible social minimum under the initial endowments after such increase, $T(\underline{c}^*(x + \kappa e), x + \kappa e)$, is maximized

[18] For a graphical illustration see Figure 3. The redistribution result is path-dependent as the blue solid line is concave.

$$\kappa^*(x) = \arg\max_{\kappa} T\left(\underline{c}^*(x+\kappa e), x+\kappa e\right)$$

Once $\kappa^*(x)$ is found, the social planner can redistribute the resources when the development stage of the society has reached $x + \kappa^*(x)\, e$ to minimize the social inequality in individual consumptions. Under uniform amount of increase in endowments, the gap between social members in their initial endowments will remain the same without further redistribution. Therefore, once the redistribution is chosen at a development stage that maximize the total contribution from the rich people, the inequality in initial endowments is also minimized universally from all possible transfer schemes a benevolent utilitarian social planner can advocate. Therefore, in the diminishing consumption inequality scenario, the redistribution at most needs to be implemented once by the social planner.

A similar approach can also be processed in the constant consumption inequality scenario that economic development is represented by a uniform *percentage* of relative increase in individual endowments, where the discrepancy between rich people and poor people in their individual endowments will enlarge along the development path but the ratio will keep constant. Following the exact same steps, the social planner can also find the highest possible social minimum $\underline{c}^*(x)$ such that the total contribution potential is no less than the resources needed to lift up poor social members to $\underline{c}^*(x)$ for any given initial endowment vector $x = (x_1, x_2, \ldots, x_n)$, as well as the corresponding total contribution potential $T\left(\underline{c}^*(x), x\right) \equiv \sum_{j\in J} \rho_j\left(\underline{c}, x\right)$. The only difference here is that any 1 unit of resource transfer from the rich people to the poor people implemented at endowments x (initial development stage) will worth κ units at a later development stage with endowments κx, $\kappa > 1$. Therefore, the social planner will search for the uniform *percentage* increase κ on everyone's endowment such that the total contribution potential divided by κ, given the highest possible social minimum under the initial endowments after such increase, $\frac{T(\underline{c}^*(\kappa x), \kappa x)}{\kappa}$, is maximized

$$\kappa^*(x) = \arg\max_{\kappa} \frac{T\left(\underline{c}^*(\kappa x), \kappa x\right)}{\kappa}$$

Once $\kappa^*(x)$ is found, the social planner can redistribute the resources when the development stage of the society has reached $\kappa^*(x)\, x$ to minimize the social inequality in individual consumptions. Under uniform *percentage* of increase in endowments, the ratio between social members in their initial endowments will remain the same without further redistribution. Therefore, once the redistribution is chosen at a development stage that maximize the total contribution from the rich people as a proportion of total endowments $\kappa\bar{x}$, the inequality in initial endowments is also minimized universally from all possible transfer schemes a benevolent utilitarian social planner can advocate. Therefore, in the constant consumption inequality scenario, the redistribution also at most needs to be implemented once by the social planner.

7 Voluntary Contribution

We now discuss the voluntary contribution case, where there is no social planner determining $t_i(m, x)$, and each social member i simultaneously decides on their amount of contribution to the social support system, t_i. Without any commitment on matching the contribution from other social members, social member i considers how an additional contribution will increase some given social minimum m^* resulting from the contribution of others. Therefore, the optimization problem for social member i is

$$\begin{aligned} &\max_{t_i} U_i(m^*, x_i, t_i) = f_i(m(m^*, t_i)\bar{x}) + u_i(x_i - t_i) \\ &s.t.\ 0 \le t_i \le \max\{x_i - m(m^*, t_i)\bar{x}, 0\} \end{aligned}$$

in which $m(m^*, t_i) = m^* + \frac{t_i}{p(m)n\bar{x}}$ is the social minimum after social member i contributes an additional t_i to the social support system assuming that no other social member will contribute to the social support system. Hence the additional t_i contribution will be equally shared among all social members that are at current social minimum level $m^*\bar{x}$, and will end up with a $\frac{t_i}{p(m)n}$ increase in their endowments, hence a $\frac{t_i}{p(m)n\bar{x}}\bar{x}$ increase in the social minimum. Assume that $p(m) = p(m^*)$, i.e., the additional contribution t_i will not increase the social minimum so much that more social members are below the new social minimum $m(m^*, t_i)$. The first order condition is

$$\frac{dU_i(m^*, x_i, t_i)}{dt_i} = f_i'(m(m^*, t_i)\bar{x})\frac{1}{p(m^*)n} - u_i'(x_i - t_i) = 0$$

The second order condition in this case is $\frac{d^2U_i}{dt_i^2} = f_i''(m(m^*, t_i)\bar{x})\left(\frac{1}{p(m^*)n}\right)^2 + u_i''(x_i - t_i) < 0$ as $f_i''(m(m^*, t_i)\bar{x}) \le 0$ and $u_i''(x_i - t_i) < 0$. From the first order condition using the implicit function technique $\frac{d}{dm^*}\left[f_i'(m(m^*, t_i)\bar{x})\frac{1}{p(m^*)n} - u_i'(x_i - t_i)\right] = 0$ and thus

$$\frac{dt_i}{dm^*} = -\frac{\bar{x}}{p(m^*)n}\frac{f_i''(m(m^*, t_i)\bar{x})}{u_i''(x_i - t_i)}$$

Note that $\frac{dt_i}{dm^*}|_{m^*=m^*_{DE}} \le 0$ for all $i \in N$ as $\frac{\partial^2 f_i}{\partial m^2} \le 0$ and $u_i''(x_i - t_i) < 0$. Let t_i^* be the value that satisfies the first order condition $f_i'(m(m^*, t_i^*)\bar{x})\frac{1}{p(m^*)n} - u_i'(x_i - t_i^*) = 0$, and $t_i(m^*) \equiv \max\{0, t_i^*\}$ be the optimal voluntary contribution of social member i. The social minimum at equilibrium $m^*_{DE} = m^* + \frac{\sum_{i=1}^{j} t_i(m^*)}{p(m^*_{DE})n\bar{x}}$ should satisfy the balanced budget condition

$$\sum_{i=1}^{j(m^*_{DE})} t_i(m^*) = \sum_{i=j(m^*_{DE})+1}^{n} (m^*_{DE}\bar{x} - x_i)$$

Such equilibrium social minimum coefficient m^*_{DE} exists and is unique due to the aforementioned first-order and second-order features. This is *not* a new result. The model configuration under volunteer contribution resembles the one used in the classical theory of public goods, and hence the results in Bergstrom et al. (1986) also apply to our model. Furthermore, the level of provision of the public good here (social minimum) can be desperately insufficient in a large society.[19] Although the social minimum here does not exactly fit into the definition of a congestible public good, the relationship between the total contribution to the social support system and the society size falls in the same line of reasoning with Chamberlin (1974). Consider expanding a given society by duplicating everyone in the original society once, so that the number of social member will become $2n$, while the distribution of initial allocation of disposable endowments unchanged. Let m^*_{DE} be the equilibrium social minimum coefficient in the original society, and $t_i(m^*_{DE})$ be the voluntary contribution (if positive) or incoming transfer (if negative) of social member i in the original society. From the same distribution of initial allocation of disposable endowments, we know that $p(m^*_{DE})$ and $\bar{x}$ will stay unchanged. Observe that the original first order condition

$$f_i'(m^*_{DE}\bar{x}) = u_i'(x_i - t_i(m^*_{DE}))\, p(m^*_{DE})\, n$$

can no longer hold in the expanded society at the initial level of m^*_{DE} and $t_i(m^*_{DE})$ since increasing n to $2n$ doubles the right hand side, while the left hand side remains unchanged, and both sides are strictly positive. According to the balanced budget condition m^*_{DE} is increasing in $t_i(m^*_{DE})$ for all $i \in N$, and hence both the voluntary contribution $t_j(m^*_{DE})$ from rich people $j \in J(m^*_{DE})$ and social minimum coefficient at equilibrium m^*_{DE} will decrease in responding to such expansion of society. The corresponding social minimum at equilibrium $m^*_{DE}\bar{x}$ will also decrease as $\bar{x}$ will stay unchanged. Specifically, when such expansion goes to its extreme that $n \to \infty$, it will be the case that $m^*_{DE}\bar{x} \to \min\{x\}$, and $\sum_{i=1}^{j(m^*_{DE})} t_i(m^*_{DE}) \to 0$. This predicts that in a sufficiently large society generated by self-replicating the original set of social members, the total voluntary contribution will be arbitrarily close to zero, and hence the social minimum at equilibrium converges to the minimum endowment, resulting in the poorest social member not receiving any positive transfer.[20] Such a low social minimum at equilibrium indicates that, at least in a sufficiently large society,

[19] It is worth mentioning that in a relatively small society, the incentive of voluntary contribution can be sufficient. In particular, in the smallest possible society of two social members, the characterized individual preference will result in transfers from the rich social member to the poor social member equivalent to the transfers that will come from an additive separable version of the social interaction model proposed by Becker (1974).

[20] The equilibrium result here is, not surprisingly, worse than the one predicted in pure public good model (Andreoni, 1988), where the cost of public goods is instead increasing in its level of provision, and hence only the average voluntary contribution converges to zero in a large economy, while the total donation, albeit converging, still increases. The average voluntary contribution can converge to a strictly positive constant if we assume that rich people can directly obtain utilities from additional contribution, as the warm-glow giving proposed by Andreoni (1989, 1990), but that is beyond our characterization of individual preferences.

voluntary contribution may not be an effective method for funding a social support system. [21]

8 Conclusion

In this paper, we have characterized a class of individual preferences that can be represented by a linear combination of a social utility function and a egoistic utility function. Surprisingly, other than the standard monotonicity axioms, the only axiom that has an unusual interpretation on individual preferences is the one regulates that a social member will be indifferent on the consumption distribution of others in society if that particular social member is the poorest person, defined as the person who enjoys the lowest consumption, in society. Such axiom with an egoistic interpretation on one's well-being leads to an equal attentiveness to the lowest consumption in the society even when that particular social member is no longer the poorest person in the society. The equal attentiveness to the lowest consumption then enters one's individual preference and later becomes the incentive to voluntarily contribute to a social fund that aims at increase the minimum consumption in the society, or the social minimum. Such social minimum at equilibrium is no longer a exogenously-given, constant level of consumption intended to be some benchmark of decent life, but an endogenous one established from the individual preferences of social members and initial endowment allocations of the society. Thus, the endogenous social minima in two societies with different sets of social members (individual preferences) and/or different endowment allocations will generally not be identical.

While the endogenous social minimum supported under voluntary contribution converges to zero as the society continuously duplicates itself, with the assistance of a benevolent utilitarian social planner, the same level of social minimum can be achieved in a larger society with same endowment distribution. Still, the relative social minimum, as a fraction of the average endowment, does not always increase with individual endowments. If the marginal utilities of the poor social members decreases faster at the social minimum than those of the rich social members at their current consumption level, the relative social minimum may also decrease. This paper then contrasts conditions under which, by the ruling of a benevolent utilitarian social planner, the social minimum converges to the average endowment versus when it converges to the lowest individual endowment. When the limits of marginal rate of contribution for all social members are not too high given a (hypothetically) small group of rich people and not too low given a large group of rich people along the development path, the social minimum shall stay between these two extremes. The redistribution outcomes are path-dependent: multiple increases in endowments of smaller amounts lead to

[21] We choose not to study the comparative statics for the voluntary contribution case because there is no explicit solution for the endogenous social minimum. Still, the first order condition under the voluntary contribution case can quickly lead to an observation that the resulted social minimum is weakly increasing in the disposable endowment of any social member $i \in N$.

higher total transfers between the rich and the poor than a single large redistribution. Even in a scenario where the social minimum converges to the lowest endowment, the social planner can optimize transfers by choosing the right development stage for redistribution, maximizing transfers from the rich to the poor and reducing consumption inequality.

There is certainly abundant room for further exploration on the endogenous social minimum that can be supported in a given society. As a starting point, weakening the current axioms will certainly improve the unobjectionableness of the axioms, and hence is always a natural extension available for following studies. In addition, it is reasonable to speculate that, with some proper modification on the threshold below which we no longer have complete separability between one's own consumption and the consumptions of others, a similar set of axioms can lead to a class of individual preferences with analogous structure in which the social utility function will be determined by some other measures. One potential contribution could be new characterizations of individual preferences with a social utility component related to consumption inequality, similar to the one used in Fehr and Schmidt (1999), as the motivation of axioms shall be quite different from existing ones in Neilson (2006) and Rohde (2010). In general, the behavioural fundamentals for the social members to focus on the minimum consumption, or some other possible measures or references, are unquestionably worth studying. We will leave that to future research.

Second, although it has been shown in this paper that the social minimum supported under voluntary contribution can be in general very low in a sufficiently large society, we might still be able to observe meaningful social minimum in a relatively small society in which the social members might not have intimate relationships (e.g., consortium, exclusive private club, parliamentary group, etc.), given that the current set of axioms does not explicitly require the social members to have much kindness to others. Investigating whether there exists some non-reciprocal actions among these social members that ensure some sort of minimum treatment for the most miserable members in such society can be very interesting, and the social minimum here can also be generalized to some other measure besides the level of consumption.

Last but not the least, the social utility function characterized in this paper is fully determined by the social minimum. When a social member becomes richer, the incentive of contributing more (under a fixed social minimum) is rooted in the diminishing marginal utility in one's own consumption. Recent experimental work, such as Andreoni et al. (2021), provides evidence supporting that high socioeconomic status people may have equal or higher incentive to conduct prosocial behaviour, controlling the diminishing marginal utility in income. Future work that incorporates wealth effect in one's social utility function in accordance with the results from experimental literature can be very promising.

Figures

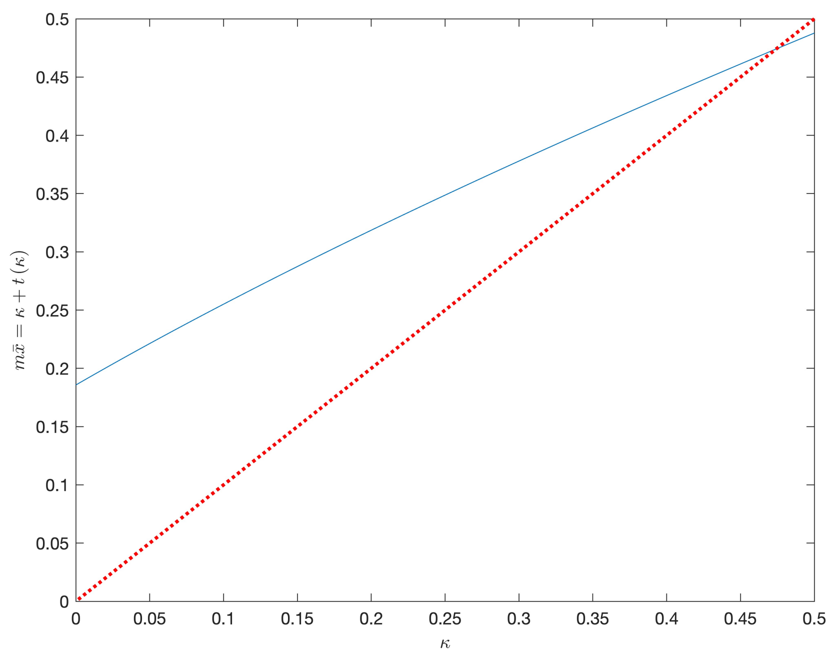

Figure 1: Social Minimum under Uniform Endowment Increase

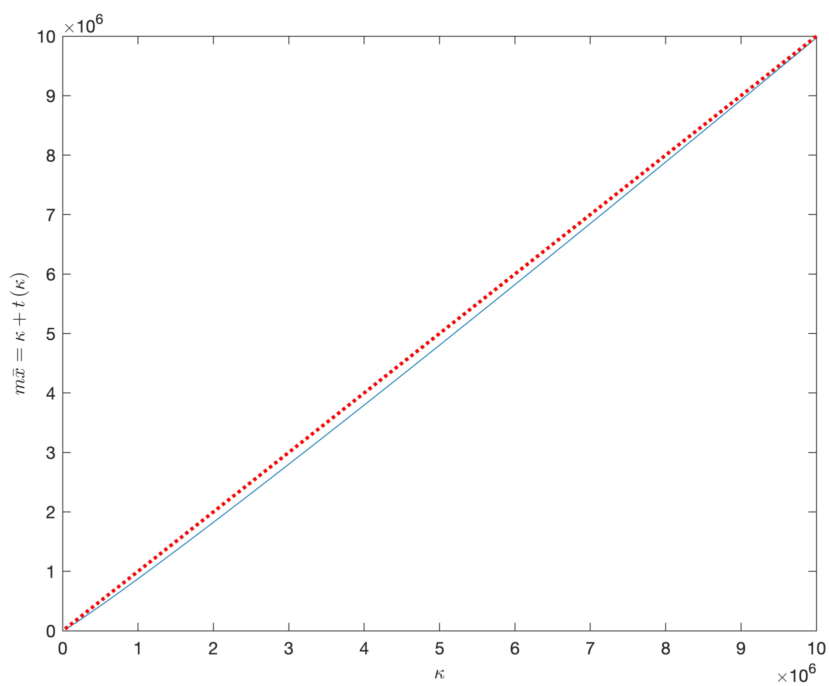

Figure 2: Social Minimum under Uniform Endowment Increase

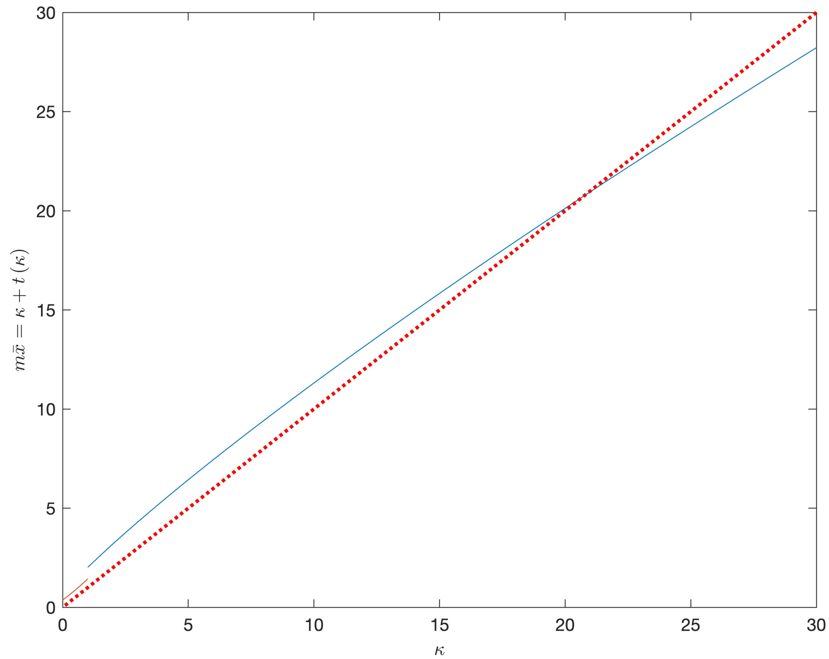

Figure 3: Path-dependent Social Minimum under Uniform Endowment Increase

Appendix

A. Incentive to Target on the Poorest People

Without any further assumptions, it might be very difficult to confine the the destination of one's voluntary contribution (or obligated contribution if there is a social planner) to being equally shared among the poorest social members. Here a possible set of additional assumptions is provided to rationalize such determination in the social utility.

Let $\mathbb{R}_+^n$ be the the nonnegative orthant of Euclidian space. Assume that the social members actually care about some poverty index $p(x; z) : \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}$ determined by the endowment vector $x = (x_1, x_2, \ldots, x_n)$ and a poverty line z, then their social utility should be positively correlated to a function $f_i(p(x; z))$ that is strictly decreasing in $p(x; z)$. Following the definition of Zheng (1994), further assume that $p(x; z)$ is constant in the permutation of the initial endowments (*symmetry*), irrelevant to the endowment level of all $x_i > z$ (*focus*), and continuous in each x_i on $[0, z]$ (*restricted continuity*), and, in accordance to the Proposition 2 in that paper, if further assume that $p(x; z)$ satisfies the transfer-sensitivity axiom, i.e., the increase in the poverty level $p(x; z)$ resulting from a transfer from a poor social member i such that $x_i \leq z$ to a richer poor social member is strictly decreasing in x_i, then (i) $p(x; z)$ must be either absolute (unchanged by a uniform addition on everyone's endowment and the poverty line) or relative (unchanged by a uniform multiplication on everyone's endowment and the poverty line), and (ii) the voluntary or obligated contribution from the rich social members must be equally shared among the poorest social members because it maximizes everyone's social utility compared to any other possible distribution methods. Note that the second property indicates that the idea of establishing a social minimum (and everyone below it should be lifted up to it) is supported by every social member.

Let $p(x; z)$ be relative as it is the prevalent way of measuring poverty in developed countries (Ravallion and Chen, 2011). In addition, let $\bar{x'} = \frac{1}{n} \sum\limits_{i=1}^{n} x'_i$ and similarly $\bar{x} = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$, and extend the measure of poverty to scenarios in which every social member is above the poverty line by assuming that $p(x; z) = p(x; \min\{x\})$ for all $z \leq \min\{x\}$, $p(x; \min\{x\}) < p(x; z')$ for all $z' > \min\{x\}$, and $p(x'; \min\{x'\}) < p(x; \min\{x\})$ for all $\min\{x'\} \left(\bar{x'}\right)^{-1} < \min\{x\} (\bar{x})^{-1}$. With these additional assumptions, the function $f_i(p(x; z))$ can be represented by a function $f_i(m)$ increasing in m, the social minimum coefficient either resulting from voluntary contribution or as a policy being chosen by the benevolent social planner. Given proper characterization, it is possible to land the individual preference of each social member on their tradeoffs between the relative social minimum and the absolute prosperity of the entire society, represented by the production of a function related to both the social minimum coefficient and the average endowment in society, e.g., $U_i = f_i(m, \bar{x})$, which we will leave to future research.

B. Omitted Proofs

B.1. Lemma 1

We prove Lemma 1 through the following proposition:

Proposition 1. *If a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies the axioms CSSM and WMOM, then for all (c_{-i}, c_i), $\left(c'_{-i}, c_i\right)$ such that $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$.*

Proof. We would like to first show that if a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies axioms CSSM and WMOM, then for all (c_{-i}, c_i), $\left(c'_{-i}, c_i\right)$ such that $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$. Assume that $\exists c_{-i}, c'_{-i}$ such that $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$ but $(c_{-i}, c_i) \prec_i \left(c'_{-i}, c_i\right)$ for all $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, then by axiom WMOM $\exists \delta > 0$ such that $(c_{-i}, c_i) \sim_i \left(c'_{-i} - \delta \mathbf{e}^{n-1}, c_i\right)$ for all $c_i > \min\left\{c_{-i}, c'_{-i} - \delta \mathbf{e}^{n-1}\right\}$, where $\mathbf{e}^{n-1}$ is an $n-1$-dimensional vector of ones. Since $\min\left\{c_{-i}, c'_{-i} - \delta \mathbf{e}^{n-1}\right\} < \min\left\{c'_{-i}\right\} \leq \min\{c_{-i}\}$, $\exists c_i^* = \min\left\{c'_{-i}\right\} - \frac{1}{3}\delta$ such that $(c_{-i}, c_i^*) \sim_i \left(c'_{-i}, c_i^*\right)$ and $(c_{-i}, c_i^*) \sim_i \left(c'_{-i} - \delta \mathbf{e}^{n-1}, c_i^*\right)$ by axiom CSSM. Again by axiom WMOM, $(c_{-i}, c_i^*) \prec_i \left(c'_{-i} - \frac{1}{2}\delta \mathbf{e}^{n-1}, c_i^*\right)$. A contradiction.

We would like to then show the only if part in the second clause. Assume that $\exists c_{-i}, c'_{-i}$ such that $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ but $\min\{c_{-i}\} < \min\left\{c'_{-i}\right\}$ for all $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, then by axiom WMOM for $\theta = \min\left\{c'_{-i}\right\} - \min\{c_{-i}\} > 0$ it must be the case that $(c_{-i}, c_i) \succ_i \left(c'_{-i} - \theta \mathbf{e}^{n-1}, c_i\right)$ for all $c_i > \min\left\{c_{-i}, c'_{-i} - \theta \mathbf{e}^{n-1}\right\}$, where $\mathbf{e}^{n-1}$ is an $n-1$-dimensional vector of ones. Similarly by axiom WMOM $\exists \sigma > 0$ such that $\left(c_{-i} - \sigma \mathbf{e}^{n-1}, c_i\right) \sim_i \left(c'_{-i} - \theta \mathbf{e}^{n-1}, c_i\right)$ for all $c_i > \min\left\{c_{-i} - \sigma \mathbf{e}^{n-1}, c'_{-i} - \theta \mathbf{e}^{n-1}\right\}$, where $\min\left\{c_{-i} - \sigma \mathbf{e}^{n-1}\right\} < \min\left\{c'_{-i} - \theta \mathbf{e}^{n-1}\right\} = \min\{c_{-i}\} < \min\left\{c'_{-i}\right\}$, $\exists c_i^* = \min\left\{c'_{-i} - \theta \mathbf{e}^{n-1}\right\} - \frac{1}{3}\sigma$ such that $(c_{-i}, c_i^*) \sim_i \left(c'_{-i} - \theta \mathbf{e}^{n-1}, c_i^*\right)$ and $\left(c_{-i} - \sigma \mathbf{e}^{n-1}, c_i^*\right) \sim_i \left(c'_{-i} - \theta \mathbf{e}^{n-1}, c_i^*\right)$ by axiom CSSM. Again by axiom WMOM,
$\left(c_{-i} - \frac{1}{2}\sigma \mathbf{e}^{n-1}, c_i^*\right) \succ_i \left(c'_{-i} - \theta \mathbf{e}^{n-1}, c_i^*\right)$. Also a contradiction.

Therefore, if a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies the axioms CSSM and WMOM, then for all (c_{-i}, c_i), $\left(c'_{-i}, c_i\right)$ such that $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$. ■

We then prove Lemma 1:

Proof. From Proposition 1 we know that if a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies the axioms CSSM and WMOM, then for all (c_{-i}, c_i), $\left(c'_{-i}, c_i\right)$ such that $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$. Observe that when $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, if in addition $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$, then it must be the case that $c_i > \min\left\{c'_{-i}\right\} = \min\left\{c'_{-i}, c_i\right\}$ and $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}\right\}$, and hence $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}, c_i\right\}$. Similarly, if $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$ and $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}, c_i\right\}$, then it must be the case that $c_i > \min\left\{c'_{-i}\right\} = \min\left\{c'_{-i}, c_i\right\}$ and $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}\right\}$, and hence $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$. Therefore, if $c_i > \min\left\{c_{-i}, c'_{-i}\right\}$, then the condition $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$ is equivalent to the condition $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}, c_i\right\}$.

Now we consider the cases in which $c_i \leq \min\left\{c_{-i}, c'_{-i}\right\}$. By the axiom CSSM if $c_i \leq \min\left\{c_{-i}, c'_{-i}\right\}$ then $(c_{-i}, c_i) \sim_i \left(c'_{-i}, c_i\right)$ for any c_{-i}, c'_{-i}. In addition, $\min\{c_{-i}, c_i\} = \min\left\{c'_{-i}, c_i\right\}$ if $c_i \leq \min\left\{c_{-i}, c'_{-i}\right\}$. Hence the statement $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}, c_i\right\}$ is trivially true.

Therefore, if a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c satisfies the axioms CSSM and WMOM, then for all (c_{-i}, c_i) and $\left(c'_{-i}, c_i\right)$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if $\min\{c_{-i}, c_i\} \geq \min\left\{c'_{-i}, c_i\right\}$. ■

B.2. Theorem 1

Proof. We first prove the sufficiency of the axioms. Pick an arbitrary constant $c^* \in \mathbb{R}_+$ and an arbitrary element $j \in N$ such that $j \neq i$, consider the case in which $\mathcal{C}_j = [0, c^*]$, $\mathcal{C}_k = [c^*, \infty)$ for all $k \in N$ such that $k \neq j$, and let $\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$ be endowed with the product topology. Then $\succeq_i$ can be regarded as a binary relation on $\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$, where $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ are connected topological spaces. Since $\succeq_i$ is complete, continuous, and transitive, it is a continuous weak order. By the configuration of the topological spaces and Lemma 1, $c_i \geq c_j = \min\{c\}$, and for all $c_k \in \mathcal{C}_k$ and $c'_k \in \mathcal{C}_k$ such that $k \neq i$ and $k \neq j$, $(c_{-k}, c_k) \sim_i (c_{-k}, c'_k)$. Since $\mathcal{C}_i, \mathcal{C}_j$ are the only two essential coordinates, let $\mathcal{C}_i \times \mathcal{C}_j$ be endowed with the product topology and construct an auxiliary binary relation $\succeq_i^j$ on $\mathcal{C}_i \times \mathcal{C}_j$ such that $(c_i, c_j) \succeq_i^j \left(c'_i, c'_j\right)$ if and only if $(c_{-i-j}, c_i, c_j) \succeq_i \left(c_{-i-j}, c'_i, c'_j\right)$. Naturally $\succeq_i^j$ is also a continuous weak order. Given these results, axiom TC indicates that if $(c_i, c_j) \preceq_i^j \left(c'_i, c'_j\right)$, $\left(c_i, c''_j\right) \succeq_i^j \left(c'_i, c'''_j\right)$, and $\left(c''_i, c_j\right) \succeq_i^j \left(c'''_i, c'_j\right)$, then $\left(c''_i, c''_j\right) \succeq_i \left(c'''_i, c'''_j\right)$. In accordance with Theorem 4.4 in Wakker (1988), there exists a continuous additive representation (up to a positive affine transformation) for $\succeq_i^j$. Again from the configuration of the topological spaces and Lemma 1, since $c_i \geq c_j = \min\{c\}$, for all (c_{-i}, c_i) and $\left(c'_{-i}, c_i\right)$, $(c_{-i}, c_i) \succeq_i \left(c'_{-i}, c_i\right)$ if and only if $\min\{c_{-i}\} \geq \min\left\{c'_{-i}\right\}$. Therefore, in each specific configuration $\succeq_i^j$ must be invariant throughout all $j \in N$ such that $j \neq i$, and hence $\succeq_i$ can be represented by a continuous utility function $U_i(c) = f_i(c_j) + u_i(c_i) = f_i(\min\{c\}) + u_i(c_i)$ for all the cases when $c_i \geq \min\{c\}$, as c^* and j are arbitrarily chosen. By axiom CSSM, for all the cases when $c_i < \min\{c_{-i}\}$, $(c_{-i}, c_i) \sim_i \left(c'_{-i}, c_i\right)$ where $c'_j = c_i$ for all $j \neq i$, and hence the same representative utility function $U_i(c) = f_i(\min\{c\}) + u_i(c_i)$ also apply to all the cases in which $c_i < \min\{c_{-i}\}$. By axioms

WMOM and MS, both f_i and u_i are strictly increasing in their arguments.

We then prove the necessity of the axioms. It is trivial that a complete, continuous, and transitive individual preference $\succeq_i$ on the consumption vector c that can be represented by a individual utility function of functional form $U_i(c) = f_i(\min\{c\}) + u_i(c_i)$ that both f and u are strictly increasing in their arguments satisfies the axioms CSSM, WMOM, and MS. We now prove that it also satisfies the axiom TC. If $(c_{-i}, c_i) \preceq_i (c'_{-i}, c'_i)$, $c_i \geq \min\{c_{-i}\}$, $(c''_{-i}, c_i) \succeq_i (c'''_{-i}, c'_i)$, $c'_i \geq \min\{c'''_{-i}\}$, and $(c_{-i}, c''_i) \succeq_i (c'_{-i}, c'''_i)$, $c'''_i \geq \min\{c'_{-i}\}$, then $f_i(\min\{c_{-i}\}) + u_i(c_i) \leq f_i(\min\{(c'_{-i}, c'_i)\}) + u_i(c'_i)$, $f_i(\min\{(c''_{-i}, c_i)\}) + u_i(c_i) \geq f_i(\min\{c'''_{-i}\}) + u_i(c'_i)$, and $f_i(\min\{(c_{-i}, c''_i)\}) + u_i(c''_i) \geq f_i(\min\{c'_{-i}\}) + u_i(c'''_i)$. Since both f is strictly increasing in its argument, $f_i(\min\{(c''_{-i}, c_i)\}) + u_i(c''_i) \geq f_i(\min\{c'''_{-i}\}) + u_i(c'''_i)$ as $f_i(\min\{(c'_{-i}, c'_i)\}) \leq f_i(\min\{c'_{-i}\})$ and $f_i(\min\{(c_{-i}, c''_i)\}) \leq f_i(\min\{c_{-i}\})$. In addition, for all $c''_i \geq \min\{c''_{-i}\}$, we have $f_i(\min\{(c''_{-i}, c''_i)\}) = f_i(\min\{c''_{-i}\}) \geq f_i(\min\{(c''_{-i}, c_i)\})$, so $f_i(\min\{(c''_{-i}, c''_i)\}) + u_i(c''_i) \geq f_i(\min\{c'''_{-i}\}) + u_i(c'''_i) \geq f_i(\min\{(c'''_{-i}, c'''_i)\}) + u_i(c'''_i)$. Hence it must be the case that $(c''_{-i}, c''_i) \succeq_i (c'''_{-i}, c'''_i)$ for all $c''_i \geq \min\{c''_{-i}\}$. ■

B.3. Theorem 2

Proof. If for all $i \in N \backslash J$ and all all $j \in J$, $|u''_i(m^*_{CU}\bar{x})| \geq |u''_j(x_j - t_j)|$, then $\Phi_k = \frac{\sum_{l=j+1}^{n} u''_l(m^*_{CU}\bar{x})}{u''_k(x_k - t_k)p(m^*_{CU})n} \geq 1$ for all $k \in J$, and hence $\frac{d(m\bar{x})}{dx_k}|_{m=m^*_{CU}} \leq \frac{1}{p(m^*_{CU})n + \sum_{i=1}^{j}(\Gamma_i + 1)} < \frac{1}{n}$. It is straightforward that $\sum_{i=1}^{j} \frac{dt_i}{dx_k} < p(m^*_{CU})$ for $k \in \{1, 2, \ldots, j\}$ as $\sum_{i=1}^{j} \frac{dt_i}{dx_k} = p(m^*_{CU})n\frac{d(m\bar{x})}{dx_k}|_{m=m^*_{CU}}$ for $k \in \{1, 2, \ldots, j\}$. ■

B.4. Proposition 2

Proposition 2. *If for all $i \in N \setminus J$ and all $j \in J$, $|u''_i(m^*_{CU}\bar{x})| \geq |u''_j(x_j - t_j)|$ and $|f''_j(m^*_{CU}\bar{x})| \geq |u''_j(x_j - t_j)|$, then $\exists \bar{\kappa} \geq 0$ such that for all $\kappa \geq \bar{\kappa}$, $x + \kappa e = (x_1 + \kappa, x_2 + \kappa, \ldots, x_n + \kappa)$, $m^*_{CU}(x + \kappa e)(\bar{x} + \kappa) \leq x_n + \kappa$, where $m^*_{CU}(x + \kappa e)$ is the social minimum coefficient under a benevolent utilitarian social planner under the initial endowment vector $x + \kappa e$.*

Proof. If for all $i \in N \setminus J$ and all all $j \in J$, $|u''_i(m^*_{CU}\bar{x})| \geq |u''_j(x_j - t_j)|$, then $\frac{d(m\bar{x})}{dx_k}|_{m=m^*_{CU}} \leq \frac{1}{p(m^*_{CU})n + \sum_{i=1}^{j}(\Gamma_i + 1)} < \frac{1}{n}$. Let $e = (1, 1, \ldots, 1)$ an n-dimensional vector of ones. Then $m^*_{CU}(x + \kappa e)(\bar{x} + \kappa) - m^*_{CU}(x)\bar{x} < \kappa$. Therefore, $p(m^*_{CU}(x + \kappa e))$ is weakly decreasing in κ as the increase in the social minimum is always smaller than the increase in the individual endowment for all social members. Furthermore, since

$\left|f_j''\left(m_{CU}^*\bar{x}\right)\right| \geq \left|u_j''\left(x_j - t_j\right)\right|$, $\Gamma_k = \frac{\sum_{l=1}^{n} f_l''\left(m_{CU}^*\bar{x}\right)}{u_k''(x_k - t_k)p\left(m_{CU}^*\right)n} \geq \frac{1}{p\left(m_{CU}^*(x)\right)}$, so $\frac{d(m\bar{x})}{dx_k}\mid_{m=m_{CU}^*} \leq \frac{1}{p\left(m_{CU}^*\right)n + \sum_{i=1}^{j}\left(\frac{1}{p\left(m_{CU}^*(x)\right)}+1\right)} = \frac{1}{n + \frac{\left[1-p\left(m_{CU}^*\right)\right]n}{p\left(m_{CU}^*(x)\right)}}$, and hence $m_{CU}^*\left(x+\kappa e\right)\left(\bar{x}+\kappa\right) - m_{CU}^*\left(x\right)\bar{x} < \frac{\kappa}{1+\frac{1-p\left(m_{CU}^*\right)}{p\left(m_{CU}^*(x)\right)}}$. Thus, for $\kappa \geq \bar{\kappa} = \max\left\{\left[1 + \frac{p\left(m_{CU}^*(x)\right)}{1-p\left(m_{CU}^*\right)}\right]\left[m_{CU}^*\left(x\right)\bar{x} - x_n\right], 0\right\} \geq 0$, it must be the case that $m_{CU}^*\left(x+\kappa e\right)\left(\bar{x}+\kappa\right) \leq x_n + \kappa$. ■

B.5. Proposition 3

Proposition 3. *If for all $j \in J_0$, $\lim_{\underline{c}\to\infty} MRC_j\left(\underline{c}, \underline{c}+d_j\left(x\right), n-\left|J_0\right|\right) < 1$, then the social minimum that can be supported by a benevolent utilitarian social planner will equal the endowment of the poorest social member after sufficiently high uniform amount of increase in endowments.*

Proof. If for all $j \in J_0$, $\lim_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i\in N\setminus J} u_i'(\underline{c})}{u_j'(\underline{c}+d_j(x))(n-|J_0|)} < 1$, then let $\eta_j = \lim_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i\in N\setminus J} u_i'(\underline{c})}{u_j'(\underline{c}+d_j(x))(n-|J_0|)}$, and it must be the case that $\eta_j < 1$. By the definition of limit for all $\varepsilon > 0$ $\exists \bar{c}\left(\varepsilon\right) > 0$ such that for all $\underline{c} > \bar{c}\left(\varepsilon\right)$, $\left|\frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i\in N\setminus J} u_i'(\underline{c})}{u_j'(\underline{c}+d_j(x))(n-|J_0|)} - \eta_j\right| < \varepsilon$. Pick ε' such that $0 < \varepsilon' \leq 1 - \eta_j$, and let $\bar{\kappa} = \bar{c}\left(\varepsilon'\right)$. Then for all $\kappa \geq \bar{\kappa}$, $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)) + \sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'\left(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)+d_j(x)\right)(n-|J_0|)} < \eta_j + \varepsilon' \leq 1$ for all $j \in J_0$. The first order condition of the redistribution requires that $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)) + \sum_{i\in N\setminus J} u_i'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'(x_j+\kappa-t_j(x+\kappa e))p\left(m_{CU}^*\right)n} = 1$ without any constraints on transfer, where we tentatively assume that J_0 is the group of rich social members with endowment vector $x + \kappa e$. With the constraints on transfer, $m_{CU}^*\left(x+\kappa e\right)\left(\bar{x}+\kappa\right) \geq x_n + \kappa$ and hence $m_{CU}^*\left(x+\kappa e\right)\left(\bar{x}+\kappa\right) + d_j\left(x\right) \geq x_j + \kappa$. Then $x_j + \kappa - t_j\left(x+\kappa e\right) \leq m_{CU}^*\left(x+\kappa e\right)\left(\bar{x}+\kappa\right) + d_j\left(x\right)$. By the concavity of egoistic utility functions, $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)) + \sum_{i\in N\setminus J} u_i'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'(x_j+\kappa-t_j(x+\kappa e))(n-|J|)} < 1$ for all j's such that $t_j\left(x+\kappa e\right) > 0$. Thus, with the constraints on transfer, for all $\kappa \geq \bar{\kappa}$, $t_j\left(x+\kappa e\right) = 0$ for all $j \in J_0$. Hence J_0 is indeed the group of rich social members with endowment vector $x + \kappa e$, and it must be the case that $m_{CU}^*\left(x+\kappa e\right)\left(\bar{x}+\kappa\right) = x_n + \kappa$, indicating that the social minimum that can be supported by a benevolent utilitarian social planner will equal the endowment of the poorest social member after sufficiently high uniform amount of increase in endowments. ■

Corollary 2. *If $\exists R \subseteq J_0$ such that for all $j \in R$, $\lim_{\underline{c}\to\infty} MRC_j\left(\underline{c}, \underline{c}+d_j\left(x\right), n-\left|R\right|\right) < 1$, then $\exists \bar{\kappa} \geq 0$ such* *that for all $\kappa \geq \bar{\kappa}$, $t_j(x+\kappa e) = 0$ for all $j \in R$.*

B.6. Proposition 4

Proposition 4. *If for all $j \in J_0$, $\lim\limits_{\underline{c}\to\infty} MRC_j(\underline{c}, \underline{c}, n-1) > 1$, then the social minimum that can be supported by a benevolent utilitarian social planner will equal the average endowment in society after sufficiently high uniform amount of increase in endowments.*

Proof. If for all $j \in J_0$, $\lim\limits_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'(\underline{c})(n-1)} > 1$, then let $\eta_j = \lim\limits_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'(\underline{c})(n-1)}$, and it must be the case that $\eta_j > 1$. By the definition of limit for all $\varepsilon > 0$ $\exists \bar{c}(\varepsilon) > 0$ such that for all $\underline{c} > \bar{c}(\varepsilon)$, $\left| \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'(\underline{c})(n-1)} - \eta_j \right| < \varepsilon$. Let $0 < \varepsilon' \leq \eta_j - 1$, $\kappa = \bar{c}(\varepsilon')$. Then for all $\kappa \geq \bar{\kappa}$, $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)) + \sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'\left(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)\right)(n-1)} > \eta_j - \varepsilon' \geq 1$ for all $j \in J_0$. The first order condition of the redistribution requires that $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)) + \sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'(x_j+\kappa-t_j(x+\kappa e))p\left(m_{CU}^*\right)n} = 1$ without any constraints on transfer, where we tentatively assume that J_0 is the group of rich social members with endowment vector $x+\kappa e$. Here, before reaching an equality-of-outcome society, the maximum number of poor people will be at most $n-1$ (anyone except the sole rich contributor). With the constraints on transfer, $x_j + \kappa - t_j(x+\kappa e) \geq m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)$. By the concavity of egoistic utility functions, $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'(x_j+\kappa-t_j(x+\kappa e))(n-1)} + \frac{\sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(x+\kappa e)(\bar{x}+\kappa))}{u_j'(x_j+\kappa-t_j(x+\kappa e))(n-1)} > 1$ for any $0 \leq t_j(x+\kappa e) \leq x_j - \bar{x}$. Thus, with the constraints on transfer, for all $\kappa \geq \bar{\kappa}$, $t_j(x+\kappa e) = x_j - \bar{x}$ for all $j \in J_0$. Hence J_0 is indeed the group of rich social members with endowment vector $x+\kappa e$, and it must be the case that $m_{CU}^*(x+\kappa e)(\bar{x}+\kappa) = \bar{x}+\kappa$, indicating that the social minimum that can be supported by a benevolent utilitarian social planner will equal the average endowment in society after sufficiently high uniform amount of increase in endowments. ■

Corollary 3. *If $\exists R \subseteq J_0$ such that for all $j \in R$, $\lim\limits_{\underline{c}\to\infty} MRC_j(\underline{c}, \underline{c}, n-1) > 1$, then $\exists \bar{\kappa} \geq 0$ such that for all $\kappa \geq \bar{\kappa}$, $t_j(x+\kappa e) = x_j - m_{CU}^*(x+\kappa e)(\bar{x}+\kappa)$ for all $j \in R$.*

B.7. Theorem 3

Proof. Since $\exists R \subseteq J_0$ such that for all $j \in R$, $\inf\limits_{\underline{c}\geq x_n} MRC_j(\underline{c}, \underline{c} + d_j(x), n - |R|) > 1$, after any $\kappa > 0$

amount of increase in everyone's endowment $MRC_j\left(m^*_{CU}\left(x+\kappa e\right)\left(\bar{x}+\kappa\right),\right.$
$\left.m^*_{CU}\left(x+\kappa e\right)\left(\bar{x}+\kappa\right)+d_j\left(x\right), n-|R|\right)>1$ for all $j\in R$. If after some $\tilde{\kappa}>0$ amount of increase in everyone's endowment the social minimum that can be supported by a benevolent utilitarian social planner ever equals the endowment of the poorest social member, then it must be the case that $m^*_{CU}\left(x+\tilde{\kappa}e\right)\left(\bar{x}+\tilde{\kappa}\right)=x_n+\tilde{\kappa}$ and $t_j\left(x+\tilde{\kappa}e\right)=0$ for all $j\in R$. The first order condition of the redistribution will induce the necessary condition $MRC_j\left(x_n+\tilde{\kappa}, x_j+\tilde{\kappa}, n-|R|\right)=1$ for all $j\in R$, which contradicts the antecedent. Since both u_i and f_i are twice differentiable for all $i\in N$, MRC_j is continuous for all $j\in R$. Hence the social minimum that can be supported by a benevolent utilitarian social planner will always be strictly higher than the endowment of the poorest social member after any $\kappa>0$ amount of increase in everyone's endowment.

Since $\exists R\subseteq J_0$ such that for all $j\in R$, $\sup_{\underline{c}\geq x_n} MRC_j\left(\underline{c},\underline{c},n-1\right)<1$, after any $\kappa>0$ amount of increase in everyone's endowment $MRC_j\left(m^*_{CU}\left(x+\kappa e\right)\left(\bar{x}+\kappa\right),\right.$
$\left.m^*_{CU}\left(x+\kappa e\right)\left(\bar{x}+\kappa\right), n-1\right)<1$ for all $j\in R$. If after some $\tilde{\kappa}>0$ amount of increase in everyone's endowment the social minimum that can be supported by a benevolent utilitarian social planner ever equals the average endowment in society, then it must be the case that $m^*_{CU}\left(x+\tilde{\kappa}e\right)\left(\bar{x}+\tilde{\kappa}\right)=\bar{x}+\tilde{\kappa}$ and $t_j\left(x+\tilde{\kappa}e\right)=\max\left\{x_j-\bar{x},0\right\}$ for all $j\in R$. The first order condition of the redistribution will induce the necessary condition $MRC_j\left(\bar{x}+\tilde{\kappa},\bar{x}+\tilde{\kappa},n-1\right)=1$ for all $j\in R$, which contradicts the antecedent. As MRC_j is continuous for all $j\in R$, the social minimum that can be supported by a benevolent utilitarian social planner will always be strictly lower than the average endowment in society after any $\kappa>0$ amount of increase in everyone's endowment. ■

B.8. Proposition 5

Proposition 5. *If for all $j\in J_0$, $\lim_{\underline{c}\to\infty} MRC_j\left(\underline{c}, r_j\left(x\right)\underline{c}, n-|J_0|\right)<1$, then the social minimum that can be supported by a benevolent utilitarian social planner will equal the endowment of the poorest social member after sufficiently high percentage of relative increase in endowments.*

Proof. If for all $j\in J_0$, $\lim_{\underline{c}\to\infty}\frac{\sum_{l=1}^{n}f_l'(\underline{c})+\sum_{i\in N\setminus J_0}u_i'(\underline{c})}{u_j'(r_j(x)\underline{c})(n-|J_0|)}<1$, then let $\eta_j=\lim_{\underline{c}\to\infty}\frac{\sum_{l=1}^{n}f_l'(\underline{c})+\sum_{i\in N\setminus J_0}u_i'(\underline{c})}{u_j'(r_j(x)\underline{c})(n-|J_0|)}$, and it must be the case that $\eta_j<1$. By the definition of limit for all $\varepsilon>0$ $\exists\bar{c}\left(\varepsilon\right)>0$ such that for all $\underline{c}>\bar{c}\left(\varepsilon\right)$, $\left|\frac{\sum_{l=1}^{n}f_l'(\underline{c})+\sum_{i\in N\setminus J_0}u_i'(\underline{c})}{u_j'(r_j(x)\underline{c})(n-|J_0|)}-\eta_j\right|<\varepsilon$. Pick ε' such that $0<\varepsilon'\leq 1-\eta_j$, and let $\bar{\kappa}=\frac{\bar{c}\left(\varepsilon'\right)}{x_n}$. Then for all $\kappa\geq\bar{\kappa}$, $\frac{\sum_{l=1}^{n}f_l'(m^*_{CU}(\kappa x)\kappa\bar{x})+\sum_{i\in N\setminus J}u_i'(m^*_{CU}(\kappa x)\kappa\bar{x})}{u_j'\left(r_j(x)m^*_{CU}(\kappa x)\kappa\bar{x}\right)(n-|J_0|)}<\eta_j+\varepsilon'\leq 1$ for all $j\in J_0$. The first order condition of the redistribution requires that $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(\kappa x)\kappa\bar{x})+\sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(\kappa x)\kappa\bar{x})}{u_j'(\kappa x_j - t_j(\kappa x))p\left(m_{CU}^*\right)n} = 1$ without any constraints on transfer, where we tentatively assume that J_0 is the group of rich social members with endowment vector κx. With the constraints on transfer, $m_{CU}^*(\kappa x)\kappa\bar{x} \geq \kappa x_n$ and hence $r_j(x)\, m_{CU}^*(\kappa x)\kappa\bar{x} \geq \kappa x_j$. Then $\kappa x_j - t_j(\kappa x) \leq r_j(x)\, m_{CU}^*(\kappa x)\kappa\bar{x}$, and $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(\kappa x)\kappa\bar{x})+\sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(\kappa x)\kappa\bar{x})}{u_j'(\kappa x_j - t_j(\kappa x))(n-|J_0|)} < 1$ for any $t_j(\kappa x) \geq 0$ by the concavity of egoistic utility functions. Thus, with the constraints on transfer, for all $\kappa \geq \bar{\kappa}$, $t_j(\kappa x) = 0$ for all $j \in J_0$. Hence J_0 is indeed the group of rich social members with endowment vector κx, and it must be the case that $m_{CU}^*(\kappa x)\kappa\bar{x} = \kappa x_n$, indicating that the social minimum that can be supported by a benevolent utilitarian social planner will equal the endowment of the poorest social member after sufficiently high percentage of relative increase in endowments. ■

Corollary 4. *If $\exists R \subseteq J_0$ such that for all $j \in R$, $\lim_{\underline{c}\to\infty} MRC_j(\underline{c}, r_j(x)\underline{c}, n-|R|) < 1$, then $\exists\bar{\kappa} \geq 0$ such that for all $\kappa \geq \bar{\kappa}$, $t_j(\kappa x) = 0$ for all $j \in R$.*

B.9. Proposition 6

Proposition 6. *If for all $j \in J_0$, $\lim_{\underline{c}\to\infty} MRC_j(\underline{c}, \underline{c}, n-1) > 1$, then the social minimum that can be supported by a benevolent utilitarian social planner will equal the average endowment in society after sufficiently high percentage of relative increase in endowments.*

Proof. If for all $j \in J_0$, $\lim_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c})+\sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'(\underline{c})(n-1)} > 1$, then let $\eta_j = \lim_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c})+\sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'(\underline{c})(n-1)}$, and it must be the case that $\eta_j > 1$. By the definition of limit for all $\varepsilon > 0$ $\exists\bar{c}(\varepsilon) > 0$ such that for all $\underline{c} > \bar{c}(\varepsilon)$, $\left|\frac{\sum_{l=1}^{n} f_l'(\underline{c})+\sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'(\underline{c})(n-1)} - \eta_j\right| < \varepsilon$. Let $0 < \varepsilon' \leq \eta_j - 1$, $\kappa = \frac{\bar{c}(\varepsilon')}{x_n}$. Then for all $\kappa \geq \bar{\kappa}$, $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(\kappa x)\kappa\bar{x})+\sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(\kappa x)\kappa\bar{x})}{u_j'\left(m_{CU}^*(\kappa x)\kappa\bar{x}\right)(n-1)} > \eta_j - \varepsilon' \geq 1$ for all $j \in J_0$. The first order condition of the redistribution requires that $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(\kappa x)\kappa\bar{x})+\sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(\kappa x)\kappa\bar{x})}{u_j'(\kappa x_j - t_j(\kappa x))p\left(m_{CU}^*\right)n} = 1$ without any constraints on transfer, where we tentatively assume that J_0 is the group of rich social members with endowment vector κx. Here, before reaching an equality-of-outcome society, the maximum number of poor people will be at most $n-1$ (anyone except the sole rich contributor). With the constraints on transfer, $\kappa x_j - t_j(\kappa x) \geq m_{CU}^*(\kappa x)\kappa\bar{x}$. By the concavity of egoistic utility functions, $\frac{\sum_{l=1}^{n} f_l'(m_{CU}^*(\kappa x)\kappa\bar{x})+\sum_{i\in N\setminus J_0} u_i'(m_{CU}^*(\kappa x)\kappa\bar{x})}{u_j'(x_j - t_j(\kappa x))(n-1)} > 1$ for any $0 \leq t_j(\kappa x) \leq \kappa(x_j - \bar{x})$.

Thus, with the constraints on transfer, for all $\kappa \geq \bar{\kappa}$, $t_j(\kappa x) = \kappa(x_j - \bar{x})$ for all $j \in J_0$. Hence J_0 is indeed the group of rich social members with endowment vector κx, and it must be the case that $m^*_{CU}(\kappa x)\kappa\bar{x} = \kappa\bar{x}$. Therefore, the social minimum that can be supported by a benevolent utilitarian social planner will equal the average endowment in society after sufficiently high percentage of relative increase in endowments. ■

Corollary 5. *If* $\exists R \subseteq J_0$ *such that for all* $j \in R$, $\lim_{\underline{c} \to \infty} MRC_j(\underline{c}, \underline{c}, n-1) > 1$, *then* $\exists \bar{\kappa} \geq 0$ *such that for all* $\kappa \geq \bar{\kappa}$, $t_j(\kappa x) = x_j - m^*_{CU}(\kappa x)\kappa\bar{x}$ *for all* $j \in R$.

B.10. Theorem 4

Proof. Since $\exists R \subseteq J_0$ such that for all $j \in R$, $\inf_{\underline{c} \geq x_n} MRC_j(\underline{c}, r_j(x)\underline{c}, n - |R|) > 1$, after any $\kappa > 1$ multiplication in everyone's endowment $MRC_j(m^*_{CU}(\kappa x)\kappa\bar{x}$, $r_j(x) m^*_{CU}(\kappa x)\kappa\bar{x}, n - |R|) > 1$ for all $j \in R$. If after some $\tilde{\kappa} > 1$ multiplication in everyone's endowment the social minimum that can be supported by a benevolent utilitarian social planner ever equals the endowment of the poorest social member, then it must be the case that $m^*_{CU}(\tilde{\kappa}x)\tilde{\kappa}\bar{x} = \tilde{\kappa}x_n$ and $t_j(\tilde{\kappa}x) = 0$ for all $j \in R$. The first order condition of the redistribution will induce the necessary condition $MRC_j(\tilde{\kappa}x_n, \tilde{\kappa}x_j, n - |R|) = 1$ for all $j \in R$, which contradicts the antecedent. Since both u_i and f_i are twice differentiable for all $i \in N$, MRC_j is continuous for all $j \in R$. Hence the social minimum that can be supported by a benevolent utilitarian social planner will always be strictly higher than the endowment of the poorest social member after any $\kappa > 1$ multiplication in everyone's endowment.

Since $\exists R \subseteq J_0$ such that for all $j \in R$, $\sup_{\underline{c} \geq x_n} MRC_j(\underline{c}, \underline{c}, n-1) < 1$, after any $\kappa > 1$ multiplication in everyone's endowment $MRC_j(m^*_{CU}(\kappa x)\kappa\bar{x}, m^*_{CU}(\kappa x)\kappa\bar{x}, n-1) < 1$ for all $j \in R$. If after some $\tilde{\kappa} > 1$ multiplication in everyone's endowment the social minimum that can be supported by a benevolent utilitarian social planner ever equals the average endowment in society, then it must be the case that $m^*_{CU}(\tilde{\kappa}x)\tilde{\kappa}\bar{x} = \tilde{\kappa}\bar{x}$ and $t_j(\tilde{\kappa}x) = \max\{\tilde{\kappa}x_j - \tilde{\kappa}\bar{x}, 0\}$ for all $j \in R$. The first order condition of the redistribution will induce the necessary condition $MRC_j(\tilde{\kappa}\bar{x}, \tilde{\kappa}\bar{x}, n-1) = 1$ for all $j \in R$, which contradicts the antecedent. As MRC_j is continuous for all $j \in R$, the social minimum that can be supported by a benevolent utilitarian social planner will always be strictly lower than the average endowment in society after any $\kappa > 1$ multiplication in everyone's endowment. ■

C. Omitted Deviations of the Comparative Statics

From the maximization problem of the social planner:

$$\max_{m,t} \sum_{i=1}^{n} f_i (m\bar{x}) + u_i (x_i - t_i)$$

$$s.t. \sum_{i=1}^{j} t_i = \sum_{i=j+1}^{n} (m\bar{x} - x_i)$$

$$t_i \leq x_i - m\bar{x} \; \forall i \in \{1, 2, \ldots, j\}$$

$$t_i = x_i - m\bar{x} \; \forall i \in \{j+1, j+2, \ldots, n\}$$

$$L (m, t, x) = \sum_{i=1}^{n} [f_i (m\bar{x}) + u_i (x_i - t_i)] + \lambda \left[\sum_{i=1}^{j} t_i - \sum_{i=j+1}^{n} (m\bar{x} - x_i) \right] + \sum_{i=1}^{n} \mu_i (x_i - m\bar{x} - t_i)$$

$$F.O.C.s \quad \frac{\partial L}{\partial t_i} = \begin{cases} -u_i' (x_i - t_i) + \lambda - \mu_i = 0 & \forall i \in \{1, 2, \ldots, j\} \\ -u_i' (m_{CU}^* \bar{x}) + \mu_i = 0 & \forall i \in \{j+1, j+2, \ldots, n\} \end{cases}$$

$$\frac{\partial L}{\partial m} = \sum_{i=1}^{n} f_i' (m_{CU}^* \bar{x}) \bar{x} - \lambda p (m_{CU}^*) n\bar{x} + \sum_{i=1}^{n} \mu_i \bar{x} = 0$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{j} t_i - \sum_{i=j+1}^{n} (m_{CU}^* \bar{x} - x_i) = 0$$

$$\frac{\partial L}{\partial \mu_i} = \begin{cases} t_i - x_i + m_{CU}^* \bar{x} \leq 0 & \forall i \in \{1, 2, \ldots, j\} \\ t_i - x_i + m_{CU}^* \bar{x} = 0 & \forall i \in \{j+1, j+2, \ldots, n\} \end{cases}$$

$$C.S.C.s \quad \mu_i (t_i - x_i + m_{CU}^* \bar{x}) = 0 \quad \forall i \in \{1, 2, \ldots, j\}$$

$$\sum_{i=1}^{n} \frac{dt_i}{dx_k} = 0$$

$$\sum_{i=1}^{j} \frac{dt_i}{dx_k} = \begin{cases} p\left(m_{CU}^*\right) n\bar{x} \frac{dm}{dx_k} \mid_{m=m_{CU}^*} + p\left(m_{CU}^*\right) m_{CU}^* & \forall k \in \{1,2,\ldots,j\} \\ p\left(m_{CU}^*\right) n\bar{x} \frac{dm}{dx_k} \mid_{m=m_{CU}^*} + p\left(m_{CU}^*\right) m_{CU}^* - 1 & otherwise \end{cases}$$

Note that these two conditions are equivalent with each other given that $t_i = x_i - m\bar{x}\ \forall i \in \{j+1, j+2, \ldots, n\}$. From the first order conditions

$$\sum_{l=1}^{n} f_l'\left(m_{CU}^*\bar{x}\right) - \left[u_i'\left(x_i - t_i\right) + \mu_i\right] p\left(m_{CU}^*\right) n + \sum_{l=1}^{n} \mu_l = 0$$

for all $i \in \{1, 2, \ldots, j\}$. Totally differentiate the equation with regard to x_i

$$\frac{dt_i}{dx_i} = \begin{cases} \left\{ -\sum_{l=1}^{n} f_l''\left(m_{CU}^*\bar{x}\right)\left(\bar{x}\frac{dm}{dx_i}\mid_{m=m_{CU}^*} + \frac{m_{CU}^*}{n}\right) - \frac{d}{dx_i}\left(\sum_{l=1}^{n}\mu_l\right) \right. & \\ \left. + \frac{d\mu_i}{dx_i} p\left(m_{CU}^*\right) n \right\} \left[u_i''\left(x_i - t_i\right) p\left(m_{CU}^*\right) n\right]^{-1} + 1 & \forall i \in \{1,2,\ldots,j\} \\ 1 - \frac{m_{CU}^*}{n} - \bar{x}\frac{dm}{dx_i}\mid_{m=m_{CU}^*} & otherwise \end{cases}$$

$$\frac{dt_i}{dx_{k\neq i}} = \begin{cases} \left\{ -\sum_{l=1}^{n} f_l''\left(m_{CU}^*\bar{x}\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*} + \frac{m_{CU}^*}{n}\right) - \frac{d}{dx_k}\left(\sum_{l=1}^{n}\mu_l\right) \right. & \\ \left. + \frac{d\mu_i}{dx_k} p\left(m_{CU}^*\right) n \right\} \left[u_i''\left(x_i - t_i\right) p\left(m_{CU}^*\right) n\right]^{-1} & \forall i \in \{1,2,\ldots,j\} \\ -\frac{m_{CU}^*}{n} - \bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*} & otherwise \end{cases}$$

Let $\Gamma_i = \frac{\sum_{l=1}^{n} f_l''\left(m_{CU}^*\bar{x}\right)}{u_i''\left(x_i - t_i\right) p\left(m_{CU}^*\right) n} > 0$. Note that here from the complementary slackness condition for $i \in \{1, 2, \ldots, j\}$ if $t_i < x_i - m\bar{x}$ then $\mu_i = 0$. Therefore, assume that $t_i < x_i - m\bar{x}$ for all $i \in \{1, 2, \ldots, j\}$, then $\mu_1 = \mu_2 = \ldots = \mu_j = 0$, $\frac{d\mu_i}{dx_k}\left[u_i''\left(x_i - t_i\right)\right]^{-1} = 0$ for $i \in \{1, 2, \ldots, j\}$, and $\lambda = -u_i'\left(x_i - t_i\right)$ for all $i \in \{1, 2, \ldots, j\}$.[22] Therefore,

$$\sum_{l=1}^{n} \mu_l = \sum_{l=j+1}^{n} u_l'\left(m_{CU}^*\bar{x}\right)$$

[22] A more general discussion on this can be found in section D of the appendix. The main results naturally extended to the general case.

$$\frac{d}{dx_k}\left(\sum_{l=1}^{n}\mu_l\right)=\sum_{l=j+1}^{n}u_l''\left(m_{CU}^*\bar{x}\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*}+\frac{m_{CU}^*}{n}\right)\quad\forall any\ k\in\{1,2,\ldots,j\}$$

Plug the equation for $\frac{d}{dx_i}\left(\sum_{l=1}^{n}\mu_l\right)$ back into the previous equations on $\frac{dt_i}{dx_k}$ for $\forall i\in\{1,2,\ldots,j\}$. Let $\Phi_i=\frac{\sum_{l=j+1}^{n}u_l''(m_{CU}^*\bar{x})}{u_i''(x_i-t_i)p\left(m_{CU}^*\right)n}>0$.

$$\frac{dt_i}{dx_k}=\begin{cases}-\left(\Gamma_i+\Phi_i\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*}+\frac{m_{CU}^*}{n}\right)+1 & i=k, i\in\{1,2,\ldots,j\}\\ -\left(\Gamma_i+\Phi_i\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*}+\frac{m_{CU}^*}{n}\right) & i\neq k, i\in\{1,2,\ldots,j\}\end{cases}$$

Therefore,

$$\sum_{i=1}^{j}\frac{dt_i}{dx_k}=\begin{cases}1-\sum_{i=1}^{j}\left(\Gamma_i+\Phi_i\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*}+\frac{m_{CU}^*}{n}\right) & \forall k\in\{1,2,\ldots,j\}\\ -\sum_{i=1}^{j}\left(\Gamma_i+\Phi_i\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m_{CU}^*}+\frac{m_{CU}^*}{n}\right) & otherwise\end{cases}$$

Combine the equations for $\sum_{i=1}^{j}\frac{dt_i}{dx_k}$ derived from the first order conditions and the balanced budget condition to solve $\frac{dm}{dx_k}\mid_{m=m_{CU}^*}$.

$$\begin{aligned}\frac{dm}{dx_k}\mid_{m=m_{CU}^*}=&\left[1-p\left(m_{CU}^*\right)m_{CU}^*-\sum_{i=1}^{j}\left(\Gamma_i+\Phi_i\right)\frac{m_{CU}^*}{n}\right]\\&\times\left[p\left(m_{CU}^*\right)n\bar{x}+\sum_{i=1}^{j}\left(\Gamma_i+\Phi_i\right)\bar{x}\right]^{-1}\end{aligned}$$

Note that $\frac{dm}{dx_k}\mid_{m=m_{CU}^*}$ is independent of k, the identity of the social member. This is not surprising as the social planner is allowed to maximize the sum of individual utilities at any social member's utility cost, provided that none of the rich social members has contributed everything above the social minimum to the social support system. The comparative statics of individual transfers can therefore have an explicit expression.

$$\frac{dt_i}{dx_k} = \begin{cases} 1 - \frac{\Gamma_i + \Phi_i}{p\left(m^*_{CU}\right)n + \sum\limits_{i=1}^{j} (\Gamma_i + \Phi_i)} & i = k, i \in \{1, 2, \ldots, j\} \\ -\frac{\Gamma_i + \Phi_i}{p\left(m^*_{CU}\right)n + \sum\limits_{i=1}^{j} (\Gamma_i + \Phi_i)} & i \neq k, i \in \{1, 2, \ldots, j\} \end{cases}$$

Therefore $\frac{dt_i}{dx_k} < 0$ if $i \neq k$ and $i \in \{1, 2, \ldots, j\}$.

D. Generalized Comparative Statics for Centralized Utilitarian Case

Now assume that for some $i \in \{1, 2, \ldots, j\}$, $t_i = x_i - m\bar{x}$. First consider the trivial case that $\mu_i > 0$ for all $i \in \{1, 2, \ldots, j\}$.[23] Then trivially $m^*_{CU} = 1$, $\frac{dc_i}{dx_i} = \frac{n-1}{n}$, and hence $\frac{dm}{dx_i} \mid_{m=m^*_{CU}} = 0$ for all $i \in \{1, 2, \ldots, n\}$. Now consider the case when $\exists i^* \in \{1, 2, \ldots, j\}$ such that $\mu_{i^*} = 0$.[24] Then from the first order condition $\lambda = u'_{i^*}(x_{i^*} - t_{i^*})$, and $\frac{d\mu_{i^*}}{dx_k} = 0$. Also,

$$\sum_{l=1}^{n} \mu_l = \sum_{l=1}^{n} f'_l(m^*_{CU}\bar{x}) + u'_{i^*}(x_{i^*} - t_{i^*}) p(m^*_{CU}) n$$

$$\frac{d}{dx_k}\left(\sum_{l=1}^{n} \mu_l\right) = \begin{cases} \sum\limits_{l=1}^{n} f''_l(m^*_{CU}\bar{x}) \left(\bar{x}\frac{dm}{dx_i} \mid_{m=m^*_{CU}} + \frac{m^*_{CU}}{n}\right) + u''_{i^*}(x_{i^*} - t_{i^*}) \left(1 - \frac{dt_{i^*}}{dx_k}\right) & \\ \quad \times p(m^*_{CU}) n + u'_{i^*}(x_{i^*} - t_{i^*}) p(m^*_{CU}) & i^* = k \\ \sum\limits_{l=1}^{n} f''_l(m^*_{CU}\bar{x}) \left(\bar{x}\frac{dm}{dx_i} \mid_{m=m^*_{CU}} + \frac{m^*_{CU}}{n}\right) - u''_{i^*}(x_{i^*} - t_{i^*}) \frac{dt_{i^*}}{dx_k} & \\ \quad \times p(m^*_{CU}) n + u'_{i^*}(x_{i^*} - t_{i^*}) p(m^*_{CU}) & i^* \neq k \end{cases}$$

For all $i \in \{1, 2, \ldots, j\}$ such that $\mu_i > 0$, since they are at the social minimum and have the same characteristics as the poor people in society, they can be regarded as the poor social members. Therefore, let $N^R = \{i : x_i > m^*_{CU}\bar{x} \; and \; \mu_i = 0\}$.

$$\frac{dt_i}{dx_i} = \begin{cases} \left\{-\sum\limits_{l=1}^{n} f''_l(m^*_{CU}\bar{x}) \left(\bar{x}\frac{dm}{dx_i} \mid_{m=m^*_{CU}} + \frac{m^*_{CU}}{n}\right) - \frac{d}{dx_i}\left(\sum\limits_{l=1}^{n} \mu_l\right) \right. & \\ \left. \quad + \frac{d\mu_i}{dx_i} p(m^*_{CU}) n \right\} A_i^{-1} + 1 & \forall i \in \{1, 2, \ldots, j\} \\ 1 - \frac{m^*_{CU}}{n} - \bar{x}\frac{dm}{dx_i} \mid_{m=m^*_{CU}} & otherwise \end{cases}$$

$$\frac{dt_i}{dx_{k \neq i}} = \begin{cases} \left\{-\sum\limits_{l=1}^{n} f''_l(m^*_{CU}\bar{x}) \left(\bar{x}\frac{dm}{dx_k} \mid_{m=m^*_{CU}} + \frac{m^*_{CU}}{n}\right) - \frac{d}{dx_k}\left(\sum\limits_{l=1}^{n} \mu_l\right) \right. & \\ \left. \quad + \frac{d\mu_i}{dx_k} p(m^*_{CU}) n \right\} A_i^{-1} & \forall i \in \{1, 2, \ldots, j\} \\ -\frac{m^*_{CU}}{n} - \bar{x}\frac{dm}{dx_k} \mid_{m=m^*_{CU}} & otherwise \end{cases}$$

[23] This is a sufficient condition for $t_i = x_i - m\bar{x}$ for all $i \in \{1, 2, \ldots, j\}$.

[24] This will be true if $\exists i \in \{1, 2, \ldots, j\}$ such that $(1 - \alpha_i) u'_i(\bar{x}) > \sum_{l=1}^{n} \alpha_l \frac{df_l(1^-)}{dm} + \sum_{l \neq i} u'_i(\bar{x})$, where $\frac{df_l(1^-)}{dm} = \lim_{m^* \to 1^-} \frac{df_l(m^*)}{dm}$. It does not have to be the case that $i = i^*$.

Also,

$$\sum_{i\in N^R}\frac{dt_i}{dx_k}=\begin{cases}p\left(m^*_{CU}\right)n\bar{x}\frac{dm}{dx_k}\mid_{m=m^*_{CU}}+p\left(m^*_{CU}\right)m^*_{CU} & \forall k\in N^R\\ p\left(m^*_{CU}\right)n\bar{x}\frac{dm}{dx_k}\mid_{m=m^*_{CU}}+p\left(m^*_{CU}\right)m^*_{CU}-1 & otherwise\end{cases}$$

It is trivial that plug the previous equations on $\frac{dt_{i^*}}{dx_k}$ will still result in

$$\frac{dt_{i^*}}{dx_k}=\begin{cases}1-\sum_{i\in N^R}\left(\Gamma_i+\Phi_i\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m^*_{CU}}+\frac{m^*_{CU}}{n}\right) & i^*=k\\ -\sum_{i\in N^R}\left(\Gamma_i+\Phi_i\right)\left(\bar{x}\frac{dm}{dx_k}\mid_{m=m^*_{CU}}+\frac{m^*_{CU}}{n}\right) & i^*\neq k\end{cases}$$

Hence similarly,

$$\begin{aligned}\frac{dm}{dx_k}\mid_{m=m^*_{CU}}=&\left[1-p\left(m^*_{CU}\right)m^*_{CU}-\sum_{i\in N^R}\left(\Gamma_i+\Phi_i\right)\frac{m^*_{CU}}{n}\right]\\&\times\left[p\left(m^*_{CU}\right)n\bar{x}+\sum_{i\in N^R}\left(\Gamma_i+\Phi_i\right)\bar{x}\right]^{-1}\end{aligned}$$

$$\frac{d\left(m\bar{x}\right)}{dx_k}\mid_{m=m^*_{CU}}=\frac{1}{p\left(m^*_{CU}\right)n+\sum_{i\in N^R}\left(\Gamma_i+\Phi_i\right)}$$

E. Calibration Results for Example 2

The social planner is then maximizing the sum of individual utilities by choosing the optimal transfer t.

$$\begin{aligned}
&\max_t \frac{1}{10}(2n)\,t^{\frac{1}{4}} + \frac{9}{10}n\sqrt{1-t} + \frac{9}{10}nt^{\frac{1}{4}} \\
&F.O.C. \frac{11}{40}t^{-\frac{3}{4}} - \frac{9}{20}(1-t)^{-\frac{1}{2}} = 0 \\
&t^* = 0.3779145436 \\
&m^*_{CU} = \frac{t^*}{\bar{x}} = 2t^*
\end{aligned}$$

For every rich social member $j \in J = \{1, 2, \ldots, n\}$,

$$\begin{aligned}
A_j &= u''_j\,(1-t^*)\,p\,(m^*_{CU})\,(2n) \\
&= \frac{9}{10} \times \left(-\frac{1}{4}\right)(1-t^*)^{-\frac{3}{2}} \times \frac{1}{2} \times 2n \\
&= -0.4585719073n \\
\Gamma_j &= \sum_{l=1}^{2n} f''_l\,(m^*_{CU}\bar{x})\,A_j^{-1} \\
&= 2n \times \frac{1}{10} \times \left(-\frac{3}{16}\right) t^{*-\frac{7}{4}} \times \frac{1}{A_j} \\
&= 0.4489365991 \\
\Phi_j &= \sum_{l=n+1}^{2n} u''_l\,(m^*_{CU}\bar{x})\,A_j^{-1} \\
&= n \times \frac{9}{10} \times \left(-\frac{3}{16}\right)(t^*)^{-\frac{7}{4}} \times \frac{1}{A_j} \\
&= 2.020214696
\end{aligned}$$

To calibrate $\frac{d(m\bar{x})}{dx_k}\,|_{m=m^*_{CU}}$, simply plug in the values of A_j, Γ_j, and Φ_j into the following equation

$$\begin{aligned}
\frac{d\,(m\bar{x})}{dx_k}\,|_{m=m^*_{CU}} &= \frac{1}{p\,(m^*_{CU})\,2n + \sum_{i=1}^{n}(\Gamma_i + \Phi_i)} \\
&= \frac{1}{\frac{1}{2} \times 2n + n \times (\Gamma_j + \Phi_j)} \\
&= 0.2882549405n^{-1}
\end{aligned}$$

Check the calibration results by adding $\Delta x_j = 1 \times 10^{-7}$ unit of resources to the endowment of every rich people in society. The social planner is now maximizing the sum of individual utilities by choosing a new optimal transfer τ.

$$\begin{aligned}
&\max_{t} \frac{1}{10}\left(2n\right)\tau^{\frac{1}{4}} + \frac{9}{10}n\sqrt{1+\Delta x_j-\tau} + \frac{9}{10}n\tau^{\frac{1}{4}} \\
F.O.C.&\frac{11}{40}\tau^{-\frac{3}{4}} - \frac{9}{20}\left(1+\Delta x_j-\tau\right)^{-\frac{1}{2}} = 0 \\
&\tau^{*} = 0.3779145724
\end{aligned}$$

Estimation of $\frac{d(m\bar{x})}{dx_k}\mid_{m=m^{*}_{CU}}$ is given by $\frac{\tau^{*}-t^{*}}{\Delta x_j n} = 0.28825493n^{-1}$, which agrees with the calibration result shown above.

F. Dynamics in the Mixed Scenarios

Now we consider a mixed scenario that resides between the previous two: the society getting sufficiently rich is representing by a repeated process consists of multiple rounds, in each of which there will be a *percentage* of relative increase in the poor people's endowments, and an increase on the endowments of the rich people that lies in between the same absolute *amount* that the poor people's endowments get increased and the same *percentage* of relative increase in the poor people's endowments. In order to discuss such class of cases in the most general sense, for any given endowment vector $x = (x_1, x_2, \ldots, x_n)$, let $\kappa_{j,m} > 1$ be the multiplier applied to the endowment of rich person $j \in J$ in round m of increase, and $\beta_m > 1$ be the multiplier applied to the endowments of all poor people in round m. The social endowments after m rounds of increase will then be $x^{(m)} \overset{def}{=} \left(x_1 \prod_{l=1}^{m} \kappa_{1,l}, x_2 \prod_{l=1}^{m} \kappa_{2,l}, \ldots, x_j \prod_{l=1}^{m} \kappa_{j,l}, x_{j+1} \prod_{l=1}^{m} \beta_l, \ldots, x_n \prod_{l=1}^{m} \beta_l \right)$. Hence $\kappa_{j,m}$ can be regarded as a random variable with distribution $f(\kappa_{j,m})$ whose support is on $\left[1 + \frac{\beta_m - 1}{r_j(x) \prod_{l=1}^{m-1} \frac{\kappa_{j,l}}{\beta_l}}, \beta_m \right]$ for $m \geq 2$, given that $r_j(x) = \frac{x_j}{x_n}$ is the *percentage* difference between the initial endowments of rich person j and the poorest person in society. Let the sequence $r_{j,m}(x) = r_j(x) \prod_{l=1}^{m} \frac{\kappa_{j,l}}{\beta_l}$ document the *percentage* difference between the endowments of rich person j and the poorest person in society after m rounds of increase without any redistribution. Note that sequence $r_{j,m}(x)$ is monotonically decreasing in m and is bounded below by 1, and hence $r_{j,m}(x)$ is convergent. Let $r_{j,\infty}(x) = \lim_{m \to \infty} r_{j,m}(x)$. For notation simplicity, we will consider two possible cases: i. $\exists \overline{m} \in \mathbb{N}$ such that $r_{j,m}(x) = r_{j,\infty}(x)$ for all $m \geq \overline{m}$; and ii. for all $m \in \mathbb{N}$, $r_{j,m}(x) > r_{j,\infty}(x)$.

Proposition 7. *If for all $j \in J_0$, either $\lim_{\underline{c} \to \infty} MRC_j(\underline{c}, r_{j,\infty}(x)\underline{c}, n - |J_0|) < 1$ in case i, or $\exists \epsilon > 0$ such that $\lim_{\underline{c} \to \infty} MRC_j(\underline{c}, [r_{j,\infty}(x) + \epsilon]\underline{c}, n - |J_0|) < 1$ in case ii, then the social minimum that can be supported by a benevolent utilitarian social planner will equal the endowment of the poorest social member after sufficiently many rounds of mixed increase in endowments.*

Proof. Since for all rich people $j \in J_0$, one of the cases must occur, we partition the set J_0 into two subsets, J_1 and J_2, as the group of rich people that satisfy case i and case ii, accordingly. We first consider case i, if for all $j \in J_1$, $\lim_{\underline{c} \to \infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i \in N \setminus J_0} u_i'(\underline{c})}{u_j'(r_{j,\infty}(x)\underline{c})(n - |J_0|)} < 1$, then let $\eta_j = \lim_{\underline{c} \to \infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i \in N \setminus J_0} u_i'(\underline{c})}{u_j'(r_{j,\infty}(x)\underline{c})(n - |J_0|)}$, and it must be the case that $\eta_j < 1$. By the definition of limit for all $\varepsilon > 0$ $\exists \bar{c}(\varepsilon) > 0$ such that for all $\underline{c} > \bar{c}(\varepsilon)$, $\left| \frac{\sum_{l=1}^{n} f_l'(\underline{c}) + \sum_{i \in N \setminus J_0} u_i'(\underline{c})}{u_j'(r_{j,\infty}(x)\underline{c})(n - |J_0|)} - \eta_j \right| < \varepsilon$. Pick ε' such that $0 < \varepsilon' \leq 1 - \eta_j$, and let $m_1^* = \max\{\min\{m : r_{j,m}(x) = r_{j,\infty}(x) \ \ \textit{for all } j \in J_1\},$ $\min\left\{ m : x_n \prod_{l=1}^{m} \beta_l > \bar{c}(\varepsilon') \right\}\right\}$. Then for all $m \geq m_1^*$, $\frac{\sum_{l=1}^{n} f_l'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right) + \sum_{i \in N \setminus J_0} u_i'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)}{u_j'\left(r_{j,m}(x) m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)(n - |J_0|)} < \eta_j +$

$\varepsilon' \leq 1$ for all $j \in J_1$. The first order condition of the redistribution requires that $\frac{\sum_{l=1}^{n} f_l'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)+\sum_{i\in N\setminus J_0} u_i'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)}{u_j'\left(x_j \prod_{l=1}^{m} \kappa_{j,l} - t_j\left(x^{(m)}\right)\right)p\left(m_{CU}^*\right)n} = 1$ without any constraints on transfer, where we tentatively assume that $J_0 \supseteq J_1$ is the group of rich social members with endowment vector $x^{(m)}$. With the constraints on transfer, $m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)} \geq x_n \prod_{l=1}^{m} \beta_l$ and hence $r_{j,m}(x)\, m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)} \geq x_j \prod_{l=1}^{m} \kappa_{j,l}$. Then $x_j \prod_{l=1}^{m} \kappa_{j,l} - t_j\left(x^{(m)}\right) \leq r_{j,m}(x)\, m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}$, and for all $j \in J_1$, $\frac{\sum_{l=1}^{n} f_l'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)}{u_j'\left(x_j \prod_{l=1}^{m} \kappa_{j,l} - t_j\left(x^{(m)}\right)\right)(n-|J_0|)} + \frac{\sum_{i\in N\setminus J_0} u_i'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)}{u_j'\left(x_j \prod_{l=1}^{m} \kappa_{j,l} - t_j\left(x^{(m)}\right)\right)(n-|J_0|)} < 1$ for any $t_j\left(x^{(m)}\right) \geq 0$ by the concavity of egoistic utility functions. Thus, with the constraints on transfer, for all $m \geq m_1^*$, $t_j\left(x^{(m)}\right) = 0$ for all $j \in J_1$.

We then consider case *ii*, if for all $j \in J_2$, $\exists \epsilon > 0$ such that $\lim_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c})+\sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'([r_{j,\infty}(x)+\epsilon]\underline{c})(n-|J_0|)} < 1$, then let $\eta_j = \lim_{\underline{c}\to\infty} \frac{\sum_{l=1}^{n} f_l'(\underline{c})+\sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'([r_{j,\infty}(x)+\epsilon]\underline{c})(n-|J_0|)}$, and it must be the case that $\eta_j < 1$. By the definition of limit for all $\epsilon > 0$ $\exists \bar{m}(\epsilon) > 0$ such that for all $m > \bar{m}(\epsilon)$, $|r_{j,m}(x) - r_{j,\infty}(x)| < \epsilon$, and for all $\varepsilon > 0$ $\exists \bar{c}(\varepsilon) > 0$ such that for all $\underline{c} > \bar{c}(\varepsilon)$, $\left| \frac{\sum_{l=1}^{n} f_l'(\underline{c})+\sum_{i\in N\setminus J_0} u_i'(\underline{c})}{u_j'([r_{j,\infty}(x)+\epsilon]\underline{c})(n-|J_0|)} - \eta_j \right| < \varepsilon$. Pick ε' such that $0 < \varepsilon' \leq 1 - \eta_j$, and let $m_2^* = \max\left\{\min\left\{m : r_{j,m}(x) < r_{j,\infty}(x) + \epsilon \text{ } for\ all\ j \in J_2\right\}, \min\left\{m : x_n \prod_{l=1}^{m} \beta_l > \bar{c}(\varepsilon')\right\}\right\}$. Then for all $m \geq m_2^*$, $\frac{\sum_{l=1}^{n} f_l'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)+\sum_{i\in N\setminus J_0} u_i'\left(m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)}{u_j'\left(r_{j,m}(x) m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)}\right)(n-|J|)} < \eta_j + \varepsilon' \leq 1$ for all $j \in J_0$. The rest part of the proof will be similar to the one presented for case *i*. With the constraints on transfer, for all $m \geq m_2^*$, $t_j\left(x^{(m)}\right) = 0$ for all $j \in J_2$.

Let $m^* = \max\{m_1^*, m_2^*\}$. Then with the constraints on transfer, for all $m \geq m^*$, $t_j\left(x^{(m)}\right) = 0$ for all $j \in J_0$. Hence J_0 is indeed the group of rich people social members with endowment vector $x^{(m)}$, and it must be the case that $m_{CU}^*\left(x^{(m)}\right)\bar{x}^{(m)} = x_n \prod_{l=1}^{m} \beta_l$, indicating that the social minimum that can be supported by a benevolent utilitarian social planner will equal the endowment of the poorest social member after sufficiently many rounds of mixed increase in endowments. ■

Corollary 6. *If* $\exists R \subseteq J_0$ *such that for all* $j \in R$, *either* $\lim_{\underline{c}\to\infty} MRC_j\left(\underline{c}, r_{j,\infty}(x)\,\underline{c}, n - |R|\right) < 1$ *in case i, or* $\exists \epsilon > 0$ *such that* $\lim_{\underline{c}\to\infty} MRC_j\left(\underline{c}, [r_{j,\infty}(x) + \epsilon]\,\underline{c}, n - |R|\right) < 1$ *in case ii, then* $\exists m^* \in \mathbb{N}$ *such that for all* $m \geq m^*$, $t_j\left(x^{(m)}\right) = 0$ *for all* $j \in R$.

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