Non-Exponential Growth Theory*

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Abstract

To explain the observed stability in real GDP growth, endogenous growth theories typically need a knife-edge degree of externality, which is not supported by microlevel observations. We develop a model where a constant number of new goods are introduced per unit of time and focus on the movement of prices and quantities after introduction. In this environment, positive real GDP growth, as measured by SNA statistics, does not necessarily mean exponential growth in the quantity, quality, or variety of final outputs. We derive the conditions under which measured growth can be sustained, which are less restrictive than typical knife-edge assumptions.

Keywords: endogenous growth theory, knife-edge condition, externality, variety expansion, product lifecycle, balanced growth.

JEL Classification Codes: O41, O31.

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(a) Log per Capita Real GDP, USA (b) Per Capita Real GDP, Linear Scale

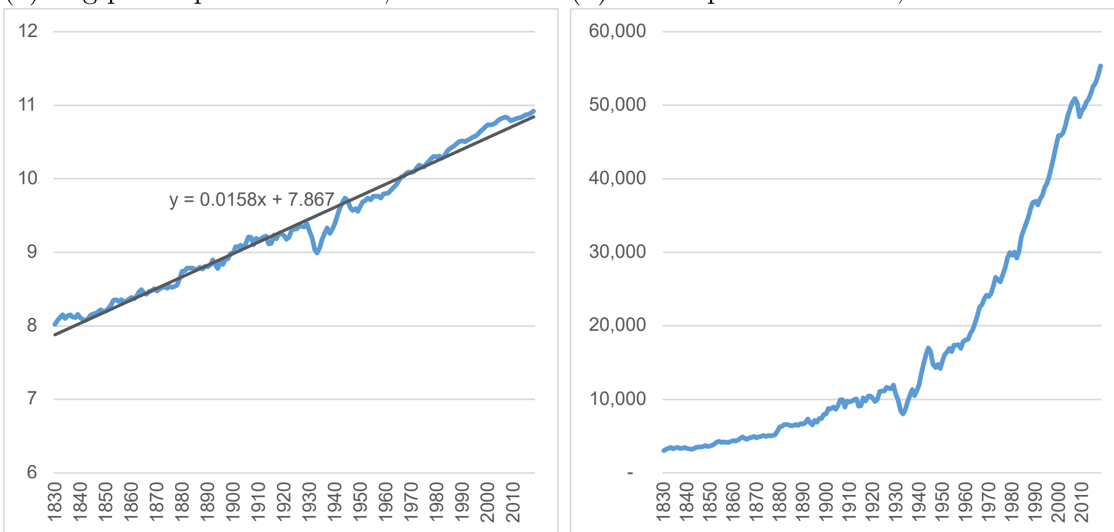


Figure 1: Long-term Evolution of Real GDP per Capita in the United States from 1830 to 2018 (2011 International Dollar). Source: Madison Project, Bolt and van Zanden (2020).

1 Introduction

Since around the time the First Industrial Revolution was completed, the growth in real GDP per capita has been surprisingly stable in the United States. Figure 1(a) depicts the time series of the real GDP on a log scale, where the slope of the series represents the growth rate. Although there have been short- to midterm fluctuations, the figure clearly shows that the log of the real GDP per capita closely follows a linear trend, implying that the long-term rate of per capita GDP growth is almost constant. Figure 1(b) shows the time path of the U.S. real GDP per capita on a linear scale without taking the logarithm. Given that the GDP growth rate is stable, it is commonly understood that the level of real GDP per capita is increasing exponentially in the long run.

Given these findings, it is natural for existing studies on endogenous growth to explain the phenomenon of long-term growth via models in which the per capita output continues to grow exponentially. Initially, this was an extremely challenging task because reproducible inputs are subject to diminishing returns, which implies that the accumulation of those factors cannot explain the exponential growth by themselves. Seminal studies in the endogenous growth theory thus overcame this challenge by assuming the presence of strong intertemporal knowledge spillovers.

In variety-expansion models (e.g. Romer, 1990; Grossman and Helpman, 1991a), the productivity of new R&D is assumed to increase as knowledge accumulates with the past stock of R&D. To sustain economic growth, the elasticity of this spillover ϕ needs to equal one. Similarly, in quality ladder models (e.g. Grossman and Helpman, 1991b; Aghion and Howitt, 1992), the increment in quality by a successful new R&D depends on the quality of the existing good, which is a result of the past stock of R&D. Sustained growth requires the increments to be proportional to the existing quality, which means that the elasticity of the externality should again be one. Finally, in AK-type growth models (e.g. Romer, 1986; Rebelo, 1991), the elasticity of production with respect to all reproducible factors and the elasticity of their externality effects must add up to one.[1] In almost all endogenous growth models, long-term growth can be sustained only when one such knife-edge condition is satisfied.[2]

Nevertheless, a puzzle still remains. Indeed, the externality and nonrivalry of knowledge play essential roles in improving productivity (e.g. Griliches, 1998). However, if we look at the spillover process more precisely, no concrete evidence supports any of these assumptions. Klenow and Rodriguez-Clare (2005, Section 3) reviewed various AK-type models. They concluded that such models are empirically implausible based on the lack of a tight enough relationship between the investment rates and growth rates in cross-country data. For the elasticity of spillover ϕ in R&D-driven growth models,

[1] When there are multiple sectors, at least one sector that produces a reproducible factor (typically physical capital or human capital) must satisfy this restriction. For example, Lucas (1988) initially introduced a human capital accumulation function $\dot{h}_t = h_t^{\phi} G(1 - u_t)$ and then made the assumption $\phi = 1$, following Uzawa (1965). After doing so, he wrote, "the feature that recommends his formulation to us is that it exhibits sustained per capita income growth," which gives a clear example of a case where such a knife-edge assumption is justified not by microlevel observations but rather by the aggregate outcome. Lucas noted that "human capital accumulation is a *social* activity," which suggests that the elasticity $\phi = 1$ includes the effect of externalities.

[2] Growiec (2007, 2010) formally proved that, with any generalization in functional form, exponential growth cannot be explained without imposing at least one knife-edge assumption in the model.

Jones (1995) clearly stated, *"$\phi = 1$ represents a completely arbitrary degree of increasing returns and... is inconsistent with a broad range of time series data on R&D and TFP growth."* He convincingly stated that $\phi = 0$ is the most natural case, and while ϕ can either be negative by the "fishing out effect" or positive by the "better tools effect," it is reasonable to assume that $\phi < 1$. Bloom et al. (2020) estimated the degree of diminishing returns $(1 - \phi)$ in research productivity in various industries and reported that ϕ is significantly less than one (even negative) for almost all industries. They concluded that improving the quality of goods at a constant exponential rate is becoming more difficult.

A possible answer to this puzzle is *semiendogenous* growth theory with $\phi \in (0, 1)$, where the long-term rate of growth is ultimately driven by population growth. However, Jones (2022) predicted that economic growth will eventually come to an end, given that there are upper limits on population, research intensity, and education attainment. Under the natural assumption of $\phi = 0$, this paper presents an alternative possibility, i.e., that the measured economic growth can continue indefinitely with a constant population, by developing a new theory.

Overview of the mechanism

This paper presents a theory that explains the stability of the observed real GDP growth rate by considering the vintages of products and their product lifecycle. In this setting, we will show that the measured GDP growth rate becomes positive under more agreeable conditions than a knife-edge level of externality, as assumed in existing endogenous growth models. Note that the focus of the paper is measured GDP growth because the notion of steady (or balanced) growth comes from the GDP data measured in System of National Account (SNA) statistics (the NIPA in the U.S.).

Recall that we first presented the (log) level of GDP in Figure 1, and then discussed real GDP growth. However, in SNA statistics, the GDP data are constructed in reverse order. Statistical agencies first calculate the real GDP growth rate by comparing quantities of various product groups in adjacent years, using the same set of prices for both years. Then, they construct the aggregate level of real GDP via the chain rule:

$$\Big[\text{Real GDP at year } T\Big] = \Big[\text{Real GDP at reference year } t_0\Big] \times \prod_{t=t_0+1}^{T} (1 + g_{t,t-1}),$$

where $g_{t,t-1}$ is the measured real GDP growth rate between year t and year $t-1$.[3] Therefore, the fact that the time series of measured per capita real GDP exhibits exponential growth only means that the series of $g_{t,t-1}$, from which the real GDP is calculated, is stationary. Because the composition of final goods differs across time, it is not evident whether the stationarity in the $g_{t,t-1}$ series implies exponential growth in the quantity or quality of any particular final good. In fact, the quantity or quality of no particular good needs to grow exponentially to sustain the $g_{t,t-1}$ series at a positive level. (See Appendixes B.1 and B.2 for the two simplest examples.) Additionally, sustained growth in the measured GDP does not necessarily correspond to similar improvements in utility or welfare in the long run.[4]

Given that there is no need to explain the exponential increase in any good, less restrictive assumptions are sufficient to explain the fact that the measured real GDP has been growing steadily. To replicate the environment where the real GDP growth rate is calculated by statistical agencies, we consider a stylized model in which new final goods are gradually introduced and explicitly focus on their prices and quantities over their lifecycle. In this multiproduct setting, we show that the measured GDP growth rate becomes a positive constant when the following is true: (i) new goods (or services) are continually introduced to the market; (ii) the quality-adjusted price of each good decreases as they get older compared to newer goods; and (iii) the expenditure share for very old goods is limited. Condition (i) does not require the number of goods to increase exponentially. Conditions (ii) and (iii) state that the price and quantity for each good should follow the well-observed pattern of the product lifecycle. This type of economic movement does not require a knife-edge level of externality; this is in contrast to existing endogenous growth models where some variables need to grow exponentially. Nevertheless, knowledge externalities are crucial for growth, as they often

[3]The real GDP at reference year t_0 can be set arbitrarily because this is simply an index.

[4]See Appendix B.3 for more discussion using examples.

work behind the quality improvements and cost reductions of existing goods, and our prototype endogenous model incorporates these. However, we show that the fall in quality-adjusted prices does not need to occur at an exponential speed. As a result, a weaker externality is sufficient for sustaining measured real GDP growth.

Some recent studies view long-term growth differently than an exponential increase in final output at the rate of measured GDP growth. León-Ledesma and Moro (2020) considered a two-sector model and calculated the growth rate via the methodology employed by the NIPA. They showed that the shift in the expenditure share from goods to services explains cross-country growth. The current paper proposes that continual shifts in expenditure shares from older goods and services to newer goods and services are behind the stability in measured GDP growth. Aghion et al. (2019) examined the possibility that the measured GDP growth rate underestimates the welfare gains from creative destruction. In addition to their study, the current paper shows another fundamental reason why the measured GDP growth may not represent a similar increase in welfare in the long run. Philippon (2022) suggested that a linear trend fits the TFP data better than an exponential trend for periods ranging from several decades to a few centuries. According his theory, long-term GDP growth can be sustained only when there are occasional changes in the linear trend (e.g., by the arrival of general-purpose technologies), and the slope of the linear trend needs to increase exponentially. This paper explores a mechanism that does not require exponential increases and a knife-edge degree of externalities, even in the very long run.

The rest of the paper is constructed as follows. Section 2 presents a stylized but fairly general theory that provides the condition under which measured real GDP growth can be sustained in a setting without exponential expansion. Based on this theory, Section 3 develops a prototype R&D-based endogenous growth model. Without requiring knife-edge conditions, the model shows that innovation continues and that the measured GDP growth remains positive. Section 4 generalizes the theory and the prototype model in several directions to demonstrate that we can obtain a positive constant real GDP growth rate in wider (even less restrictive) situations. Section 5 concludes the paper.

2 Theory

This section theoretically derives the condition under which the real GDP growth rate, as measured by the SNA, can be sustained. In a setting where new goods are continually introduced, but not at an exponential speed, we show that the sustainability of measured GDP growth depends on the pattern of changes in prices and quantities in the product lifecycle. The results suggest various possibilities in constructing general equilibrium models where measured GDP growth can be sustained under less restrictive assumptions than typical knife-edge settings. A simple prototype model is presented in Section 3.

2.1 Measuring GDP Growth with Vintages of Goods

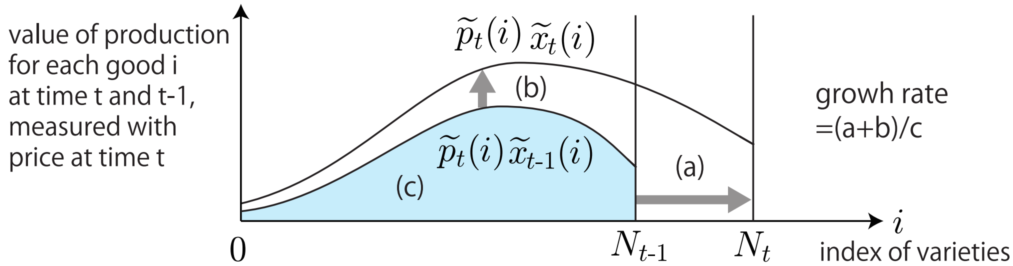
Let us consider an economy with a constant population and many goods. While we follow a convention in the variety-expansion model by calling them goods, it is more suitable to think of each good in theory as a group of products or services based on the same technology. Each good is indexed by $i \in [0, N_t]$, where $i = 0$ is the oldest, and $i = N_t$ is the most recently introduced good. The number of goods N_t increases whenever new goods are introduced.[5]

Let $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ denote the price and quantity of each good i at time t. We normalize the price level at each instant to keep the nominal expenditure (per capita) constant in the long run. As in SNA statistics, we define $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ as quality-adjusted values. For example, if the quality of good i is doubled (so that consumers receive the same utility from half the quantity), then our measure of $\widetilde{x}_t(i)$ is doubled, whereas that of $\widetilde{p}_t(i)$ is halved.

In this stylized environment, we follow the SNA statistics methods to calculate the real GDP growth rate. This can be done by comparing the values of all final outputs between two consecutive years, e.g., year $t-1$ and year t. Their values are measured via the common set of prices, which is usually the set of observed prices in a given base year. Because the base year is frequently updated in official statistics and because this paper is interested in long-term dynamics, we suppose that there is no gap between the base

[5] N_t includes the number of goods that are no longer produced.

Case 1: When $\widetilde{x}_t(i)$ is always increasing in t.



Case 2: When $\widetilde{x}_t(i)$ decreases with t sometime after introduction.

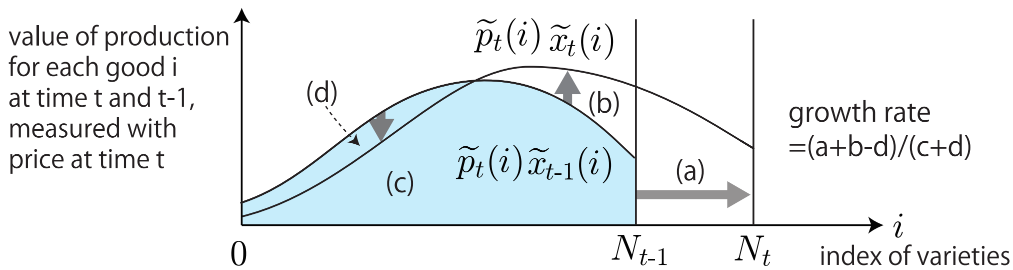


Figure 2: Calculation of the Real GDP Growth Rate: Two Cases.

year and the year in which the growth rate is computed.[6] Then, the real GDP growth rate between years $t-1$ and t can be written as follows:

$$g_{t-1,t} = \frac{\int_{N_{t-1}}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di + \int_0^{N_{t-1}} \widetilde{p}_t(i)\left(\widetilde{x}_t(i) - \widetilde{x}_{t-1}(i)\right)di}{\int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)di}. \tag{1}$$

This equation is composed of the integrals of two functions: $\widetilde{p}_t(i)\widetilde{x}_t(i)$ and $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. Figure 2 depicts the curves of these two functions against the index of varieties i for two cases, i.e., when the quantity of existing goods always increases with time (Case 1) and when the quantity of existing goods shrinks in some part of their lifecycle (Case 2). In Case 1, area (a) represents the sum of the values of new goods introduced between time $t-1$ and time t, evaluated by the prices at time t. Similarly, area (b) represents the value of the increased production of goods that already existed at time $t-1$. These two areas

[6] In the U.S., the NIPA computes the growth rate in two ways, i.e., by setting the base year to t and by setting it to $t-1$. Then, the agency calculates the geometric average of the two values. Here, we only calculate the growth rate in which the base year is t; the difference disappears at the limit where the period length approaches 0, as we see in the next subsection.

measure how economic activity has increased from time $t-1$ to time t and correspond to the two terms in the numerator of (1). Area (c) represents the value of total production at time $t-1$, evaluated again by the prices at time t. This area corresponds to the denominator of Equation (1). In this way, the real GDP growth rate can be understood as the ratio of area (a)+(b) to area (c), which measures the rate at which the economic activity at time t increases from time $t-1$.

This procedure can be generalized to the case where the output quantity $\widetilde{x}_t(i)$ is not monotonic in t. Case 2 in Figure 2 illustrates an example where the production of a certain range of goods declines between periods $t-1$ and t. A portion of curve $\widetilde{p}_t(i)\widetilde{x}_t(i)$ then falls below curve $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. In this case, the real GDP growth rate is given by the ratio of area (a)+(b)−(d) to area (c)+(d).

2.2 Non-Exponential Steady State with Product Lifecycle

The fact that the measured U.S. real GDP growth rate has been stable for almost two centuries suggests that N_t, $\widetilde{p}_t(i)$, and $\widetilde{x}_t(i)$ in Equation (1) may have some steady-state properties in the long run. This subsection presents a simple notion of a steady state in the environment explained thus far. In particular, we focus on the steady-state dynamics where neither variety, quantity, nor quality expands exponentially. For ease of analysis, we describe the economy in continuous time throughout the rest of the paper.

Suppose that, in the long run, N_t increases by a positive constant n per unit of time as follows:

$$\dot{N}_t \to n > 0 \quad \text{as} \quad t \to \infty. \tag{2}$$

Recall that existing variety-expansion models require a strong and exact degree of knowledge spillover to maintain the exponential expansion of varieties, where $\dot{N}_t/N_t$ is constant. In contrast, the linear increase in N_t in Equation (2) does not require such strong knowledge spillovers within the R&D sector, as we will see in the general equilibrium model in Section 3.

Let $s(i)$ denote the time when good i is developed. It is convenient to label each good by its age, $\tau = t - s(i)$, i.e., the time passed from its introduction. Given that n new goods are introduced per unit of time, an age τ good is the $n\tau$th newest good. This means that the index of a good i and its age τ are related by the following:

$$i = N_t - n\tau, \text{ or equivalently, } \tau = t - s(i) = \frac{N_t - i}{n}. \tag{3}$$

With this notation, let us say that the economy has reached a steady state if every good's price and quantity follow the same time evolution with respect to τ. Formally, the economy can be said to be converging to a steady state if there exist time-invariant functions $p(\tau)$ and $x(\tau)$ such that

$$\widetilde{p}_t(i) \to p(t - s(i)) \equiv p(\tau), \quad \widetilde{x}_t(i) \to x(t - s(i)) \equiv x(\tau) \quad \text{as} \quad t \to \infty. \tag{4}$$

Let $T > 0$ denote the age beyond which the product is never produced. In typical variety-expansion endogenous growth models, goods never retire from the market. In this case, $T = \infty$. However, in practice, many products disappear after some time. Our theory can be applied to both cases where T is finite or infinite. We assume that $p(\tau)$ and $x(\tau)$ satisfy the following properties:

Assumption 1.
(i) Both $p(\tau)$ and $x(\tau)$ are nonnegative and continuous for all $0 \le \tau \le T$, where T is such that $x(\tau) = 0$ for all $\tau \ge T$. Additionally, they are differentiable for all $0 < \tau < T$.
(ii) T can be infinite, but $p(\tau)$ and $x(\tau)$ do not grow exponentially: $\lim_{\tau\to\infty} p'(\tau)/p(\tau) \le 0$ and $\lim_{\tau\to\infty} x'(\tau)/x(\tau) \le 0$ if $T = \infty$.[7]
(iii) The newest good's price and quantity are both positive: $p(0) > 0$ and $x(0) > 0$.[8]

With Assumption 1(i), the present paper focuses on the continuous setting because it is mathematically less demanding and does not sacrifice intuitions. Since $x(\tau)$ represents the quality-adjusted quantity, Assumption 1(ii), combined with Equation (2), guarantees that neither quantity, quality, nor variety grows exponentially in this economy. Assumption 1(iii) is an obvious assumption. When a new good appears in the market, it should imply that the expenditure for the good, $p(0)x(0)$, is positive.

[7] Note that the time derivative of the quantity in the steady state is $\dot{\widetilde{x}}_t(i) = \frac{d}{dt}x(t-s(i)) = x'(t-s(i)) = x'(\tau)$. Therefore, $x'(\tau)/x(\tau) = \dot{\widetilde{x}}_t(i)/\widetilde{x}_t(i)$ represents the growth rate of the quantity of age τ good, or equivalently, that of index $i = N_t - n\tau$ good. Similarly, $p'(\tau)/p(\tau) = \dot{\widetilde{p}}_t(i)/\widetilde{p}_t(i)$ in the steady state.

[8] In this paper, we use the term "positive" to mean greater than (not including) zero.

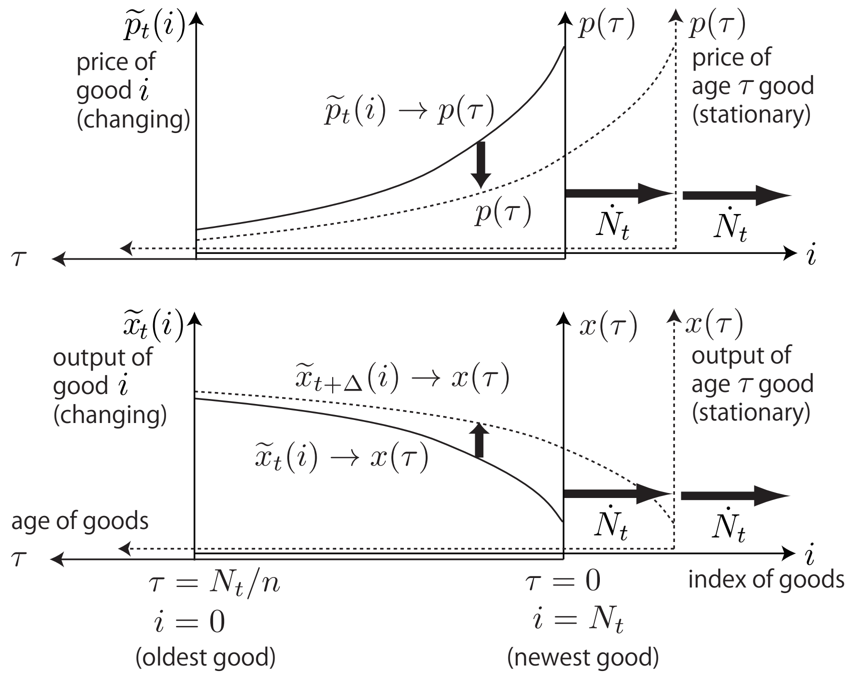


Figure 3: Evolution of Prices and Quantities in a Non-Exponential Steady State.

Definition 1. *A non-exponential asymptotic steady state is a situation in which the number of goods follows Equation (2), while the paths of quality-adjusted prices and quantities of goods, i.e.,* $\widetilde{p}_t(i)$ *and* $\widetilde{x}_t(i)$*, respectively, satisfy Condition (4) and Assumption 1.*

In the remainder of the paper, we use the term "steady state" unless doing so leads to confusion. Figure 3 intuitively depicts the evolution of the quality-adjusted prices and quantities in the above definition of the steady state. The graphs can be viewed in two ways, i.e., either drawn against the i-axis (index of goods) running from left to right or drawn against the τ-axis (age of goods) running in the opposite direction. The two variables, i and τ, are related according to Equation (3); however, the relationship changes over time as N_t increases. At time t, the origin of the τ-axis coincides with the point of $i = N_t$ on the i-axis because the newest good $i = N_t$ is age $\tau = 0$ at time t. As time passes, the origin of the τ-axis moves to the right with the speed of the introduction of new goods, $\dot{N}_t = n$, as does the position of the graph drawn against τ.

The upper panel of Figure 3 illustrates the schedule of quality-adjusted price $p(\tau)$, assuming that it decreases with age τ either because a product becomes cheaper or has higher quality as time passes after its introduction. Then, $\widetilde{p}_t(i)$ is increasing in i at any given time t since the newer goods have a larger index i. The figure also explains the movement of the price of each good $\widetilde{p}_t(i)$ over time. Even in the steady state where function $p(\tau)$ is stationary, the price of individual good $\widetilde{p}_t(i)$ shifts downward to the dotted curve because the position of function $p(\tau)$ continues to move to the right as new goods are developed.[9]

The lower panel of Figure 3 explains the evolution of quality-adjusted quantities of goods over time. The panel is drawn assuming that $x(\tau)$ is increasing in τ, which naturally matches our example in which the older goods have lower quality-adjusted prices. In this case, the demand for each good $\widetilde{x}_t(i)$ increases over time as the $x(\tau)$ function shifts to the right. Note that, however, Assumption 1(ii) rules out exponential growth in the quantity of any good. Even when $T = \infty$, the growth rate of $x(\tau)$ must be either zero or negative as $\tau \to \infty$.

Similar to Case 2 of Figure 2, we can also consider a steady state in which the quantity may decrease with age, even though older goods are less expensive. Such a pattern will emerge when consumers do not like outdated goods or if newer goods replace parts of functions that are provided by older goods, as we discuss later in Subsection 4.1.

2.3 Measured Real GDP Growth Rate in the Steady State

Now, we examine whether the non-exponential steady state explained in Section 2.2 implies a positive and constant real GDP growth rate. Note that the conventional definition of real GDP growth in Equation (1) gives the average growth rate between two discrete periods. To map this definition to a continuous-time growth model, it is convenient to consider the instantaneous growth rate g_t at time t. This can be obtained

[9] Although this is a convenient way to explain the steady-state dynamics, note that the economic environment, such as technology, preference, and market structure, first determines the evolution of the price of individual goods $\widetilde{p}_t(i)$ in equilibrium. Then, the long-term pattern of movement in $\widetilde{p}_t(i)$ shapes the stationary $p(\tau)$ function.

by replacing $t-1$ in Equation (1) with $t-\Delta$ and taking the limit of $\Delta \to 0$ in $g_{t-\Delta,t}/\Delta$.[10]

$$g_t = \lim_{\Delta \to 0} \frac{g_{t-\Delta,t}}{\Delta} = \frac{\dot{N}_t \cdot \widetilde{p}_t(N_t)\widetilde{x}_t(N_t) + \int_0^{N_t} \widetilde{p}_t(i)\dot{\widetilde{x}}_t(i)di}{\int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}. \tag{5}$$

Suppose that the economy converges to a steady state, as defined in Definition 1. The number of goods grows linearly, and the evolution of prices and quantity in terms of age becomes stationary. Given that $\int_0^T p(\tau)x(\tau)d\tau$ is finite, the long-term growth rate can be obtained by substituting Equations (2)-(4) into Equation (5).[11]

$$g_t \to g \equiv \lim_{\overline{T} \to T} \frac{np(0)x(0) + n\int_0^{\overline{T}} p(\tau)x'(\tau)d\tau}{n\int_0^{\overline{T}} p(\tau)x(\tau)d\tau} \quad \text{as} \quad t \to \infty. \tag{6}$$

The interpretation of the growth rate in Equation (6) is essentially the same as that in Equation (1), except that growth is now represented in terms of age and in continuous time. In the numerator, $np(0)x(0)$ represents the value of newly introduced goods, whereas $n\int_0^{\overline{T}} p(\tau)x'(\tau)d\tau$ represents the value of changes in quantities of existing goods given price function $p(\tau)$. Both terms are multiplied by n because there are n goods per unit of age. The sum of these terms reflects the speed of increase in economic activity. The denominator of Equation (6), $n\int_0^{\overline{T}} p(\tau)x(\tau)d\tau$, gives the total value of

[10]SNA statistics use the cumulative output of good i for a given period (e.g., a year or a quarter) when constructing the growth rate between adjacent periods. Specifically, to match this definition precisely, we need to integrate $\widetilde{x}_t(i)$ for the duration of the period and apply the result to Equation (1). As we take the limit where the duration of one period is almost zero, we confirm that this exact GDP growth rate converges to the expression in Equation (5):

[11]Equation (6) can be obtained from Equation (5) as follows. First, we substitute $p(\tau)$ and $x(\tau)$ for $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$. Similarly, $\dot{\widetilde{x}}_t(i)$ can be written as $x'(\tau)$ (see footnote 7). Next, we change the integration variable from di in Equation (5) to $d\tau$. By differentiating Equation (3) for a given t, we obtain $di = nd\tau$. We also need to change the integration interval. From Equation (3), $i = 0$ and $i = N_t$ correspond to $\tau = N_t/n$ and $\tau = 0$, respectively, as illustrated in Figure 3. As $t \to \infty$, N_t/n also approaches ∞. From these, $\lim_{t\to\infty} \int_0^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di = \lim_{t\to\infty} \int_{N_t/n}^0 p(\tau)x(\tau)(-n)d\tau \to n\int_0^\infty p(\tau)x(\tau)d\tau$. However, since $x(\tau) = 0$ for $\tau \geq T$, the limit becomes $n\int_0^T p(\tau)x(\tau)d\tau$. Similarly, the limit of the numerator of Equation (5) is $np(0)x(0) + n\int_0^T p(\tau)x'(\tau)d\tau$. If T is finite, then both limits are finite; therefore, we can substitute these limits into Equation (5). However, when $T = \infty$, then the limits may be infinite. In this case, we use a large but finite $\overline{T}$ in place of T before substituting them into Equation (5) and take the limit of $\overline{T} \to T = \infty$ for the whole fraction, as shown in Equation (6):

existing production, i.e., the nominal GDP of the economy given prices $p(\tau)$. The ratio of the two yields the real GDP growth rate.

The following proposition provides a simpler formula for the long-term GDP growth rate in the steady state.

Proposition 1. *Suppose that the economy converges to a non-exponential asymptotic steady state, as defined by Definition 1. Then, the real GDP growth rate g_t asymptotes to g in the long run, where g is given as follows:*
(i) If $\int_0^T p(\tau)x(\tau)d\tau$ is finite (which is always true when T is finite), then[12]

$$g = \frac{-\int_0^T x(\tau)dp(\tau)}{\int_0^T p(\tau)x(\tau)d\tau}. \tag{7}$$

(ii) If $\int_0^T p(\tau)x(\tau)d\tau = \infty$, then $g = 0$.

Proof. (i) When $\int_0^T p(\tau)x(\tau)d\tau$ is finite, we can take away $\lim_{\overline{T}\to T}$ in the RHS of Equation (6) and replace $\overline{T}$ with T. In its numerator, integration by parts implies that $\int_0^T p(\tau)x'(\tau)d\tau = p(T)x(T) - p(0)x(0) - \int_0^T p'(\tau)x(\tau)d\tau$, where $p(0)x(0)$ cancels out. When T is finite, $p(T)x(T) = 0$. When $T = \infty$, the finiteness of $\int_0^T p(\tau)x(\tau)d\tau$ implies that $\lim_{\tau\to\infty} p(\tau)x(\tau) = 0$ (i.e., $p(T)x(T) = 0$). Therefore, we obtain Equation (7).
(ii) In this case, T is necessarily ∞. If $\int_0^\infty p(\tau)x'(\tau)d\tau$ is finite, then the result directly follows from Equation (6). Now, suppose that $\int_0^\infty p(\tau)x'(\tau)d\tau$ is either $+\infty$ or $-\infty$. Since both the numerator and the denominator in Equation (6) are infinite, we apply L'Hospital's rule to Equation (6) to obtain the following:

$$g = \lim_{\overline{T}\to\infty} \frac{p\left(\overline{T}\right) x'\left(\overline{T}\right)}{p\left(\overline{T}\right) x\left(\overline{T}\right)} = \lim_{\overline{T}\to\infty} \frac{x'\left(\overline{T}\right)}{x\left(\overline{T}\right)} \leq 0, \tag{8}$$

where the last inequality follows from Assumption 1(ii). In the following, we show that $g < 0$ does not occur by contradiction. For g to be strictly negative, $x(\tau)$ needs to shrink exponentially, which also means that $x'(\tau)$ must shrink exponentially. However, from $\lim_{\tau\to\infty} p'(\tau)/p(\tau) \leq 0$ in Assumption 1(ii), $\int_0^T p(\tau)x'(\tau)d\tau$ is finite since $p(\tau)x'(\tau)$ should shrink exponentially. Therefore, $g < 0$ contradicts the initial assumption that $\int_0^T p(\tau)x'(\tau)d\tau$ is either $+\infty$ or $-\infty$. □

[12] Note that $\int_0^T x(\tau)dp(\tau)$ is equivalent to $\int_0^T p'(\tau)x(\tau)d\tau$ given that $p'(\tau)$ exists.

Although Equation (7) has a simple form, it includes the contribution from the new goods since it is mathematically equivalent to Equation (6) as long as $\int_0^T p(\tau)x(\tau)d\tau < \infty$. Additionally, it accounts for any negative effect on g from disappearing goods when T is finite.[13] Proposition 1 immediately implies the requirements for positive long-term GDP growth.

Corollary 1. *The long-term real GDP growth rate g is a positive and finite constant if and only if the following two conditions are satisfied:*[14]

$$-\int_0^T x(\tau)dp(\tau) \text{ is positive and finite, and} \tag{9}$$

$$\int_0^T p(\tau)x(\tau)d\tau \text{ is finite.} \tag{10}$$

The expression in Condition (9), $-\int_0^T x(\tau)dp(\tau)$, is the numerator of Equation (7). It represents the cumulative reduction in the quality-adjusted price of a good during its product lifecycle. When the quality-adjusted price of goods declines, consumers have more purchasing power, thus improving their utility. This income effect from price reductions is more significant when the quantity of the good is greater. Therefore, in Condition (9), the price reduction $-dp(\tau)$ is weighted by quantity $x(\tau)$ and then integrated. The integrated sum gives the total income effect that one product generates over its product lifecycle. The expression $\int_0^T p(\tau)x(\tau)d\tau$ in Condition (10) is the denominator of Equation (7). It is the cumulative expenditure that one product attracts over its lifecycle. Proposition 1 says that if every product follows the same lifecycle pattern, then the real GDP growth rate in the economy is given by the ratio of the two. If both values are positive and finite in a non-exponential steady state, as defined in Definition 1, this indicates that the real GDP growth rate, as measured by the SNA, can be sustained even when no variable grows exponentially. We first provide three examples in the following

[13]Note that Assumption 1(i) assumes that $x(\tau)$ is continuous in τ up to $\tau = T$, where $x(T) = 0$. Therefore, $x(\tau)$ must fall continuously to zero at the end of its lifecycle. This negative effect on g is included in the second term in the numerator of Equation (6) and hence in Equation (7).

[14]Note that $\int_0^T p(\tau)x(\tau)d\tau$ is always positive from Assumption 1; therefore, we require only finiteness in Condition (10).

subsection and then discuss the implications of Conditions (9) and (10) in more detail in Subsection 2.5.

2.4 Graphical Examples

Proposition 1 shows that the real GDP growth rate depends only on functions $p(\tau)$ and $x(\tau)$. We can represent the growth rate graphically via the shapes of these two functions. Figure 4 provides three examples.

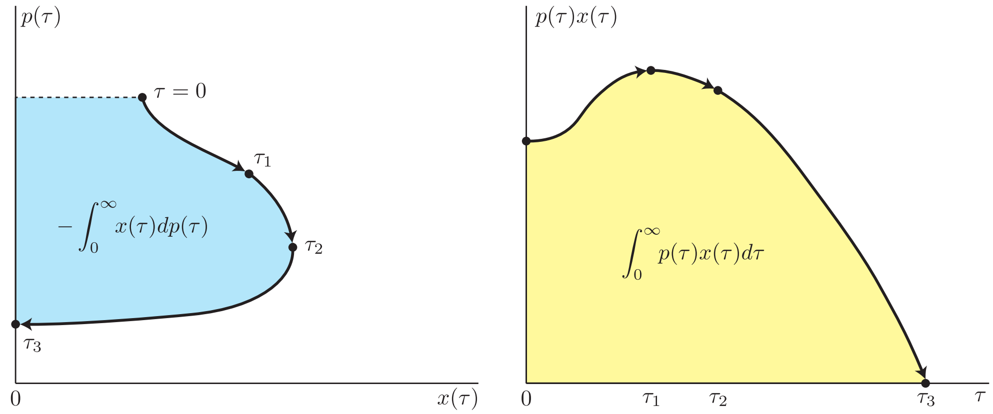
Example 1 shows the simplest case, where the quality-adjusted price (weakly) falls with age throughout the product lifecycle. The left panel depicts the evolution of $\{x(\tau), p(\tau)\}$ in the x-p diagram. T is finite in this example. The good enters the market at the point $\{x(0), p(0)\}$ and continues to be produced until its age reaches $T = \tau_3$. Then, the numerator, $-\int_0^T x(\tau)dp(\tau)$, can be expressed by the area that is encompassed by the locus of $\{p(\tau), x(\tau)\}$ and the vertical axis in the x-p diagram (shown in blue). This graphical representation can be interpreted as follows. Whenever the quality-adjusted price falls by $dp(\tau)$, either through cost reductions or through quality improvements, consumers can save the purchasing power by the amount of $-x(\tau)dp(\tau)$. The blue area shows the cumulative benefits of this good throughout its lifetime. The area is positive and finite as long as $p(0) < p(T)$.[15]

The right panel shows the evolution of expenditure for a good against its age, $p(\tau)x(\tau)$. The area below the curve (shown in yellow) gives the denominator, $\int_0^T p(\tau)x(\tau)d\tau$. According to Assumption 1, the expenditure for the good is positive at the time of introduction, and it evolves in the nonnegative region during its lifetime. Since expenditure $p(\tau)x(\tau)$ falls to zero at finite $T = \tau_3$, this area is positive and finite. Proposition 1 says that the ratio of the blue area to the yellow area gives the real GDP growth rate. Therefore, we can conclude that the real GDP growth rate in this example is positive and finite.

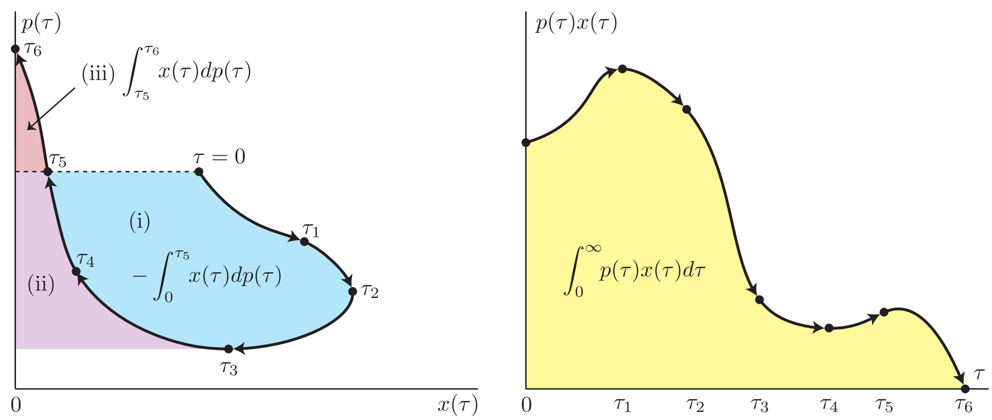
Next, Example 2 considers a case where $p(\tau)$ is not monotonic. As shown in the left panel, the quality-adjusted price begins to increase after τ_3 years. When the price of the good (relative to the newest good) rises during a part of its lifecycle (from $\tau = \tau_3$ to τ_6),

[15] $p(0) < p(T)$ requires the price to fall strictly with age in some part of a good's life.

Example 1: When T is finite and $p(\tau)$ is weakly decreasing



Example 2: When T is finite and $p(\tau)$ is nonmonotonic



Example 3: When $T = \infty$ and $p(\tau)$ is decreasing

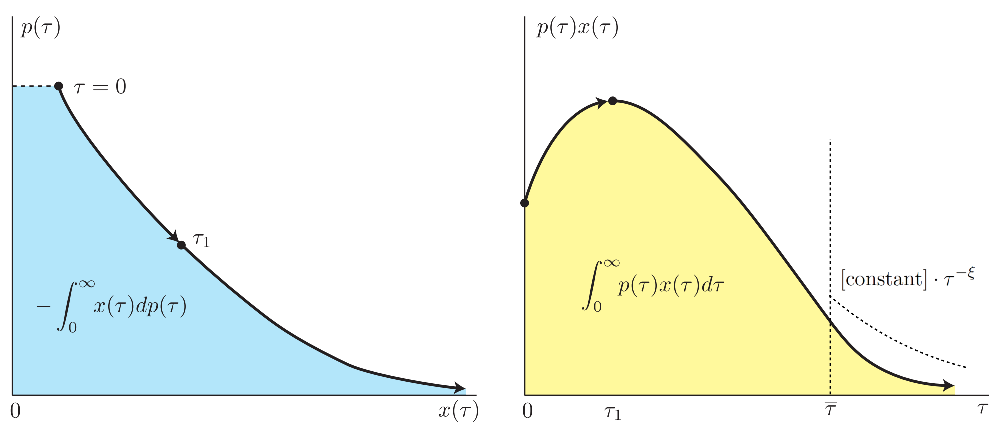


Figure 4: Graphical Representation of the Real GDP Growth Rate. The growth rate is measured by the ratio of the areas of the two panels.

the area between this part of the x-p locus and the vertical axis (marked as (ii) and (iii)) represents the loss of the purchasing power of consumers. This area needs to be deducted from the benefits of the fall in quality-adjusted prices from $\tau = 0$ to τ_3. Therefore, the numerator, $-\int_0^T x(\tau)dp(\tau)$, is given by area (i) minus area (iii) because area (ii) cancels out. It can be either positive or negative but is always finite since $T = \tau_6$ is finite. The yellow area in the right panel gives the denominator, $\int_0^T p(\tau)x(\tau)d\tau$, which is positive and finite. Therefore, the real GDP growth rate is finite, which is given by the ratio of the blue minus red area to the yellow area. Additionally, note that the growth rate becomes zero only by coincidence, only when the blue and red areas are the same size.

Finally, Example 3 shows a case in which the good stays in the market forever ($T = \infty$). The price $p(\tau)$ (relative to the newest good) falls throughout the lifecycle, and the quantity $x(\tau)$ remains positive as $\tau \to \infty$. For the yellow area to be finite, the expenditure on very old goods has to shrink. More concretely, Condition (10) is satisfied if expenditure for old goods is bounded by a polynomial function of age with a power of less than -1:[16]

$$p(\tau)x(\tau) \leq [\text{constant}] \cdot \tau^{-\xi} \text{ for all } \tau \geq \overline{\tau}, \tag{11}$$

for some $\xi > 1$ and $\overline{\tau} > 0$. The dotted curve in the right panel gives an example of such an upper bound. While we need a concrete model to determine whether Condition (11) is satisfied, let us note that the condition does not require an exponential decrease in expenditure. The RHS of Equation (11) decreases with age at the rate of ξ/τ for $\tau > \overline{\tau}$. The rate of decline in the quality-adjusted price, ξ/τ, can be arbitrarily close to zero when we choose a large $\overline{\tau}$. Therefore, there is no minimum rate at which the expenditure needs to decrease.

The blue area is positive, given that the quality-adjusted price falls throughout the product lifecycle. Combined with Condition (11), the GDP growth rate is also positive. The growth rate is finite if $p(\tau)$ is bounded away from 0 as $\tau \to \infty$.[17] If $p(\tau)$ falls to

[16] Suppose that Condition (11) is satisfied. Then, the denominator of Equation (7) is $\int_0^\infty p(\tau)x(\tau)d\tau \leq \int_0^{\overline{\tau}} p(\tau)x(\tau)d\tau + \int_{\overline{\tau}}^\infty [\text{constant}] \cdot \tau^{\xi} d\tau$. The first term is finite, and the second term becomes $[\text{constant}] \cdot \overline{\tau}^{1-\xi}/(\xi - 1)$, which is also finite:

[17] In this case, $x(\tau)$ must be finite as $\tau \to \infty$ since otherwise, $p(\tau)x(\tau)$ becomes infinite, contradicting

0 as $\tau \to \infty$, then the finiteness depends on the relationship between $p(\tau)$ and $x(\tau)$. Specifically, if the quantity depends only on price, then the area becomes finite if the price elasticity of the demand is less than one as the price approaches 0 from above.[18]

2.5 Discussion: Two Conditions for Sustained GDP Growth

The previous three examples illustrate that the measured real GDP growth rate in the steady state becomes positive and finite in various scenarios. Here, we discuss more generally when the two required conditions in Corollary 1 hold.

Condition (9): the quality-adjusted price falls during its product lifecycle

For this condition to be satisfied, $p(\tau)$ must fall with τ at least for a portion of its product lifecycle. Recall that we normalize the price level so that the nominal expenditure in the steady state is constant. This normalization also implies that the price of the newest goods when they appear does not change over time in the steady state. Therefore, Condition (9) only requires the quality-adjusted prices of older goods to decrease relative to those of newer goods, and it is not essential for the prices of individual goods measured in a currency to decrease. In terms of actual currencies, we can determine that $p(\tau)$ is decreasing if the quality-adjusted currency prices of individual goods lag behind the growth of the nominal per capita GDP.[19]

With this definition, the quality-adjusted price of a good may decrease with age for

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Condition (11). Given this, the blue area is finite.

[18]Suppose that we can define a static inverse demand function $P(x)$. Focusing on the case of $x \to \infty$ and $P(x) \to 0$, the blue area can be written as $p(0)x(0) + \int_{x(0)}^{\infty} P(x)dx$. If the price elasticity of the demand as $p \to 0$ is less than one, then the elasticity of $P(x)$ with respect to x as $x \to \infty$ is greater than one. This means that $P(x)$ is bounded by [constant] $\cdot\, x^{-\xi' x}$ for some $\xi' > 1$ for large x. Therefore, the integral is finite.

[19]Suppose that the per capita nominal GDP growth rate in dollars is $g^{\$}$. Note that in price normalization in our theory, nominal per capita expenditure is constant, which means that there is a $g^{\$}$ difference in the inflation rate between the prices in theory and in dollars. Then, in dollars, the rate of price change of age τ good is $p'(\tau)/p(\tau) + g^{\$}$. Therefore, we can determine that $p'(\tau)$ is negative if the quality-adjusted dollar prices of individual goods are increasing more slowly than $g^{\$}$.

several reasons. For example, the cost of production falls through learning-by-doing and knowledge spillovers. In this case, time and production experience will contribute to price reduction. Apart from cost reduction, changes in the form of competition may also lower prices because older goods are typically less protected from competition by patents and trade secrets than newer goods are.

Price reductions also occur in the form of quality improvements. For example, the effective price of computers has been declining for decades, not only because computers have become cheaper but also because the average performance of computers has drastically improved. SNA statistics record such changes as a decline in the quality-adjusted price.

Notably, our theory does not require an exponential decrease in the quality-adjusted price. If the quality improvements are exponential, then economic growth can easily be maintained, e.g., as in usual quality-ladder models. According to "Moore's law," the quality of computers has been improving at a constant rate; however, this trend of exponential improvement is expected to slow. In fact, computers are a remarkable exception in terms of continued improvements in performance. Most other products experience tapering in the rate of productivity improvement as they mature. Our theory shows that slowdowns in productivity increases in individual goods are consistent with a sustained rate of measured GDP growth, as long as a constant number of new products are introduced per unit time.

Finally, let us discuss the case in which the quality-adjusted price of the good increases for some part of its lifecycle, as we discuss in Example 2 of Figure 4. Although we need a concrete model to analyze how this happens and whether Condition (9) is satisfied, we discuss two possibilities here. One possibility is when products have antique or scarce value as they become very old. In this scenario, $p(\tau)$ increases only when $x(\tau)$ becomes considerably smaller than when it is newer. Another possibility is that producing a good in small lots costs more. This happens, for example, when a particular good continues to be produced to meet a niche demand, typically near the end of the product lifecycle.

The numerator of the equation, $-\int_0^T x(\tau)dp(\tau)$, is the weighted sum of the price changes, $dp(\tau)$, where the weights are the quantities, $x(\tau)$. Therefore, if the quantity

$x(\tau)$ tends to be small when $p(\tau)$ increases, then the negative effect of such movements on the GDP growth rate is likely to be limited. Therefore, even when the price at the end of the lifecycle $p(T)$ is higher than the initial price $p(0)$, the lifetime contribution of this good to the real GDP growth rate may well be positive, as in the case of Example 2.

Condition (10): The cumulative expenditure for a single good is finite

This condition requires the expenditure for older goods $p(\tau)x(\tau)$ to decrease so that they are effectively retired from the market in terms of expenditure share. The condition is always satisfied if the representative good ceases to be produced at finite age T. Even when the good stays in the market forever ($T = \infty$), the condition is satisfied if the expenditure decreases with age reasonably quickly (condition 11). Notably, the decline in the speed of expenditure does not need to be exponential.

The expenditure for the good can decrease with age for several reasons. One possibility is that the price decreases when the price elasticity of demand is less than one, at least for older goods. To illustrate this possibility, suppose that the demand for a good is determined solely by its price $p(\tau)$, and the price falls toward zero. Even when the good becomes almost free, it is unrealistic to expect consumers to demand an infinite amount of any particular product. This consideration suggests that the price demand elasticity of a product tends to be less than one when the price becomes sufficiently low, and the expenditure for the good will eventually vanish as $p(\tau) \to 0$. Section 3 presents a full endogenous growth model on the basis of this idea.

The expenditure for older goods can also decrease for other reasons. Sometimes, consumers are attracted by the novelty of new goods, but they become less interested as time passes. Advertisements for newer goods increase the speed of the obsolescence of older goods. Changes in the underlying economic environment may also make older goods useless. When these effects are present, Condition (10) may be satisfied regardless of the elasticity of demand. We extend the model to include obsolescence in Subsection 4.1.

3 A Prototype Non-Exponential Growth Model

This section presents a general equilibrium model that yields non-exponential steady-state dynamics. While the theory in the previous section suggests many ways to construct a model that achieves non-exponential growth while capturing various aspects of reality, this section presents the simplest prototype model to convey the substance of the non-exponential growth theory as clearly as possible. We discuss the generalizability of the prototype model in Section 4.

3.1 Consumers

Consider an economy with infinitely lived representative consumers of constant population L. At each point in time, each consumer supplies one unit of labor. The wage level is normalized to one.[20]

The lifetime utility function of the representative consumer is given by the following:

$$\int_0^\infty \left[\int_0^{N_t} u(\widetilde{c}_t(i))di\right] e^{-\rho t}dt, \tag{12}$$

which is separable across both time and goods. Note that the subutility function is symmetric across goods; thus, we do not consider the obsolescence of older goods in this simplest prototype model.

When demand depends only on price, as discussed in Example 3 of Section 2.4, positive GDP growth requires the price elasticity of demand for individual goods to be less than one, at least for older and cheaper goods. This setting is reasonable when the price is close to zero because it is not realistic for the expenditure for a single good to become infinite, as $p \to 0$. At the same time, it is reasonable to assume that the price elasticity is greater than one when the price is very high. Otherwise, the expenditure for a single good increases without bound as $p \to \infty$, which is also unrealistic. To incorporate these considerations in the simplest way, we consider a subutility function

[20] In the steady state where the fraction of consumption out of labor income is constant, this normalization implies that the nominal expenditure is constant, which is consistent with the theory in the previous section.

in which the elasticity changes at a threshold level $\widehat{c} > 0$:

$$u(\widetilde{c}_t(i)) = \begin{cases} \frac{\widetilde{c}_t(i)^{1-1/\varepsilon}}{1-1/\varepsilon} + \overline{u} & \text{for } \widetilde{c}_t(i) \geq \widehat{c}, \quad (0 < \varepsilon < 1) \\ \underline{u}\frac{\widetilde{c}_t(i)^{1-1/\widehat{\varepsilon}}}{1-1/\widehat{\varepsilon}} & \text{for } 0 \leq \widetilde{c}_t(i) < \widehat{c}, \quad (\widehat{\varepsilon} > 1) \end{cases} \tag{13}$$

where we specify parameters $\underline{u} = \widehat{c}^{1/\widehat{\varepsilon}-1/\varepsilon} > 0$ and $\overline{u} = (1/(1-1/\widehat{\varepsilon}) + 1/(1/\varepsilon-1))\widehat{c}^{1-1/\varepsilon} > 0$ so that both $u(c)$ and $u'(c)$ are continuous at $c = \widehat{c}$. Note that $u(c)$ is an increasing, continuously differentiable, and concave function of c, originating from $u(0) = 0$.[21]

The dynamic budget constraint of the representative consumer is given by the following:

$$\dot{k}_t = r_t k_t + 1 - \int_0^{N_t} \widetilde{p}_t(i)\widetilde{c}_t(i)di. \tag{14}$$

In equilibrium, the aggregate asset holding, Lk_t, should equal the value of all firms in the economy. Consumers maximize their lifetime utility (12) subject to the budget constraint (14), given interest rate r_t, prices of goods $\widetilde{p}_t(i)$ for $i \in [0, N_t]$, initial asset holding k_0, and the standard non-Ponzi game condition.

From the above, we obtain a piecewise isoelastic demand function for individual goods by the representative consumer:

$$\widetilde{c}_t(i) = \begin{cases} \lambda_t^{-\varepsilon}\widetilde{p}_t(i)^{-\varepsilon} & \text{if } \widetilde{p}(i) \leq \widehat{c}^{-1/\varepsilon}/\lambda_t, \\ (\lambda_t/\underline{u})^{-\varepsilon}\,\widetilde{p}_t(i)^{-\widehat{\varepsilon}} & \text{if } \widetilde{p}(i) > \widehat{c}^{-1/\varepsilon}/\lambda_t. \end{cases} \tag{15}$$

The shadow price of the budget constraint λ_t evolves according to the Euler equation $\dot{\lambda}_t = (\rho - r_t)\lambda_t$. Its initial value is determined so that the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$ is satisfied given the evolution of k_t in Equation (14).

3.2 R&D and Production Technologies

Each consumer works either as a production worker or as a researcher. A researcher succeeds in developing a new good with a Poisson probability of a per unit of time.

[21] In the following analysis, we focus mostly on the case in which all existing goods satisfy $\widetilde{c}_t(i) \geq \widehat{c}$. Nevertheless, having the second line in Equation (13) is theoretically important. If the elasticity of $u(c)$ is $0 < \varepsilon < 1$ for all $c \geq 0$, then $u(0)$ will necessarily be $-\infty$. This would be incompatible with a variety-expansion model, where the range of the integration (0-N_t) changes endogenously.

Let L_t^R denote the number of researchers in the economy, which is to be determined in equilibrium. Over time, the number of goods increases according to the following:

$$\dot{N}_t = aL_t^R. \tag{16}$$

Equation (16) is similar to standard variety expansion models, except that there is no spillover term from the stock of past R&D.

Once developed, each individual good is produced with a linear production technology that requires only labor. The output of good i is given by the following:

$$\widetilde{x}_t(i) = \widetilde{q}_t(i)\widetilde{l}_t(i), \tag{17}$$

where $\widetilde{l}_t(i)$ is the labor input and $\widetilde{q}_t(i)$ is the marginal product of labor in producing good i. Alternatively, we can interpret $\widetilde{x}_t(i)$ as the quality-adjusted output and $\widetilde{q}_t(i)$ as the quality of good i. In this case, one unit of labor produces one unit of good i with quality $\widetilde{q}_t(i)$. In either interpretation, we call $\widetilde{q}_t(i)$ the productivity for good i.

When any good is first developed, the productivity is normalized to 1. Then, as the production of this good proceeds, the productivity increases according to[22]

$$\dot{\widetilde{q}}_t(i) = I(\widetilde{x}_t(i)) \cdot \beta\widetilde{q}_t(i)^{\psi}, \quad 0 < \psi < 1, \tag{18}$$

where $I(\widetilde{x}_t(i))$ is an indicator function that takes a value of 1 when $\widetilde{x}_t(i) > 0$ and 0 otherwise. This means that productivity increases as long as production takes place. The specification in Equation (18) has a similarity to those in quality ladder models. There are knowledge spillovers from the past productivity of technology to the current productivity increments. Parameter $\psi \in (0, 1)$ specifies the degree of such spillovers. While quality ladder models need to assume that $\psi = 1$ to achieve an exponential increase in productivity (or quality), we do not make this knife-edge assumption. For the moment, we consider the case of $\psi \in (0, 1)$ and later compare the result to the case

[22] For simplicity, we assume that only experience in terms of time matters for productivity improvement. Alternatively, we can consider experience in terms of the cumulative production amount. Horii (2012) analyzed a model in the latter setting and derived a GDP growth rate defined in the same way as in Equation (1); however, it is a semiendogenous growth model that requires an exponentially growing population (c.f. Jones, 1995):

of $\psi = 1$. The parameter $\beta > 0$ represents other possible factors that affect the speed at which the productivity increases.

As long as $\widetilde{x}_t(i) > 0$, then Equation (18) is an autonomous differential equation in $\widetilde{q}_t(i)$. Similar to Section 2, let $\tau \equiv t - s(i)$ denote the age of the good. Then, the solution to the differential Equation (18) can be written as follows

$$q(\tau) = \kappa_1 \left(\tau + \kappa_0\right)^{\theta}, \tag{19}$$

where $\theta \equiv 1/(1-\psi) > 1$, $\kappa_0 \equiv \theta/\beta > 0$, and $\kappa_1 \equiv (\beta/\theta)^{\theta} > 0$. Given that $\psi \in (0, 1)$, the productivity improvement is less than exponential. The rate of increase in productivity is given by the following:

$$g_q(\tau) = \frac{q'(\tau)}{q(\tau)} = \frac{\theta}{\tau + \kappa_0} = \frac{\beta}{(1-\psi)\beta\tau + 1}. \tag{20}$$

In this specification, $g_q(\tau)$ takes the highest value at the time of introduction ($g_q(0) = \beta$) and then then falls to 0 as a good becomes older ($g_q(\infty) = 0$). This rules out the trivial possibility that the exponential increase in the productivity of individual goods explains the sustained GDP growth.

3.3 Behavior of Firms

Let us now turn to the behavior of production firms. While any product is protected by a patent forever, the patent *breadth* is limited (e.g. O'Donoghue, Scotchmer, and Thisse, 1998). This means that while other producers are prohibited from using the identical technology as the original inventor, they are allowed to produce similar products if they use a technology that is sufficiently different from the original. Alternatively, we may also think that a part of the technology is kept secret by the inventor and that the outsiders need to rely on less efficient technologies. In either case, outsiders face lower productivity than the original firm does.

To formalize this idea, let us assume that there are potentially many outside firms. These firms have partial access to the technology of the original inventor $\widetilde{q}_t(i)$ to produce the same good i. However, their productivity is $1/(1+\mu)$ times lower, where parameter μ represents the patent breadth or the strength of the trade secret. For simplicity, we assume that $0 < \mu < 1/(\widehat{\varepsilon} - 1)$. In this case, the profit-maximizing strategy is to set the limit price, which is $(1+\mu)$ times higher than the marginal cost.[23] Given the production function (17) and the fact that the wage is normalized to one, the pricing by a firm that has τ years of experience is as follows:

$$p(\tau) = \frac{1+\mu}{q(\tau)}. \tag{21}$$

3.4 Steady-State Equilibrium

Now, we derive the long-term property of the equilibrium dynamics in this prototype model. The following defines a notion of long-term equilibrium suitable for our model.

Definition 2. *An equilibrium path that satisfies the following properties as $t \to \infty$ is called the asymptotic steady-state equilibrium (ASSE).*

1. *The speed of the introduction of new goods converges to a positive and finite constant: $\dot{N}_t \to n^* > 0$.*
2. *The Lagrange multiplier of the budget constraint, λ_t, converges to a positive and finite constant: $\lambda_t \to \lambda^* > 0$.*

In the steady state, the equilibrium output of an age τ good is determined by Equations (15) and (21) with $\lambda_t = \lambda^*$ and does not depend on t:

$$x(\tau) = \begin{cases} D(\lambda^*)q(\tau)^{\varepsilon} & \text{if } q(\tau) \geq (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}, \\ \widehat{D}(\lambda^*)q(\tau)^{\widehat{\varepsilon}} & \text{if } q(\tau) < (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}, \end{cases} \tag{22}$$

where demand shifters $D(\lambda) = L((1+\mu)\lambda)^{-\varepsilon}$ and $\widehat{D}(\lambda) = L((1+\mu)\lambda/\underline{u})^{-\widehat{\varepsilon}}$ are decreasing functions of λ. The following lemma gives the condition under which the production of all existing goods is determined by the first line of Equation (22), where the price elasticity of demand is $\varepsilon < 1$.

[23] If the patent breadth was infinite, then the firms would choose monopoly pricing. In that case, the profit-maximizing markup would be $1/(\widehat{\varepsilon}-1)$ if the demand elasticity is $\widehat{\varepsilon} > 1$ and infinity if the elasticity is $\varepsilon < 1$. Since μ is lower than both, the firms set the limit price.

Lemma 1. *Suppose that $\widehat{c}$ is smaller than $\left(a\mu L \int_0^\infty q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau\right)^{-1}$. Then, in the ASSE, $q(\tau) \geq (1+\mu)\lambda^* \widehat{c}^{1/\varepsilon}$ for all $\tau \geq 0$.*
Proof: In Appendix A.1.

In the main text, we focus on the simple case where $\widehat{c}$ is sufficiently small so that the assumption in Lemma 1 is satisfied; we leave the analysis of the general case for Appendix A.2. Then, the profit of an age-τ firm is

$$\pi(\tau) = \mu D(\lambda^*) q(\tau)^{\varepsilon-1}. \tag{23}$$

The equilibrium values of n^* and λ^* are determined by the free entry condition for R&D and the labor market clearing condition. Let us first focus on the R&D condition. Recall that the Euler equation is $\dot{\lambda}_t/\lambda_t = \rho - r_t$. Since λ_t is stationary in the ASSE, the interest rate necessarily converges to $r_t \to \rho$. Using interest rate $r_t = \rho$ and the profit function (23), we can calculate the present value of a new firm just after it has succeeded in developing a new good:

$$V(\lambda^*) = \mu D(\lambda^*) \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau. \tag{24}$$

From the R&D function (16), the expected cost of developing a new good is $1/a$. Therefore, given that there is a positive flow of R&D, $n > 0$, and that the financial market is complete, the value of the new firm (24) should be equalized to the expected cost of development: $V(\lambda^*) = 1/a$. This condition gives the equilibrium value of $D(\lambda^*)$ in the ASSE:

$$D(\lambda^*) = \frac{1}{a\mu}\left(\int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1} \equiv D^*. \tag{25}$$

By substituting Equation (19) into Equation (25), we can calculate the value for D^*, which is always positive and finite.[24] We also obtain $\lambda^* = \frac{1}{1+\mu}\left(L/D^*\right)^{1/\varepsilon}$ from the definition of $D(\lambda) = L((1+\mu)\lambda)^{-\varepsilon}$.

[24] Let $\Gamma(\cdot,\cdot)$ denote the upper incomplete Gamma function, defined as $\Gamma(s,z) \equiv \int_z^\infty t^{s-1}e^{-t}dt$. The values of $\Gamma(s,z)$ are available in most programming platforms. The function $\Gamma(s,z)$ is positive and finite for all $s \in (-\infty,\infty)$ and $z \in (0,\infty)$. By changing the variable of integration from τ to $\widetilde{\tau} = (\tau+\kappa_0)/\rho$ and utilizing Equation (19), Equation (25) implies the following:

$$D^* = \frac{\kappa_1^{1-\varepsilon}\rho^{1+\theta(1-\varepsilon)}}{a\mu e^{\rho\kappa_0}\Gamma(1-\theta(1-\varepsilon),\rho\kappa_0)} > 0, \tag{26}$$

Next, let us turn to the labor market. First, Equation (16) implies that the number of research workers in the ASSE is $L^{R*} = n^*/a$. Second, according to functions (17) and (22), the aggregate demand for production workers in the ASSE is as follows:[25]

$$L^{P*} = \lim_{t\to\infty} \int_0^{N_t} \widetilde{l}_t(i)di \to\ n^* \int_0^{\infty} \frac{x(\tau)}{q(\tau)} d\tau = n^* D^* \int_0^{\infty} q(\tau)^{\varepsilon-1} d\tau. \tag{27}$$

The labor supply is given by population L. Therefore, the labor market clearing condition is as follows:

$$L = L^{R*} + L^{P*} = \frac{n^*}{a} + n^* D^* \int_0^{\infty} q(\tau)^{\varepsilon-1} d\tau. \tag{28}$$

From Equation (19), the integral in the RHS, $\int_0^\infty q(\tau)^{\varepsilon-1}d\tau$, becomes finite if and only if $\theta(1-\varepsilon) > 1$. Using the definition $\theta \equiv 1/(1-\psi)$, the condition reduces to $\psi \in (\varepsilon, 1)$, where ψ is the degree of knowledge spillover from past productivity to its increments. If $\psi < \varepsilon$, then the integral is infinite; therefore, Equation (28) implies that $n^* = 0$. Since we are interested in the ASSE with $n^* > 0$, the remaining analysis focuses on the case of $\psi \in (\varepsilon, 1)$.

Then, from Equation (28), we obtain the equilibrium research intensity in the ASSE as follows:

$$n^* = \frac{aL}{1 + aD^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau}. \tag{29}$$

From (29), $L^{R*} = n^*/a$ and $L^{P*} = L - L^{R*}$ are also obtained. We can calculate the explicit value of n^* as follows. Using Equation (25) and then Equation (19), the equilibrium ratio of the two types of labor is as follows:

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\int_0^\infty q(\tau)^{\varepsilon-1} d\tau}{\mu \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau}, \tag{30}$$

the value of which can be expressed via the Gamma function.[26] Using $\left(L^P/L^R\right)^*$, the ASSE research intensity can be written as follows:

$$n^* = aL^{R*} = \frac{aL}{1 + (L^P/L^R)^*}, \tag{32}$$

[25] In Equation (27), the variable of integration is changed from i to τ via Equation (3).

[26] Using Equation (26), the value of (30) can be calculated as follows:

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\kappa_0^{1-\theta(1-\varepsilon)} \rho^{1+\theta(1-\varepsilon)}}{\mu(\theta(1-\varepsilon)-1)e^{\rho\kappa_0}\Gamma(1-\theta(1-\varepsilon), \rho\kappa_0)} \text{ if } \psi > \varepsilon, \quad \left(\frac{L^P}{L^R}\right)^* = \infty \text{ otherwise.} \tag{31}$$

which becomes a positive and finite constant given that $\psi \in (\varepsilon, 1)$.

The pair of $D^* = D(\lambda^*)$ in Equation (25) and n^* in Equation (32) characterizes the long-term equilibrium of this economy. These equations also explain how parameters affect long-term dynamics. For example, a larger μ means that the breadth of patents is wider (or that trade secrets are better maintained). A higher value of a means that R&D requires less labor. In these cases, innovation intensity n^* increases because of greater profitability, whereas the output of each good, proportional to D^*, decreases because there are more production firms to which the aggregate labor needs to be divided.[27] The opposite occurs when the time preference ρ is greater because it raises the interest rate, reducing the present value of profits.

When population L is larger, the research intensity n^* is multiplied proportionally to L. However, the production of each good (proportional to D^*) does not change because both the number of products introduced each year and the number of total production workers are multiplied by the same factor. This outcome resembles the mechanism of the second-generation endogenous growth models, where the horizontal number of sectors is adjusted proportionally to the total population.[28]

Before closing this subsection, let us briefly compare those results against the case of $\psi = 1$. When $\psi = 1$, the solution to the differential equation (18) is exponential: $q(\tau) = e^{\beta\tau}$. Then, we can calculate n^* and D^* in the ASSE as follows:

$$n^* = \frac{\mu(1-\varepsilon)\beta a L}{(1+\mu)(1-\varepsilon)\beta + \rho}, \quad D^* = \frac{(1-\varepsilon)\beta + \rho}{a\mu}. \tag{33}$$

The comparative static properties with respect to μ, ρ, L and a are the same as those in the case of $\psi \in (\varepsilon, 1)$. Therefore, the exponential growth in productivity ($\psi = 1$) can be viewed as a particular case of our model, although we do not focus on it because it is a knife-edge case.

[27]The derivative of the upper incomplete Gamma function with respect to the second argument, $\partial\Gamma(s,z)/\partial z = -z^{s-1}e^{-z}$, is always negative. Using this, the properties in the text can be confirmed from Equations (26), (31) and (32).

[28]However, note that the long-term growth in these models is typically maintained by the exponential increase in the productivity (or quality) in each sector, whereas this paper focuses on the case where such exponential improvements cannot be sustained ($\psi < 1$ in Equation 18).

3.5 Measured Real GDP Growth Rate

Now, we are ready to examine the long-term GDP growth rate, as measured as by the SNA, in this prototype model.[29] In this subsection, we assume that $\psi \in (\varepsilon, 1)$ so that the economy has an ASSE with finite $n^* > 0$ and $\lambda^* > 0$. This ASSE satisfies the definition of a non-exponential asymptotic steady state in Definition 1. In this definition (in Section 2), prices are normalized so that the per capita expenditure is constant. We can confirm that the per capita expenditure in the ASSE is also constant.[30] In addition, using Equations (19), (21) and (22), we can confirm that $p(\tau)$ and $x(\tau)$ satisfy Conditions (9) and (10) given that $\psi \in (\varepsilon, 1)$.[31] Therefore, we can apply Proposition 1 to calculate the measured real GDP growth rate in the ASSE. The result is as follows:[32]

$$g^* = \frac{-\int_0^\infty p'(\tau)x(\tau)d\tau}{\int_0^\infty p(\tau)x(\tau)d\tau} = \frac{\psi - \varepsilon}{1-\varepsilon}\beta \quad \text{for } \varepsilon < \psi \leq 1. \tag{34}$$

Note that Equation (34) also applies to the special case of $\psi = 1$, where the output of all goods increases exponentially at the rate of β ($g^* = \beta$).

Equation (34) shows that the measured growth rate takes a positive and finite value whenever $\psi \in (\varepsilon, 1]$. The requirement $\psi > \varepsilon$ can be understood in terms of Condition (10) in Corollary 1. Given that $\psi < 1$, the expenditure for an age τ good in the ASSE can be written as $p(\tau)x(\tau) = [\text{constant}] \cdot (\tau + \kappa_0)^{-(1-\varepsilon)\theta}$. For $\int_0^\infty p(\tau)x(\tau)d\tau$ to be finite, the power of $(\tau + \kappa_0)^{-(1-\varepsilon)\theta}$ must be less than -1. This is a particular case of

[29] We continue to focus on the case where $\widehat{c}$ is sufficiently small so that Lemma 1 holds. We examine the general case in Appendix A.3 and show that the measured GDP growth rate becomes positive under the same conditions as in the main text.

[30] The per capita expenditure in the ASSE is $\int_0^\infty \widetilde{p}_t(i)\widetilde{c}_t(i)di \;\to\; (n^*/L)\int_0^\infty p(\tau)x(\tau)d\tau = (n^*D^*/L)\,(1+\mu)\,\int_0^\infty q(\tau)^{\varepsilon-1}d\tau$. Using the definitions of $q(\tau)$ in Equation (19) and $\theta \equiv 1/(1-\psi) > 1$, it becomes $(n^*D^*/L)\,(1+\mu)(1-\psi)\,\kappa_0^{1-(\psi-\varepsilon)/(1-\psi)}/(\psi-\varepsilon)$, which is a positive and finite constant given that $\psi \in (\varepsilon, 1)$.

[31] Similar to the calculation in footnote 30, we find that the denominator of the formula is $\int_0^\infty p(\tau)x(\tau)d\tau = D^*(1+\mu)(1-\psi)\kappa_0^{1-(\psi-\varepsilon)/(1-\psi)}/(\psi-\varepsilon)$. Using $p'(\tau) = -(1+\mu)g_q(\tau)/q(\tau)$ and Equation (20), the value of the numerator is $-\int_0^\infty p'(\tau)x(\tau)d\tau = D^*(1+\mu)\kappa_0^{-(\psi-\varepsilon)/(1-\psi)}/(1-\varepsilon)$. Both are positive and finite given that $\psi \in (\varepsilon, 1)$.

[32] The calculations from footnote 31 Implies that $-\int_0^\infty p'(\tau)x(\tau)d\tau/\int_0^\infty p(\tau)x(\tau)d\tau = (\psi-\varepsilon)/(1-\varepsilon)(1-\psi)\kappa_0$. Using definitions $\kappa_0 \equiv \theta/\beta$ and $\theta \equiv 1/(1-\psi)$, we obtain Equation (34).

Condition (11) in Section 2. Intuitively, for the expenditure on existing goods to be finite, the expenditure for a single good must decline reasonably fast with age. In this prototype model environment, the condition is accomplished if the degree of spillover in the productivity increase, ψ, is greater than ε. Otherwise, $\int_0^\infty p(\tau)x(\tau)d\tau$ becomes infinite, and Proposition 1 implies that the long-term GDP growth rate is zero.

Given that the markup ratio μ is constant and that Condition (10) is satisfied, the growth formula (7) in the ASSE can also be represented as follows:

$$g^* = \int_0^\infty g_q(\tau)\sigma(\tau)d\tau, \text{ where } \sigma(\tau) = \frac{p(\tau)x(\tau)}{\int_0^\infty p(\tau')x(\tau')d\tau'} \tag{35}$$

is the expenditure share for age τ goods, and $g_q(\tau)$ is the rate of productivity increase for those products, as defined in Equation (20). The growth formula in this form clarifies that real GDP growth is the weighted average of the rate of productivity increase among goods of various ages. Recall that, in our specification of the technology, the newest goods have the fastest rate of productivity improvement, β, whereas the rate of improvement is lower for the older goods because $g_q'(\tau) < 0$ (see Equation 20). In particular, the rate of productivity improvement $g_q(\tau)$ is almost zero for very old goods with large τ. Therefore, it is natural that the aggregate GDP growth rate in Equation (34) is between zero and β because the economy consists of goods of all ages.

Now, it is clear why the growth rate g^* in Equation (34) is decreasing in the price elasticity of demand, ε. Recall that ε also represents the elasticity of substitution across goods. With a higher ε, consumers spend more on old and low-priced goods and less on new and expensive goods. Since the rate of productivity increase in Equation (20) is lower for older goods (with high age τ), the weighted average will also be low.

Equation (34) also shows that the growth rate g^* increases with ψ, the degree of knowledge spillover in production. When $\psi \leq \varepsilon$, $\left(L^P/L^R\right)^*$ in Equation (30) becomes infinity, which means that $n^* = 0$. Without introducing new goods, the distribution of product ages simply moves up, and the growth rate decreases to $g_q(\infty) = 0$. As ψ increases between ε and 1, the schedule of the $g_q(\tau)$ function in Equation (20) moves up, as does the real GDP growth rate. When ψ reaches 1, the long-term growth rate increases to β. This is an anticipated result; when $\psi = 1$, the productivity of all goods, both the new and the old, increases with a common constant exponential rate of β. Therefore, the case of $\psi = 1$ corresponds to conventional growth theory, where labor productivity increases exponentially and uniformly. However, the main finding is that even when the productivity of each product does not rise exponentially (i.e., with $\psi < 1$), the economy as a whole can exhibit a constant measured growth rate, although it is lower than β. Finally, $\psi > 1$ is unrealistic because it means that the productivity of individual goods increases more than exponentially and that $g_q(\tau) \to \infty$ as $\tau \to \infty$.

Notably, in this simple prototype setting, the equilibrium long-term rate of growth in Equation (34) does not depend on the equilibrium values of n^* and D^* as long as they are positive.[33] When the research intensity n^* is high, more economic activity is added per unit of time. However, at the same time, there is also proportionally more "stock" of existing activities. The real GDP growth rate expresses the ratio between the two, which is unchanged.[34] Similarly, when D^* is larger, each good has more demand. This means that the production of new goods, as well as the increase in the production of other goods as time passes, is greater. At the same time, however, the total value of existing products is also higher, exactly canceling out the effects on g^*.[35] As a result, even when changes in population L, R&D productivity a, or patent policy μ affect n^* and D^*, they do not affect the real GDP growth rate. This result contrasts with the implications of existing R&D-based growth models, where g^* follows directly from n^*. Although this result depends on the simplified specification of the prototype model, it might provide a possible interpretation of why the measured GDP growth rates in the U.S. and some other developed countries have been relatively stable, even though the

[33]This property depends on the simplistic settings in this prototype model. For example, when the aggregate R&D intensity n^* has some positive spillovers on the rate of productivity increases in individual goods $g_q(\tau)$, then n^* will affect g^*. Additionally, when the amount of production has some effect on $g_q(\tau)$, g^* will depend on D^*.

[34]Nonetheless, it is essential that there is a positive flow of new innovations $n^* > 0$, since otherwise, g^* becomes 0.

[35]This can also be seen in Example 3 of Figure 4. When D^* is increased, the left panel is stretched horizontally (along the $x(\tau)$ axis), whereas the right panel is stretched vertically (along the $p(\tau)x(\tau)$ axis) by the same magnification ratio. As a result, the growth rate, given by the ratio of the two areas, is unaffected.

underlying parameters seem to have significantly changed over long periods.

3.6 Aggregate Variables and Balanced Growth

The ASSE in this model works very differently from the balanced growth path (BGP) in existing growth models. Nonetheless, we show that when aggregate variables are measured in a conventional way, this model exhibits balanced growth in those measured aggregate variables.

Note that the total labor income for production is L^{P*} since the wage rate is normalized to one. All goods are sold at $(1+\mu)$ times the labor cost, as shown in Equation (21). Therefore, the aggregate value of production, which equals the aggregate value of consumption, is $C^* = (1+\mu)L^{P*}$. In our model, investments take the form of R&D, and the total value of R&D outputs is $I^* = n^*V(\lambda^*) = L^{R*}$. The GDP in our model can be calculated as the sum of the value of production and the value of investments: $Y^* = C^* + I^* = (1+\mu)L^{P*} + L^{R*}$. Similarly, we can derive the steady-state value of aggregate capital, K^*, which is defined as the value of all firms in the economy (knowledge capital).[36]

Note that those aggregate variables are measured under the price normalization of our model, in which the nominal wage is set to 1. We now calculate their real values in the same spirit as the SNA.[37] Let $\bar{t}$ be the reference year, and let $Y_{\bar{t}}^{\$}$ be the dollar value of the GDP in year $\bar{t}$, which we assume is known to the researcher. Since the real GDP growth rate is constant at g^* in the ASSE, the real GDP level in t is as follows:

[36] K^* can be calculated as the sum of the present value of the future profits from all firms that exist today. In v years from now, the present value of the profit from those firms will be $e^{-\rho v}\int_v^\infty \pi(\tau)n^* d\tau$, since the profits from firms less than v years old at that time will not be part of the value of today's firm. By aggregating all v and using the profit function (23), we have $K^* = \mu n^* D^* \int_0^\infty e^{-\rho v}\int_v^\infty q(\tau)^{\varepsilon-1} d\tau dv$, which is constant under price normalization in the model.

[37] The NIPA publishes two series of real GDP. One is the quantity index, which is 100 in the reference year (2012 as of the time of writing). The values for other years are obtained by chaining the real GDP growth rate. The other is the chained (2012) dollar series, the values of which are calculated as the product of the quantity index and the 2012 current dollar value of the corresponding series divided by 100. See U.S. Bureau of Economic Analysis, "Table 1.1.6. Real Gross Domestic Product, Chained Dollars." In this paper, we use the latter.

$Y_t^{\text{real}} = Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}$. Since the ratios among Y^*, C^*, I^* and K^* are constant, their real values increase in the same proportion. Specifically,

$$C_t^{\text{real}} = \frac{C^*}{Y^*} Y_t^{\text{real}} = \frac{1+\mu}{1+\mu+\left(L^R/L^P\right)^*} Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}, \tag{36}$$

$$I_t^{\text{real}} = \frac{I^*}{Y^*} Y_t^{\text{real}} = \frac{1}{(1+\mu)\left(L^P/L^R\right)^*+1} Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}, \tag{37}$$

where $\left(L^R/L^P\right)^*$ is given by the inverse of Equation (30).

The interest rate $r^* = \rho$ is also defined under our normalization of prices. Since the nominal GDP growth rate in the steady state is zero, the steady-state inflation rate is $-g^*$ in our price normalization. Then, the real interest rate in the steady state is $r^{\text{real}} = r^* + g^* = \rho + g^*$. We can also derive other real aggregate variables in similar ways, and their growth rates are constant. Therefore, if the statistical agency were to measure the aggregate variables in our model economy, then those observed variables would grow exponentially along the BGP, even though neither the quantity, quality, nor the variety of individual goods were growing exponentially.

3.7 Welfare

Finally, let us discuss the welfare of the representative consumer in the ASSE. As shown by Equation (12), the welfare (lifetime utility) of the consumer is $\int_0^\infty U_t e^{-\rho t} dt$, where $U_t = \int_0^{N_t} u(\widetilde{c}_t(i))di$ is the instantaneous utility. Using Equations (13), (22) and $c(\tau) = x(\tau)/L$, the instantaneous utility in the ASSE can be written as follows:

$$U_t = n^* \overline{u} t - \frac{\varepsilon n^*}{1-\varepsilon} \left(\frac{L}{D^*}\right)^{(1-\varepsilon)/\varepsilon} \int_0^t q(\tau)^{\varepsilon-1} d\tau. \tag{38}$$

When $t \to \infty$, the second term converges to a finite value.[38] Therefore, asymptotically, the instantaneous utility increases by $n^* \overline{u}$ per unit time. This result can be interpreted as follows. In the ASSE, the schedule for consumption against the age of goods does not change with t, with the only difference being that the economy at time $t+1$ has n^* more oldest goods than at time t. When $t \to \infty$, the quality-adjusted amount of consumption for each of those oldest goods approaches infinity ($c(\tau) \to \infty$ as $\tau \to \infty$).

[38] When the ASSE exists (i.e., when $\psi \in (\varepsilon, 1)$), the integral $\int_0^\infty q(\tau)^{\varepsilon-1} d\tau$ becomes a finite constant.

Therefore, the difference in the instantaneous utility is $n^* c(\infty) = n^* \overline{u}$. According to Equation (34), n^* is not related to the long-term real GDP growth rate g^*. Nonetheless, Equation (38) demonstrates that R&D and innovations are important for improving the welfare of consumers.

4 Generalizations

This section generalizes the prototype model and the underlying theory to show that positive and steady long-term GDP growth, as observed in SNA data, can be explained under more relaxed conditions. In the first subsection, we introduce obsolescence to the prototype model and show that sustained GDP growth does not require $\varepsilon < 1$. In the second subsection, we extend the non-exponential growth theory of Section 2 to include multiple types of goods that follow different patterns of $p(\tau)$ and $x(\tau)$.

4.1 Obsolescence

In the prototype model of Section 3, we considered an environment where goods stay in the market forever ($T = \infty$) and consumers have symmetric preference across goods (12). Sustained growth in the measured GDP then requires the price elasticity of demand ε to be less than one, at least when the price is very low. The condition $\varepsilon < 1$ was necessary to induce consumers to spend less on older (and cheaper) goods. However, even without such an assumption, consumers may spend more on new goods simply because they prefer them to older ones. Here, we show that condition $\varepsilon < 1$ can be relaxed once we include obsolescence.

We now consider a generalized version of the lifetime utility function (12):

$$\int_0^\infty \left[\int_0^{N_t} \left[\delta(t - s(i)) u(\widetilde{c}_t(i)) + (1 - \delta(t - s(i)))\widehat{u} \right] di \right] e^{-\rho t} dt, \tag{39}$$

where $t - s(i) = \tau$ is the age of good i. The function $\delta(\tau)$ is decreasing in τ with $\delta(0) = 1$ and $\delta(T) = 0$, where $T > 0$ can be either finite or infinite. Its steepness represents the speed of obsolescence, or equivalently, consumers' taste for recently developed goods. Obsolescence may occur for different reasons and will have varied effects on the utility of individuals. The constant $\widehat{u} \in [0, \overline{u}]$ controls for those differences, although it does not affect the following analysis.[39] Alternatively, we may specify obsolescence as a function of $N_t - i$, i.e., the number of goods newer than i. In the ASSE, where n^* new goods are developed per unit time, $\delta(N_t - i)$ becomes $\delta(n^*\tau)$, which shows that obsolescence is faster when R&D is more active. Since the implications are similar, we focus on the simpler case of Equation (39).[40] We keep all other settings in Section 3 except that we allow any $\varepsilon > 0$ in the subutility function (13). If $\varepsilon > 1$, then we can choose $\widehat{c} = 0$ so that the price elasticity of demand is constant for all $\widetilde{c}_t(i) > 0$. If $\varepsilon \leq 1$, we again assume that $\widehat{c}$ is small enough that all existing goods satisfy $\widetilde{c}_t(i) \geq \widehat{c}$.

In this setting, the ASSE exists if and only if $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is finite.[41] In the ASSE, the expenditure for an age τ good is $e(\tau) = p(\tau)x(\tau) = (1+\mu)D^*\delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$. This equation illustrates that even when $\varepsilon > 1$, expenditures for older goods decrease with age if obsolescence is fast enough. Proposition 1 continues to apply in an environment with obsolescence, and the formula for the GDP growth rate (35) shows that the growth rate becomes a positive constant, given that $\int_0^T e(\tau)d\tau$ is finite, which is equivalent to the finiteness of $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$.

When goods retire from the market at a certain age (i.e., when T is finite), $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is obviously finite. Therefore, we always obtain a positive long-term GDP growth rate. When T is infinite and the rate of obsolescence is constant at $\overline{\delta} > 0$ per year, function $\delta(\tau)$ can be expressed as $\exp(-\overline{\delta}\tau)$. In this case, $\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ becomes finite because

[39] When $\widehat{u} = 0$, U_t may remain constant on the ASSE while $g^* > 0$. This specification is suitable, for example, when considering fashion cycles. Such cycles may generate positive measured economic growth but do not necessarily improve the utility of consumers in the long run. In some other scenarios, the utility of consumers may increase as older goods become obsolete. One example is when newer products replace some of the functionalities of older goods almost for free. In this case, $\widehat{u} = \overline{u}$ $(= u(\infty))$ would be appropriate.

[40] An additional implication when obsolescence is a function of $N_t - i$ is that policies that promote horizontal R&D may increase the measured GDP growth rate. A higher n^* will make function $\delta(n^*\tau)$ steeper as a function of τ. As we show below, faster obsolescence accelerates measured growth.

[41] Similar to the derivation of Equation (25), we obtain $D^* = \left(a\mu \int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1}$, which is always positive and finite because of the $e^{-\rho\tau}$ term. Using this value of D^*, the speed of innovation is $n^* = aL\left(1 + aD^* \int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau\right)^{-1}$. The value of n^* is positive if and only if $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is finite.

$\delta(\tau)^{\varepsilon}$ is falling exponentially and $q(\tau)^{\varepsilon-1}$ is not growing exponentially. Therefore, a constant rate of obsolescence always sustains positive measured GDP growth regardless of ε. Positive GDP growth can also be maintained with slower, non-exponential obsolescence. Consider an example where $\delta(\tau)$ is a negative power function of τ: $\delta(\tau) = \delta_0^{\omega}(\tau + \delta_0)^{-\omega}$ where ω and δ_0 are positive constants.[42] Then, $\int_0^{\infty} \delta(\tau)^{\varepsilon} q(\tau)^{\varepsilon-1} d\tau$ becomes finite if and only if[43]

$$\varepsilon < \begin{cases} \frac{\psi}{1-\omega(1-\psi)} & \text{if } \omega < \frac{1}{1-\psi} \ (\equiv \theta), \\ \infty & \text{if } \omega \geq \frac{1}{1-\psi}. \end{cases} \tag{40}$$

In a particular case of $\delta_0 = \kappa_0$, where κ_0 is defined in Equation (19), we obtain an explicit expression for the long-term GDP growth rate:[44]

$$g^* = \frac{\psi - \varepsilon + (1-\psi)\varepsilon\omega}{1 - \varepsilon + (1-\psi)\varepsilon\omega}\beta, \tag{41}$$

which is positive when Condition (40) holds. Figure 5 depicts the relationship between ε and g^* for various values of ω. As we have seen in Section 3, sustained GDP growth requires $\varepsilon < \psi$ when obsolescence is not present. With $\omega = 0$, Condition (40) and Equation (41) reduce to Equation (34). When obsolescence is faster (ω is higher), Condition (40) is easier to satisfy. In particular, when $\omega > 1$, the first line of Condition (40) is larger than one, which means that $\varepsilon < 1$ is not necessary for $g^* > 0$. When ω is greater than $1/(1-\psi)$, the long-term GDP growth rate g^* is positive regardless of ε.[45]

[42]We need a constant $\delta_0 > 0$ in $(\tau + \delta_0)^{-\omega}$ because otherwise, $\tau^{-\omega}$ cannot be defined when $\tau = 0$ and $\omega > 0$. The δ_0^{ω} term normalizes the $\delta(\tau)$ function so that $\delta(0) = 1$.

[43]Using Equation (19), $\int_0^{\infty} \delta(\tau)^{\varepsilon} q(\tau)^{\varepsilon-1} d\tau = \delta_0^{\varepsilon\omega} \kappa_1^{\varepsilon-1} \int_0^{\infty} (\tau + \delta_0)^{-\omega\varepsilon} (\tau + \kappa_0)^{\theta(\varepsilon-1)} d\tau$. The integral becomes finite if and only if the sum of the powers of the integrand, $-\omega\varepsilon + \theta(\varepsilon - 1)$, is less than minus one. From $\theta = 1/(1-\psi)$, this condition is equivalent to Condition (40).

[44]Equation (41) is obtained from formula (35), which shows that the GDP growth rate is the expenditure-weighted average of $g_q(\tau)$, where $g_q(\tau)$ is the rate of productivity increase for individual goods. When ψ is larger, $g_q(\tau)$ is higher given age τ. Nevertheless, Equation (41) shows that g^* is decreasing in ψ if $\varepsilon > 1$. When $\varepsilon > 1$, a larger ψ will induce consumers to spend more on cheaper, older goods. As a result, the expenditure is skewed more toward older goods, where $g_q(\tau)$ is small, reducing the expenditure-weighted average of $g_q(\tau)$:

[45]Interestingly, the measured GDP growth rate increases with ε when $\omega > 1/(1-\psi)$. A higher ε means

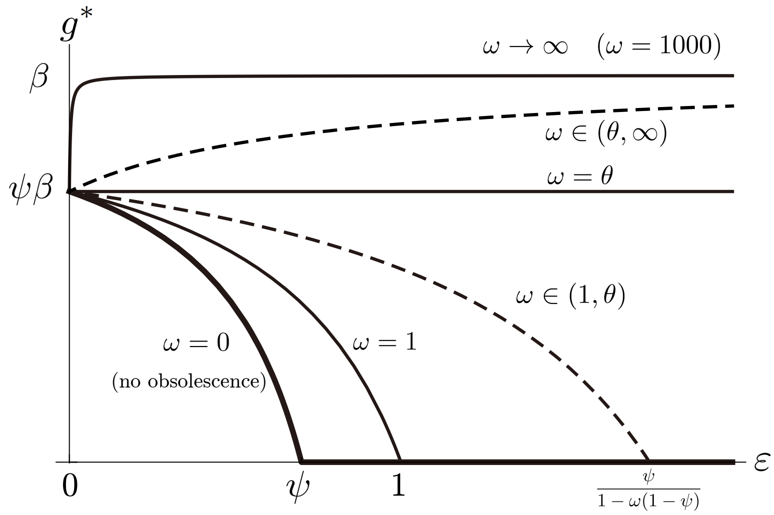

Figure 5: Price Elasticity of Individual Goods and the Measured Long-term GDP Growth Rate Under Different Speeds of Obsolescence.

Figure 5 also shows that when ω is increased, the entire curve for g^* moves upward. Faster obsolescence not only makes sustained GDP growth more likely but also accelerates the measured rate of economic growth. Intuitively, obsolescence skews expenditures toward newer goods. Since newer goods have more margins for productivity increases, the overall growth rate increases with obsolescence. This result has important policy implications. When the government tries to protect obsolete companies (or industries), it will reduce the GDP growth rate, not only because of efficiency loss but also because of the way the GDP growth rate is calculated. Conversely, advertisements and marketing practices that attract consumers to newer goods enhance GDP growth, even when the attractiveness of the newer goods is illusionary.

4.2 Multiple sectors

In the non-exponential growth theory, we define the steady state as the situation wherein the paths of quality-adjusted prices and quantities, $p(\tau)$ and $x(\tau)$, follow the same pattern

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that consumers are more willing to move from old and obsolete goods to newer goods, thus enhancing the positive effect of obsolescence on growth.

in terms of their age (See Definition 1 in Section 2.2). This definition allows the prices and quantities of individual goods at a given time to differ depending on their age. In this sense, our definition of the steady state is more flexible than in most endogenous growth models where goods are symmetric in the steady state. Nevertheless, once we look at the data, it is immediately apparent that goods in different categories follow distinct lifecycle patterns. For example, while the product lifecycle is relatively fast in electronics, some basic goods (e.g., grains) show little sign of lifecycle movements.

In this subsection, we further extend the notion of the steady state by allowing $p(\tau)$ and $x(\tau)$ to follow different patterns. We categorize goods into groups (which we call sectors) so that goods in a sector have the same pattern of movements in terms of quality-adjusted price and quantity with respect to their age, at least in the long run. More specifically, suppose that there are $J > 0$ sectors (or categories) of goods and label each by $j \in \{1, \dots, J\}$. $N_{j,t}$ denotes the index of the newest good in sector $j \in \{1, \dots, J\}$. The number of new goods introduced per unit time, $\dot{N}_{j,t} \geq 0$, can differ across sectors. The quality-adjusted price of the ith good in sector j and its quality-adjusted quantity are denoted by $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$. In this setting, we define the asymptotic steady state as follows.

Definition 3. *A non-exponential asymptotic steady state with multiple sectors is the situation where $\dot{N}_{j,t}$, $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$, for all $j \in \{1, \dots, J\}$, satisfy the following conditions:*

(a) $\dot{N}_{j,t}$ converges to a constant, i.e., $\dot{N}_{j,t} \to n_j \geq 0$.

(b) $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$ converge to time-invariant functions of $\tau = t - s(i)$, i.e., $\widetilde{p}_{j,t}(i) \to p_j(\tau)$ and $\widetilde{x}_{j,t}(i) \to x_j(\tau)$.

(c) Assumption 1 holds, where $p(\tau)$, $x(\tau)$ and T are replaced by $p_j(\tau)$, $x_j(\tau)$ and T_j, respectively.

(d) The expenditure share of the sector, which is defined by

$$\alpha_{j,t} = \frac{\int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}{\sum_{j'=1}^{J} \int_0^{N_{j',t}} \widetilde{p}_{j',t}(i)\widetilde{x}_{j',t}(i)di}, \tag{42}$$

converges to a constant value, i.e., $\alpha_{j,t} \to \alpha_j \geq 0$.

Definition 3 says that the economy is in a steady state if the composition of sectors in terms of expenditure share is stationary, and each sector satisfies the requirement for the steady state in Definition 1. In addition, Definition 3 does not require n_j to be positive, thus including the possibility where the introduction of goods eventually stops in some sectors. Additionally, α_j may be zero for some j, allowing for the possibility that some sectors disappear in the long run.

Similar to Equation (5), the instantaneous GDP growth rate in this multisector economy at any given time t can be defined as follows:

$$g_t = \frac{\sum_{j=1}^{J} \dot{N}_{j,t}\widetilde{p}_{j,t}(N_{j,t})\widetilde{x}_{j,t}(N_{j,t}) + \sum_{j=1}^{J} \int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\dot{\widetilde{x}}_{j,t}(i)di}{\sum_{j=1}^{J} \int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}. \tag{43}$$

Here, the denominator gives the expenditure for all goods, the first term in the numerator is the value of all new goods introduced at time t, and the second term is the value of the changes in the production of existing goods. Using the sectoral expenditure share defined by Equation (42), Equation (43) can be expressed as the share-weighted average of the sectoral GDP growth rate.

$$\begin{aligned} g_t &= \sum_{j=1}^{J} \alpha_{j,t} g_{j,t}, \text{ where,} \\ g_{j,t} &= \frac{\dot{N}_{j,t}\widetilde{p}_{j,t}(N_{j,t})\widetilde{x}_{j,t}(N_{j,t}) + \int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\dot{\widetilde{x}}_{j,t}(i)di}{\int_0^{N_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}. \end{aligned} \tag{44}$$

Since Equation (44) takes the same form as Equation (5), we can utilize Proposition 1 to obtain the long-term GDP growth rate in a steady state.

Proposition 2. *Suppose that the multisector economy converges to an asymptotic steady state, as defined by Definition 3. Then, the real GDP growth rate g_t asymptotes to the following:*

$$g = \sum_{j=1}^{J} \alpha_j g_j, \tag{45}$$

where g_j is given by Proposition 1, in which $p(\tau)$, $x(\tau)$ and g are replaced by $p_j(\tau)$, $x_j(\tau)$ and g_j, respectively.

Proposition 2, combined with Proposition 1, implies that if there is a category of goods (a sector) with a positive GDP share where Conditions (9) and (10) hold in the long run, the economy-wide long-term GDP growth rate can be positive and finite.

Similar to Figure 4 in Section 2.4, we can draw the evolution of $\{x_j(\tau), p_j(\tau)\}$ in the quantity–price space and the evolution of $p_j(\tau)x_j(\tau)$ against τ. The numerator and denominator of g_j are then graphically represented as the blue and yellow areas. If $\alpha_j > 0$ and both areas are positive and finite, then sector j contributes positively to the long-term GDP growth rate. As in Example 2 of Figure 4, g_j could be negative if the prices of older and disappearing goods in that sector are higher than those of new goods in the same sector. Nonetheless, aggregate GDP growth becomes zero only by coincidence; therefore, nonzero long-term growth rates will be the norm rather than the exception. This result contrasts with existing endogenous growth models, where the growth rate can be nonzero only under strict knife-edge conditions.

As a final note, observe that g_js in Proposition 2 are the sectoral output growth rates measured according to their own sectoral price indices. They do not coincide with the sectoral output growth calculated using the general price levels (e.g., the GDP deflator). In the long run, the expenditures to all the surviving sectors (those with positive α_j values) will grow at the same rate. Even the sectors with $g_j = 0$ will record real income growth of g.

5 Concluding Remarks

Non-exponential growth theory provides a novel interpretation of observed stability in the measured GDP growth rate by focusing on the movement of quantities and prices of individual goods and calculating the GDP growth rate on the basis of SNA statistics (e.g., the NIPA). This paper has shown that sustained GDP growth can be explained without exponential growth in quantity, quality, or variety. This finding enables researchers to build endogenous growth models under less restrictive assumptions than the knife-edge conditions that are required in existing full endogenous growth models.

In standard variety-expansion models, all goods are symmetric and receive the same expenditure share. Therefore, as the number of goods increases, the share of the expenditure given to a single new good dilutes. This means that profits obtained from a single successful R&D also decrease. Therefore, to give firms enough incentives to do R&D in equilibrium, those models require a strong degree of externality in the R&D process so that the cost of inventing new goods declines exponentially. Moreover, GDP growth can be maintained only when the number of goods increases exponentially because the contribution of each new good to the economic growth rate decreases toward 0.

In contrast, if the expenditure for older goods decreases as they age, then newly introduced goods can receive a constant proportion of the total expenditure. Then, the incentive to innovate can be maintained without such strong externalities. In such a case, a constant flow of new goods, as well as improvements in the productivity of producing existing goods, constitutes a significant addition of economic activity relative to all existing activities. This enables the measured GDP growth rate to remain positive in the long run without accelerating R&D.

On the basis of this insight, we present a simple prototype model in which the GDP growth rate can be sustained if the price elasticity of demand for older goods is less than one and the price decreases with age (Section 3). If goods become obsolete as they age, then the above condition can be further relaxed (Subsection 4.1). We also extended the theory (Subsection 4.2) to show that if a group of goods with a nonzero expenditure share satisfies the required condition, the long-term GDP growth rate can be positive. Therefore, positive long-term GDP growth is consistent even when some goods do not lose expenditures forever.

This paper suggests that an endogenous growth theory can be applied to data with much weaker restrictions than before. Nevertheless, we make simplifying assumptions for expositional simplicity and ease of understanding. Notably, while existing variety-expansion endogenous growth models assume that the elasticity of spillover from R&D activity is exactly at $\phi = 1$, we assume that there is none, i.e., $\phi = 0$. In a working paper,[46] we confirm that the intuitions from the non-exponential growth theory continue to hold when $\phi < 1$, although the analysis becomes significantly intricate because the number of new goods introduced per unit time is no longer constant. Additionally, this paper abstracts from population growth and decline by assuming a constant population. While it is standard to make this assumption in existing (full-)endogenous growth mod-

[46]See Horii (2024), where the theory is extended to include $\phi \in (-\infty, 1)$ and nonzero population growth.

els, studies by Jones (1995, 2002, 2022) have shown that positive population growth is essential in sustaining growth when $\phi < 1$. An important difference from semiendogenous growth theory is that sustained GDP growth does not require positive population growth even when $\phi < 1$. This concerns the grave question of whether humanity can continue growing in the future. This paper, which is based on $\phi = 0$ and an asymptotically constant population, shows the possibility that the measured rate of GDP growth continues to be positive in a straightforward setting.

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Online Appendix for "Non-Exponential Growth Theory"

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Appendix A Analysis of the General Case

A.1 Proof of Lemma 1

The proof goes by a "guess and verify" method. Suppose that $\lambda^* < ((1+\mu)\widehat{c}^{1/\varepsilon})^{-1}$, which means that $(1+\mu)\lambda^*\widehat{c}^{1/\varepsilon} < 1$. Then, $q(\tau) > (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ holds for all $\tau \geq 0$, since $q(0) = 1$ and $q'(0) > 0$ for all $\tau > 0$.

Below, we verify that the initial guess is correct under the assumption in the lemma. Since $q(\tau) > (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ holds for all $\tau \geq 0$, we can calculate the steady-state value of λ^* as in Equation (25). Using the assumption of the lemma, $\widehat{c} < \left(a\mu L \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1}$, Equation (25) implies the following:

$$\lambda^* = \frac{1}{1+\mu}\left(a\mu L \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{1/\varepsilon} \leq \frac{1}{1+\mu}\widehat{c}^{-1/\varepsilon}, \tag{46}$$

which confirms that the initial guess is correct.

In Appendix A.2, we show that the steady-state value of λ^* is unique. Therefore, we are assured that the unique value of λ^* satisfies $\lambda^* < ((1+\mu)\widehat{c}^{1/\varepsilon})^{-1}$; thus, $q(\tau) > (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ for all $\tau \geq 0$.

A.2 Steady-state Equilibrium when $\widehat{c}$ is not Small

In Section 3.4, we assume that $\widehat{c}$ is sufficiently small so that $q(\tau) \geq (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ holds for all τ. Here, we analyze the steady-state equilibrium without this assumption. The threshold age of goods is defined as follows:

$$\widehat{\tau}(\lambda^*) = \max\left[0, \frac{\theta}{\beta}\left(\left((1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}\right)^{1/\theta} - 1\right)\right]. \tag{47}$$

Then, from Equation (19), $q(\tau) \geq (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ if and only if $\tau \geq \widehat{\tau}(\lambda^*)$.

Using Equation (22), the profit of an age-τ firm in the steady state can be written as follows:

$$\pi(\tau) = \begin{cases} \mu D(\lambda^*) q(\tau)^{\varepsilon-1} & \text{for } \tau \geq \widehat{\tau}(\lambda^*), \\ \mu \widehat{D}(\lambda^*) q(\tau)^{\widehat{\varepsilon}-1} & \text{for } \tau \leq \widehat{\tau}(\lambda^*). \end{cases} \tag{48}$$

Using Equations (47) and (48), the value of a new firm in the steady state can be written as a function of λ^*:

$$V(\lambda^*) = \mu \widehat{D}(\lambda^*) \int_0^{\widehat{\tau}(\lambda^*)} q(\tau)^{\widehat{\varepsilon}-1} e^{-\rho\tau} d\tau + \mu D(\lambda^*) \int_{\widehat{\tau}(\lambda^*)}^{\infty} q(\tau)^{-(1-\varepsilon)} e^{-\rho\tau} d\tau. \tag{49}$$

The equilibrium value of λ^* is determined by the free entry condition, $V(\lambda^*) = 1/a$. From $D(\lambda) = L((1+\mu)\lambda)^{-\varepsilon}$ and $\widehat{D}(\lambda) = L((1+\mu)\lambda/\underline{u})^{-\widehat{\varepsilon}}$, we can confirm that function $V(\lambda)$ is continuous and strictly decreases with λ.[47] Additionally, $\lim_{\lambda\to 0} V(\lambda) = \infty$ and $\lim_{\lambda\to\infty} V(\lambda) = 0$. Therefore, there is a unique value of positive and finite λ^* that solves the free entry condition. This is the steady-state value of λ^*.

Next, let us turn to the labor market. From functions (17) and (22), the total number of production workers in the ASSE can be written as $L^{P*} = n^*\ell(\lambda^*)$, where

$$\ell(\lambda^*) \equiv D(\lambda^*) \int_0^{\widehat{\tau}(\lambda^*)} q(\tau)^{\widehat{\varepsilon}-1} d\tau + D(\lambda^*) \int_{\widehat{\tau}(\lambda^*)}^{\infty} q(\tau)^{-(1-\varepsilon)} d\tau. \tag{50}$$

Note that the first integral in Equation (50) is finite because $\widehat{\tau}(\lambda^*)$ is finite. The second integral is finite if the power of $q(\tau)^{-(1-\varepsilon)} \propto (\tau+\kappa_0)^{-\theta(1-\varepsilon)}$ is less than 1, which means that $\theta(1-\varepsilon) > 1$, or equivalently $\psi > \varepsilon$. In the following, we assume that $\psi > \varepsilon$ holds. The function $\ell(\lambda^*)$ is a decreasing and continuous function of λ^*, with $\lim_{\lambda\to 0}\ell(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\ell(\lambda) = 0$. Since λ^* is positive and finite, $\ell(\lambda^*)$ is also positive and finite. Using Equation (50), the equilibrium condition for the labor market is written as follows: $n^*\ell(\lambda^*) + (n^*/a) = L$. From this, we obtain the following:

$$n^* = \frac{aL}{1 + a\ell(\lambda^*)}. \tag{51}$$

Since $\ell(\lambda^*)$ is positive and finite, n^* is also positive and finite.

[47] To calculate $V'(\lambda)$, we need to use Leibniz's rule because the range of the integration depends on λ. However, at $\tau = \widehat{\tau}(\lambda)$, we can confirm that $\widehat{D}(\lambda)q(\widehat{\tau}(\lambda))^{\widehat{\varepsilon}-1} = D(\lambda)q(\widehat{\tau}(\lambda))^{\varepsilon-1}$. Therefore, a marginal change in $\widehat{\tau}(\lambda)$ does not affect $V'(\lambda)$.

A.3 Measured Real GDP Growth Rate when $\widehat{c}$ is not Small

As shown in Appendix A.2, the economy has an ASSE with a positive and finite pair of n^* and λ^* whenever $\psi \in (\varepsilon, 1)$. In this ASSE, we now calculate the real GDP growth rate, as measured by the SNA. From Equations (21) and (22), the expenditure for an age τ good can be written as follows:

$$p(\tau)x(\tau) = \begin{cases} (1+\mu)D(\lambda^*)q(\tau)^{-(1-\varepsilon)} & \text{for } \tau \geq \widehat{\tau}(\lambda^*), \\ (1+\mu)\widehat{D}(\lambda^*)q(\tau)^{1-\widehat{\varepsilon}} & \text{for } \tau < \widehat{\tau}(\lambda^*). \end{cases} \tag{52}$$

Using Equation (52), we can calculate the expenditure shares for the goods of each age:

$$\sigma(\tau) = \begin{cases} D(\lambda^*)q(\tau)^{-(1-\varepsilon)}/\ell(\lambda^*) & \text{for } \tau \geq \widehat{\tau}(\lambda^*), \\ \widehat{D}(\lambda^*)q(\tau)^{1-\widehat{\varepsilon}}/\ell(\lambda^*) & \text{for } \tau < \widehat{\tau}(\lambda^*). \end{cases} \tag{53}$$

The measured real GDP growth rate is obtained via the growth formula (35):

$$g^* = \frac{1}{\ell(\lambda^*)}\left(\widehat{D}(\lambda^*)\int_0^{\widehat{\tau}(\lambda^*)} q(\tau)^{\widehat{\varepsilon}-1}g_q(\tau)d\tau + D(\lambda^*)\int_{\widehat{\tau}(\lambda^*)}^{\infty} q(\tau)^{-(1-\varepsilon)}g_q(\tau)d\tau\right). \tag{54}$$

Using Equations (19) and (20), the growth rate can be written as follows:

$$\begin{aligned} g^* = \frac{\theta}{\ell(\lambda^*)}\Bigg(&\widehat{D}(\lambda^*)\kappa_1^{\widehat{\varepsilon}-1}\int_0^{\widehat{\tau}(\lambda^*)}(\tau+\kappa_0)^{\theta(\widehat{\varepsilon}-1)-1}d\tau \\ &+ D(\lambda^*)\kappa_1^{-(1-\varepsilon)}\int_{\widehat{\tau}(\lambda^*)}^{\infty}(\tau+\kappa_0)^{-\theta(1-\varepsilon)-1}d\tau\Bigg). \end{aligned} \tag{55}$$

The two integrals in Equation (55) are both finite, and their sum is positive. Additionally, as discussed in Section A.2, $\ell(\lambda^*)$ is positive and finite. Therefore, given $\psi \in (\varepsilon, 1)$, the measured real GDP growth rate is positive and finite.

Appendix B Simplest Examples of Non-Exponential Growth

B.1 When Goods become Free in Two Periods

Consider an economy in discrete time with overlapping generations of products. One new good is introduced every year. When the good is introduced, the price is 2; it falls to 1 in the next period and to 0 thereafter. The output quantity is 1 when it is introduced, 2 in the next period, and 3 thereafter. For example, we can consider each good as a medication for a particular disease. In this example, after two periods, generic drugs with the same effect become available (almost) for free. The pattern of movements of quantities and prices is summarized below.

index	$i = N_t - 3$	$i = N_t - 2$	$i = N_t - 1$	$i = N_t$	$i = N_t + 1$
$\widetilde{x}_{t-1}(i)$	3	2	1	N/A	N/A
$\widetilde{x}_t(i)$	3	3	2	1	N/A
$\widetilde{x}_{t+1}(i)$	3	3	3	2	1
$\widetilde{p}_t(i)$	0	0	1	2	N/A
$\widetilde{p}_{t+1}(i)$	0	0	0	1	2

In the table, i is the index of goods, N_t is the index of the newest good in period t, $\widetilde{x}_t(i)$ is the quantity of good i, and $\widetilde{p}_t(i)$ is the price of good i. Note that the newest good in period $t+1$ is $i = N_t + 1$. Therefore, the values in the $\widetilde{x}_{t+1}(i)$ and $\widetilde{p}_{t+1}(i)$ rows are shifted to the right by one column. The opposite holds for the $\widetilde{x}_{t-1}(i)$ row.

In SNA statistics, the real GDP growth rate from years $t-1$ to t is defined as the growth of the value of output when the value is evaluated by the prices in the base year. In practice, the base year is frequently updated; thus, we assume that the base year is updated every year to the year of evaluation (i.e., year t). Then, we have the following:

$$g_{t-1,t} = \frac{\sum_{i=0}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i) - \sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)}{\sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)}. \tag{56}$$

Using the number in the table, the value of the output in t using the prices in t is $\sum_{i=0}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i) = 1 \times 2 + 2 \times 1 = 4$. Similarly, the value of output in $t-1$ using the prices in t is $\sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i) = 1 \times 1 = 1$. Therefore, the GDP growth rate is $g_{t-1,t} = (4-1)/1 = 3 = 300\%$. Similarly, We can calculate the GDP growth rate for period $t+1$ as $g_{t,t+1} = \left(\sum_{i=0}^{N_t+1} \widetilde{p}_{t+1}(i)\widetilde{x}_{t+1}(i) - \sum_{i=0}^{N_t} \widetilde{p}_{t+1}(i)\widetilde{x}_t(i)\right) / \sum_{i=0}^{N_t} \widetilde{p}_{t+1}(i)\widetilde{x}_t(i) = (4-1)/1 = 300\%$. We always obtain the same growth rate as long as this pattern of quantities and prices continues. Therefore, the measured GDP growth in this steady state is constant and positive, even though the output of any good does not grow at an exponential rate.

B.2 When Goods become Obsolete in Two Periods

Similar to the previous example, one new good is introduced every year. When the new good is introduced, its price is 2, and it falls to 1 thereafter. The good is produced only for the period when it is introduced and one period afterward. The output quantity is 1 for both periods and then 0 thereafter. One can think of each good as a medication for a particular infectious disease. Owing to medication, the disease is eradicated in two years, and the good is no longer in demand. The pattern is summarized below.

index	$i = N_t - 3$	$i = N_t - 2$	$i = N_t - 1$	$i = N_t$	$i = N_t + 1$
$\widetilde{x}_{t-1}(i)$	0	1	1	N/A	N/A
$\widetilde{x}_t(i)$	0	0	1	1	N/A
$\widetilde{x}_{t+1}(i)$	0	0	0	1	1
$\widetilde{p}_t(i)$	1	1	1	2	N/A
$\widetilde{p}_{t+1}(i)$	1	1	1	1	2

Again, we can calculate the GDP growth rate via Equation (56). The value of output in t using the prices in t is $\sum_{i=0}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i) = 1 \times 1 + 2 \times 1 = 3$. Similarly, the value of output in $t-1$ using the prices in t is $\sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i) = 1 \times 1 + 1 \times 1 = 2$. Therefore, the GDP growth rate for year t is $g_{t-1,t} = (3-2)/2 = 1/2 = 50\%$. We can also calculate $g_{t,t+1}$, which is again 50%. The measured growth rate does not change as long as the same pattern continues.

When comparing the output quantities in periods t and $t-1$, the difference is that we have one unit of the newest good (whose value is 2), and we lose one unit of the 2-year-old good (whose value is 1). Since the price of the new good is higher than that of the old, disappearing good, the numerator is positive. Based on the observed prices, the GDP growth rate attributes a greater value to newly appearing goods than to old disappearing goods.

B.3 Interpretation

Both examples satisfy the required conditions for positive GDP growth explained in the Introduction: (i) new goods are introduced over time, (ii) the price of goods decreases with age, and (iii) the expenditure for old goods is limited (it becomes zero after two years).

These examples also illustrate that the GDP growth rate may not directly correspond to welfare improvements. In the first example, the economy will have a (linearly) greater variety of goods as time passes. With respect to the interpretation of medications, we have become able to cure more diseases. Therefore, it is natural to expect that the welfare of the consumer will improve over time. Nevertheless, there is no component of consumption that grows exponentially.

In the second example, the economy in any year has only two kinds of products, and the quantity is always one for each good. Therefore, the economy might seem stationary, despite the positive measured GDP growth rate. Whether the welfare of consumers improves depends on the reason why the goods become obsolete. If a medication becomes obsolete because the disease is eradicated, then obsolescence is beneficial for consumers, and their welfare will improve. However, if goods become obsolete because of changes in taste or fashion, then the welfare of consumers may well be unchanged over time. Unfortunately, the notion of GDP growth does not distinguish these two scenarios because it is constructed only from the series of prices and quantities of various goods. It does not have information on why prices and quantities have changed or why products appear and disappear.

This paper does not try to say that the GDP growth rate is wrong. The GDP growth rate is just an index and has limitations, similar to any economic index. It is among the best available indices by which we can analyze the total amount of economic activity; however this paper suggests that we need to be cautious in interpreting the GDP growth rate. This is especially true in the long run, when the relative price of goods changes and the base year needs to be updated. By explicitly considering the changes in relative prices and the introduction of new goods, this paper shows that we can interpret more flexibly the fact that the real GDP growth rate is historically stationary. One concrete benefit is that we can build endogenous growth models that require less restrictive assumptions than existing models do.