Supplement to

"Strategic Supply Function Competition with Private Information": Proofs, Simulations, and Extensions

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This appendix provides proofs, extensions, complementary simulations, and connections to the literature of the analysis and results presented in the main text. Section S.1 provides the convergence concepts, proofs and further connections with the literature of replica markets (Section 4). Section S.2 develops the analysis of Bayesian Cournot equilibrium as well as its asymptotic properties as the market grows large. Section S.3 contains a development and details of the simulations of the model in the paper (S.3.1 corresponding to Section 3.2 in the paper, the basic model, S.3.2 corresponding to Section 4, replica markets, and S.3.3 to the comparison with the Cournot model). Section

S.4 analyzes information acquisition, Section S.5 deals with demand uncertainty (Section 5.4), and Section S.6 provides the proof of Proposition 9 of the model with a public signal (Section 5.5).

S.1 Replica markets

This section provides the convergence concepts for replica markets (Section 4), proofs, and further connections with the literature.

S.1.1 Measures of speed of convergence and notation for large markets.

We say that the sequence (of real numbers) b_n is of the *order* n^{υ}, with υ a real number, whenever $n^{-\upsilon} b_n \xrightarrow[n]{} k$ for some nonzero constant k. We say that the sequence of random variables $\{y_n\}$ converges in *mean square* to zero at the rate $1/\sqrt{n^r}$ (or that y_n is of the order $1/\sqrt{n^r}$) if $E\left[(y_n)^2\right]$ converges to zero at the rate $1/n^r$ (i.e. $E\left[(y_n)^2\right]$ is of the order $1/n^r$). Given that $E\left[(y_n)^2\right] = \left(E[y_n]\right)^2 + \operatorname{var}[y_n]$, a sequence $\{y_n\}$ such that $E[y_n] = 0$ and $\operatorname{var}[y_n]$ is of the order of $1/n$ and converges to zero at the rate $1/\sqrt{n}$. We use the subscript n to emphasize the dependence of n of average random variables $\tilde{\theta}_n = \left(\sum_{i=1}^{n} \theta_i\right)/n$, $\tilde{s}_n = \left(\sum_i s_i\right)/n = \tilde{\theta}_n + \left(\sum_i \varepsilon_i\right)/n$, and $\tilde{t}_n = E\left[\tilde{\theta}_n | \tilde{s}_n\right]$.

S.1.2. Proof of Proposition 7: Recall that the n subscript denotes the n-replica market.

(i) I show that $E\left[\left(p_n - p_n^{PT}\right)^2\right]$ tends to 0 at the rate of $1/n^2$. Note first that from the equation $g_n(c;M) = 0$ in the n-replica market defining c_n (just replace β by β/n in $g(c;M) = 0$) we have that $c_n \xrightarrow[n]{} \left(\lambda^{-1} - \beta^{-1} M_\infty\right)\left(M_\infty + 1\right)^{-1}$, $M \xrightarrow[n]{} M_\infty \equiv \sigma_\varepsilon^2 \left((1-\rho)\sigma_\theta^2\right)^{-1}$, if $\rho > 0$, and $c_n \xrightarrow[n]{} \lambda^{-1}$ if $\rho = 0$. It is immediate then that the order of the distortion $d_n = \left(\beta^{-1} n + (n-1) c_n\right)^{-1}$ is $1/n$. Let us show first the order for the price difference $E\left[\left(p_n - p_n^{PT}\right)^2\right]$. Recall that the price-taking allocation coincides with the full information efficient one. We know that $p_n - p_n^{PT} = \beta\left(\tilde{x}_n^o - \tilde{x}_n\right)$ and from the proof of Proposition 4 that $E\left[\left(\tilde{x}_n - \tilde{x}_n^o\right)^2\right] = \left(\left(\beta+\lambda\right)^{-1} - \left(\beta+\lambda+d_n\right)^{-1}\right)^2 E\left[\left(\alpha - \tilde{t}_n\right)^2\right]$, where $E\left[\left(\alpha-\tilde{t}_n\right)^2\right] = \left(\alpha - \overline{\theta}\right)^2 + \frac{\left(\left(1+\left(n-1\right)\rho\right)\sigma_\theta^2\right)^2}{\left(\left(1+\left(n-1\right)\rho\right)\sigma_\theta^2 + \sigma_\varepsilon^2\right)n}$ is of the order of a constant. The order of $E\left[\left(p_n - p_n^{PT}\right)^2\right]$ and of $E\left[\left(\tilde{x}_n - \tilde{x}_n^o\right)^2\right]$ is the order of $\left(\left(\beta+\lambda\right)^{-1} - \left(\beta+\lambda+d_n\right)^{-1}\right)^2$ which is $1/n^2$ since d_n is of the order of $1/n$.

(ii) With regard to efficiency, from the proof of Proposition 4 (and with analogous notation), we know that $E\left[\left(u_{in} - u_{in}^o\right)^2\right] = \left(\lambda^{-1} - \left(\lambda + d_n\right)^{-1}\right)^2 E\left[\left(t_{in} - \tilde{t}_n\right)^2\right]$ where $E\left[\left(t_i - \tilde{t}\right)^2\right] = \frac{\left(1-\rho\right)^2\left(n-1\right)\sigma_\theta^4}{n\left(\sigma_\theta^2\left(1-\rho\right)+\sigma_\varepsilon^2\right)}$ is of the order of a constant. It follows that $E\left[\left(u_{in} - u_{in}^o\right)^2\right]$ is of order $1/n^2$ since $\left(\lambda^{-1} - \left(\lambda+d_n\right)^{-1}\right)^2$ is. We conclude that $\left(ETS_n^o - ETS_n\right)/n = \left(\left(\beta+\lambda\right)E\left[\left(\tilde{x}_n - \tilde{x}_n^o\right)^2\right] + \lambda E\left[\left(u_{in} - u_{in}^o\right)^2\right]\right)/2$ is of order $1/n^2$. ♦

S.1.3 Remarks

Remark S.1: If we want to keep aggregate uncertainty ($\mathrm{var}\left[\tilde{\theta}_n\right]$) constant when lowering market concentration (increasing n, which will mean to decrease σ_θ^2 appropriately since $\mathrm{var}\left[\tilde{\theta}_n\right]$ is decreasing in n), then c_n will be smaller than when we allow $\mathrm{var}\left[\tilde{\theta}_n\right]$ to vary (this is so since M increases by more when we keep $\mathrm{var}\left[\tilde{\theta}_n\right]$ constant). This will mean that the distortion d_n will be larger.

Remark S.2: In Biais et al. (2000) increasing the number of market makers reduces market power but the limit market has features of a monopolistically competitive equilibrium in which market makers charge a positive mark-up but make zero profits because they trade an infinitesimal amount. There is a spread even for small trades. The reason is that, under discriminatory pricing, market makers do not know whether the informed trader will want to buy more than the marginal unit. An infinitesimal order has a discrete impact on the price because it conveys a non-infinitesimal amount of information. This is why the bid-ask spread subsists even for very small orders (see Section 5.3.2 in Vives (2008)). This is not the case under uniform pricing as in our equilibrium

S.2. Cournot competition

In this section we study Bayesian Cournot competition. We characterize first equilibrium (in its strategic and price-taking versions) and consider then replica markets and asymptotic results.

S.2.1 Equilibrium

Consider the market as in Section 2, with $\rho \in [0,1]$, but now seller i sets a quantity contingent on his information $\{s_i\}$.[1] The seller has no other source of information and, in particular, does not condition on the price. The expected profits of seller i conditional on receiving signal s_i and assuming seller j, $j \neq i$, uses strategy $X_j(s_j)$, are

$$E[\pi_i | s_i] = x_i \left(P\left(\sum_{j \neq i} X_j(s_j) + x_i \right) - E[\theta_i | s_i] \right) - \frac{\lambda}{2} x_i^2 .$$

From the F.O.C. of the optimization of seller i we obtain

$$p - \left(E[\theta_i | s_i] + \lambda x_i \right) = \beta x_i .$$

Given that the profit function is strictly concave and the information structure symmetric, equilibria will be symmetric.)

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1 See Vives (2002) for related results when cost parameters are i.i.d. and Vives (1988) for the common value case.

We can define also a price-taking Bayesian Cournot equilibrium in which each seller sets a quantity but he does not realize his influence on the price. In this case seller i chooses x_i to maximize

$$E\left[\pi_i \middle| s_i\right] = x_i\left(p - E\left[\theta_i \middle| s_i\right]\right) - \frac{\lambda}{2}x_i^2,$$

yielding a F.O.C.

$$p - \left(E\left[\theta_i \middle| s_i\right] + \lambda x_i\right) = 0.$$

The following proposition characterizes the Bayesian Cournot equilibrium (denoted by a superscript C) and the price-taking Bayesian Cournot equilibrium (denoted by a superscript CPT). Both equilibria are different from their supply function counterparts (except in the knife-edge case for which $c = 0$ at the SFE) since there is no conditioning in the market price. Note that the Bayesian Cournot equilibrium exists even if $\rho = 1$ since there is no learning from prices.

Proposition S.1. Let $\rho \in [0,1]$. There is a unique Bayesian Cournot equilibrium and a unique price-taking Bayesian Cournot equilibrium. They are symmetric and affine in the signals. Letting $\xi \equiv \sigma_\theta^2 / \left(\sigma_\theta^2 + \sigma_\varepsilon^2\right)$ the strategies of the sellers are given (respectively) by:

$$X^C\left(s_i\right) = b^C\left(\alpha - \bar{\theta}\right) - a^C\left(s_i - \bar{\theta}\right),$$

where $a^C = \dfrac{\xi}{2\beta + \lambda + \beta(n-1)\rho\xi}$, and $b^C = \dfrac{1}{\lambda + \beta(1+n)}$; and

$$X^{CPT}\left(s_i\right) = b^{CPT}\left(\alpha - \bar{\theta}\right) - a^{CPT}\left(s_i - \bar{\theta}\right),$$

where $a^{CPT} = \dfrac{\xi}{\beta + \lambda + \beta(n-1)\rho\xi}$, and $b^{CPT} = \dfrac{1}{\lambda + \beta}$.

Proof: See the proof of Proposition 2.1 in Vives (2008).

S.2.2 Replica markets and convergence

We study convergence to price taking and its speed as the economy is replicated. Consider thus the replica market as in Section 4 with inverse demand $P_n(y) = \alpha - \beta y/n$.

The equilibria are then given as in Proposition S.1 replacing β by β/n. The following proposition characterizes the convergence of the Bayesian Cournot equilibrium to a price-taking equilibrium. ETS_n^C denotes here the expected total surplus at the (price-taking) Bayesian Cournot equilibrium in the $n-$replica market and, ETS_n^{CPT} in the price-taking Bayesian Cournot equilibrium. It is worth to remark that ETS_n^{CPT} does not attain in general the expected total surplus at the efficient (full information) allocation ETS_n^o since the price-taking Bayesian Cournot equilibrium with $\lambda > 0$ is not full information efficient except if $\rho = 0$ (Vives (2002)).[2]

In general we have that $ETS_n^o > ETS_n^{CPT}$ since the price-taking Bayesian Cournot equilibrium does not aggregate information, and as the market grows large there is no convergence to a full information equilibrium for $\rho > 0$. A consequence of the result is that for a given $\rho > 0$ and for large enough n we have always that $\left(ETS_n^{SF} - ETS_n^C\right)/n > 0$. This is so since as n grows the SFE, but not the Cournot equilibrium, converges to the (full information) first best. However, Proposition 7 holds for the (Bayesian) Cournot equilibrium (with price p_n^C and expected total surplus ETS_n^C) replacing ETS_n^o with ETS_n^{CPT} where ETS_n^{CPT} stands for the expected total surplus at the price-taking (Bayesian) Cournot equilibrium.

<u>Proposition S.2</u>. Let $\rho \in [0,1]$. As the market grows large the market price p_n^C at the Bayesian Cournot equilibrium converges in mean square to the price-taking Bayesian Cournot price p_n^{CPT} at the rate of $1/n$. (That is, $E\left[\left(p_n^C - p_n^{CPT}\right)^2\right]$ tends to 0 at the rate of $1/n^2$.) The difference $\left(ETS_n^{CPT} - ETS_n^C\right)/n$ is of the order of $1/n^2$.

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2 With constant marginal costs the Bayesian Cournot equilibrium does replicate the full information outcome under some regularity conditions (see Palfrey (1985) and Vives (1988)). A price-taking Bayesian Cournot equilibrium is team optimal (i.e. maximizes total expected surplus subject to the constraint that sellers use decentralized -quantity- strategies in information, see Vives (1988)).

Proof: Let us show first the order for the price difference $E\left[\left(p_n^C - p_n^{PT}\right)^2\right]$. We know that $p_n^C - p_n^{PT} = \beta\left(\tilde{x}_n^{CPT} - \tilde{x}_n^C\right)$ and, letting $k_i \equiv E\left[\theta_i \middle| s_i\right]$ and $\tilde{k}_n = \left(\sum_{i=1}^n k_i\right)\Big/n$ it is easily checked that

$$E\left[\left(\tilde{x}_n^{CPT} - \tilde{x}_n^C\right)^2\right] = \left(\left(\beta+\lambda\right)^{-1} - \left(\beta+\lambda+\beta n^{-1}\right)^{-1}\right)^2 E\left[\left(\alpha - \tilde{k}_n\right)^2\right],$$

where $\tilde{k}_n = \xi \tilde{s}_n + \left(1-\xi\right)\overline{\theta}$. It follows that $E\left[\left(\alpha - \tilde{k}_n\right)^2\right]$ is of the order of a constant since $\text{var}\left[\tilde{k}_n^2\right] = \xi^2 var\left[\tilde{s}_n\right]$ is. The order of $E\left[\left(p_n^C - p_n^{CPT}\right)^2\right]$ and of $E\left[\left(\tilde{x}_n^{CPT} - \tilde{x}_n^C\right)^2\right]$ is the order of $\left(\left(\beta+\lambda\right)^{-1} - \left(\beta+\lambda+\beta n^{-1}\right)^{-1}\right)^2$ which is $1\big/n^2$.

Similarly as in Section 3 (See Lemma 1 in Vives (2002)) we can decompose the deadweight loss at the Bayesian Cournot equilibrium in relation to the price-taking allocation letting $u_{in}^C \equiv x_{in}^C - \tilde{x}_n^C$ and $u_{in}^{CPT} \equiv x_{in}^{CPT} - \tilde{x}_n^{CPT}$ as:

$$\left(ETS_n^{CPT} - ETS_n^C\right)\Big/n = \left(\left(\beta+\lambda\right)E\left[\left(\tilde{x}_n^{CPT} - \tilde{x}_n^C\right)^2\right] + \lambda E\left[\left(u_{in}^{CPT} - u_{in}^C\right)^2\right]\right)\Big/2.$$

It is easily checked also that $E\left[\left(u_{in}^{CPT} - u_{in}^C\right)^2\right] = \left(\lambda^{-1} - \left(\lambda + \beta n^{-1}\right)^{-1}\right)^2 E\left[\left(k_i - \tilde{k}_n\right)^2\right]$. The order of $E\left[\left(k_i - \tilde{k}_n\right)^2\right]$ will be the same as $\xi^2 E\left[\left(s_i - \tilde{s}_n\right)^2\right]$ which is the order of a constant. Since the order of $\left(\lambda^{-1} - \left(\lambda + \beta n^{-1}\right)^{-1}\right)^2$ is $1\big/n^2$, it follows that the order of $E\left[\left(u_{in}^{CPT} - u_{in}^C\right)^2\right]$ is also $1\big/n^2$. We conclude that $\left(ETS_n^{CPT} - ETS_n^C\right)\Big/n$ is of order $1\big/n^2$. ♦

S.3. Simulations

This section provides details and further results on the simulations performed with the basic and with the replica models, as well as providing a comparison with the Cournot model.

S.3.1 The basic model

This subsection presents details and a complementary illustration of the welfare simulations in Section 3.2 of the basic model (Section 2). Simulations have been performed in the following base case parameter grid: β in $\{.5,1\}$, λ in $\{.5,1,5\}$, $\rho \in [0,.99]$ with step size .01, σ_ε^2 in $[0,10]$ and σ_θ^2 in $[.01,10]$ with step size 1, and $n \in [2,5,10,15,20,25]$. In this range of simulations and with $\bar{\theta} = 20$, $\alpha = 200$ the probability of a negative output is at most 13% in either the strategic or price-taking equilibrium, as well as in the Cournot equilibrium. We have than 97% of points in the grid have a maximal probability of negative output of less than 1%.

Figure S.1a depicts the evolution of $E[DWL]$ as ρ increases for different numbers of sellers and complements Figure 2. Note that $E[DWL]$ is higher for $n=3$ than for $n=2$ when ρ is close to 1. We have in fact in this case that $d(n=3) > d(n=2)$. Indeed, when ρ is close to 1, $c<0$ and d need not be decreasing in n. When $c<0$, the simulations show that a possible pattern is for $(n-1)c$ to have a U-shaped form with n (and therefore $d = \left(\beta^{-1} + (n-1)c\right)^{-1}$ a hump-shaped form). Simulations have been extended to the range of parameters β and λ in $[1,10]$, with step size 1, and n up to 30.

$E[DWL]$

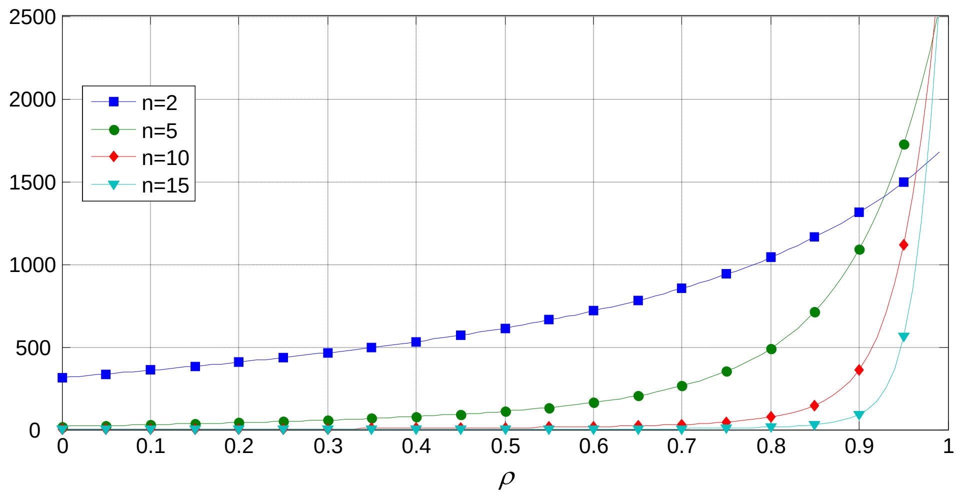

Figure S.1a. $E[DWL] \equiv ETS^o - ETS$ as a function of ρ for different values of n (with parameters $\beta = \lambda = 1$, $\sigma_\theta^2 = \sigma_\varepsilon^2 = 1$).

Increasing ρ may decrease $E[DWL]$ when σ_ε^2 is small for a range of ρ bounded away from 1 (see Figure S.1b) and increasing σ_ε^2 may decrease $E[DWL]$ when ρ is small (see Figure S.1c).

$E[DWL]$

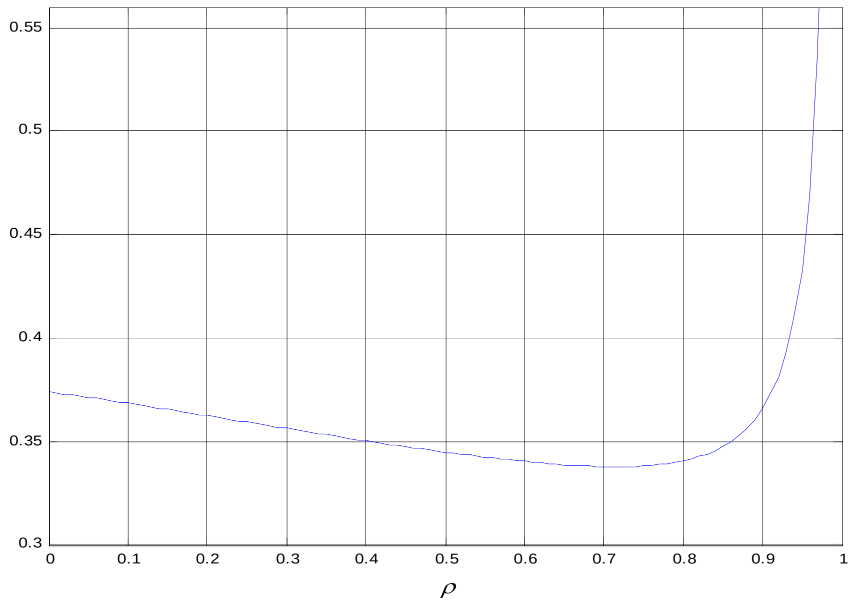

Figure S.1b. $E[DWL]$ as a function of ρ (with parameters $\sigma_\varepsilon^2 = .01$, $\beta = \lambda = 1$, $\sigma_\theta^2 = 1$, $n = 4$, $\alpha = 20$, $\overline{\theta} = 5$).

$E[DWL]$

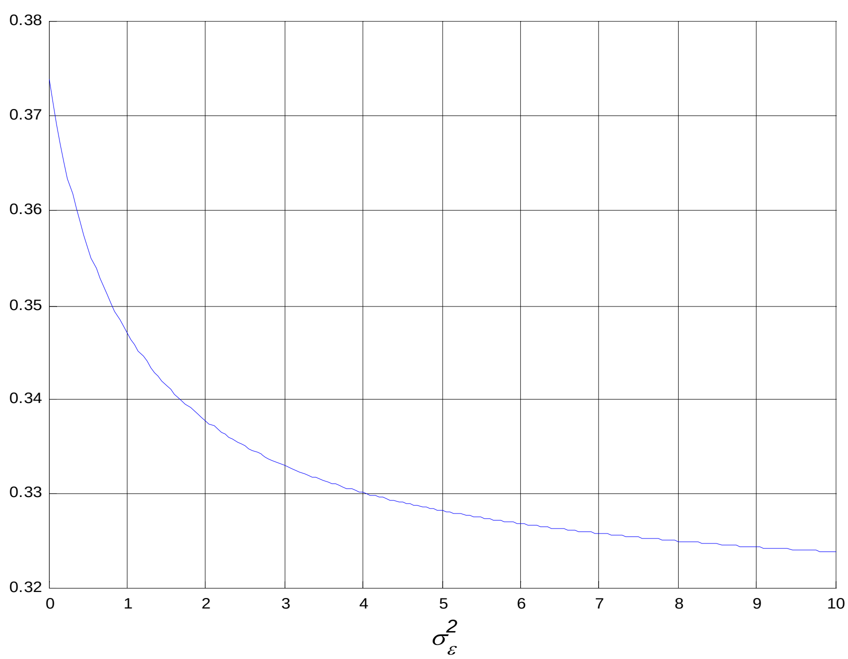

Figure S.1c. $E[DWL]$ as a function of σ_ε^2 (with parameters $\rho = .01$, $\beta = \lambda = 1$, $\sigma_\theta^2 = 1$, $n = 4$, $\alpha = 20$, $\overline{\theta} = 5$).

The welfare loss due to private information $ETS^f - ETS$ is typically increasing in ρ or σ_ε^2 except possibly for small values of those parameters (see Figure S.1d). An example where $ETS^f - ETS$ decreases in σ_ε^2 when ρ is small is the parameter constellation $\rho = .01$, $\lambda = 1$, $\beta = 5$, $\sigma_\theta^2 = 8$, $n = 10$ (and $\bar{\theta} = 20$, $\alpha = 200$) for σ_ε^2 large enough.

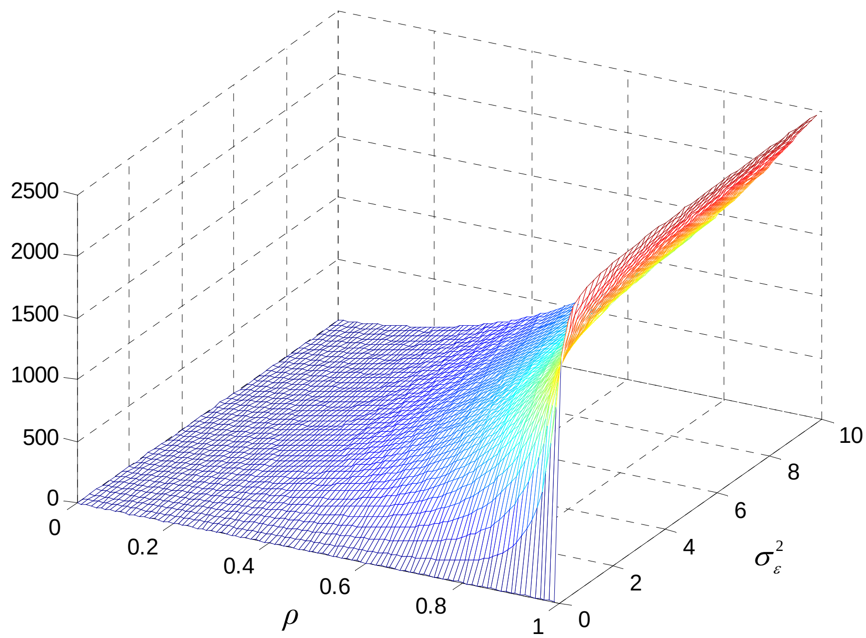

Figure S.1d. $ETS^f - ETS$ as a function of ρ and σ_ε^2 (with parameters $\beta = \lambda = 1$, $\sigma_\theta^2 = 1$).

Expected profits $E[\pi_i]$ increase in ρ (when $\sigma_\varepsilon^2 > 0$) or in σ_ε^2 (when $\rho > 0$) provided ρ or σ_ε^2 are not too small (see Figure S.2). Otherwise $E[\pi_i]$ may decrease in ρ or σ_ε^2, and this will tend to be so for σ_θ^2 large (see Figure S.3). Recall that $E[\pi_i]$ decrease in ρ when $\sigma_\varepsilon^2 = 0$ and in σ_ε^2 when $\rho = 0$ (Proposition 4(iv)).

Figure S.2 depicts the outcome of a typical simulation of $E[\pi_i]$. We can also check that when close to $(0,0)$, $E[\pi_i]$ decrease in ρ and σ_ε^2.

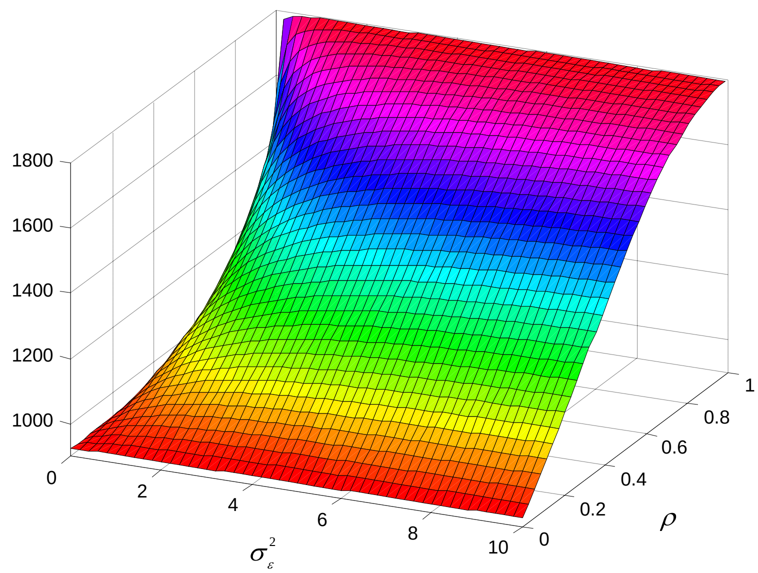

Figure S.2. $E[\pi_i]$ as a function of ρ and σ_ε^2 (with parameters $\beta = \lambda = 1$, $\sigma_\theta^2 = 1$, $n = 4$).

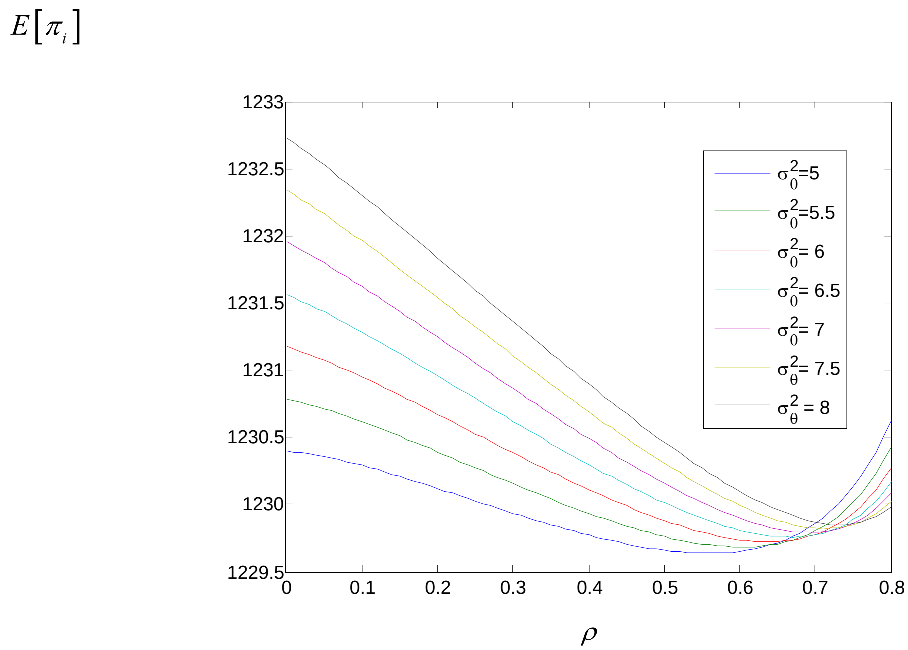

Figure S.3. $E[\pi_i]$ as a function of ρ for different values of σ_θ^2 (with parameters $\beta = \lambda = .5$, $\sigma_\varepsilon^2 = .01$ $n = 5$).

Furthermore, increasing σ_ε^2 may decrease $E\left[\pi_i\right]$ when ρ is small (always when $\rho = 0$, Proposition 4(iv)) (for example this happens when $\beta = 1$, $\lambda = 5$, $\rho = .01$, $\sigma_\theta^2 = 5$ and $n = 2$, $\alpha = 200$, $\bar{\theta} = 20$).

S.3.2 Replica markets

This subsection presents details and complementary results on the welfare simulations of the replica market model (Section 4).

Simulations have been performed for the base case β and λ in $\{1,5\}$, $\rho \in [.01,.99]$ with step size 0.01, σ_ε^2 and σ_θ^2 in $[0.01,10.01]$ with step size 2, and $n \in [2,82]$ with step size 20. (When needed we extend the simulations to ρ in $[.001,.999]$ with step size .001 and σ_ε^2 in $[0,10]$ with step size .01, and to large n) In this range of simulations and with $\bar{\theta} = 30$, $\alpha = 50$ the probability of a negative output is at most 15% in either the SFE (strategic or competitive versions) or the Cournot equilibrium and the upper bounds are attained only when $\beta = 5$ and $\lambda = 1$. Otherwise the probabilities of negative output tend to be very low.

Figures S.4a and S.4b provide the counterpart of Figure S.1 and Figure 2 for the replica market. Figure S.4a displays $E\left[DWL_n\right]/n \equiv \left(ETS_n^o - ETS_n\right)/n$ as a function of ρ for different values of n. It is worth noting that $E\left[DWL_n\right]/n$ is monotone in n, this is a general feature of the simulations.

$E\left[DWL_n\right]/n$

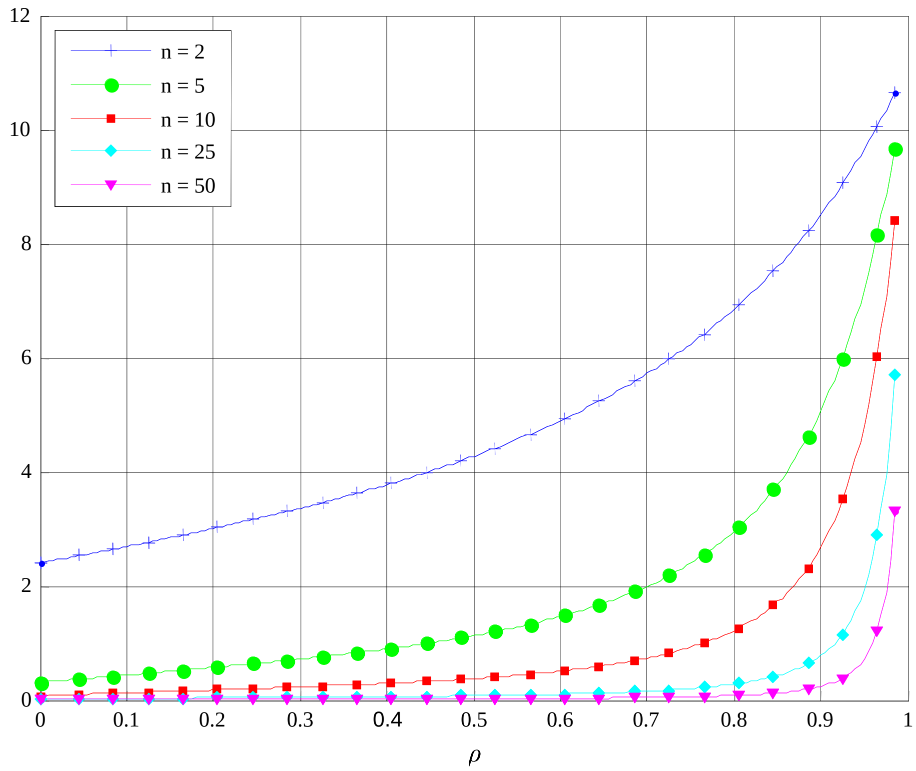

Figure S.4a $E\left[DWL_n\right]/n \equiv \left(ETS_n^o - ETS_n\right)/n$ as a function of ρ for different values of n (with parameters $\beta = \lambda = 1$, $\sigma_\theta^2 = \sigma_\varepsilon^2 = 1$, $\overline{\theta} = 30$, $\alpha = 50$).

Figure S.4b displays $E\left[DWL_n\right]/n$ as a function of σ_ε^2 for different values of ρ. Note that the effect of increases in σ_ε^2 are small when ρ is small.

$E[DWL_n]/n$

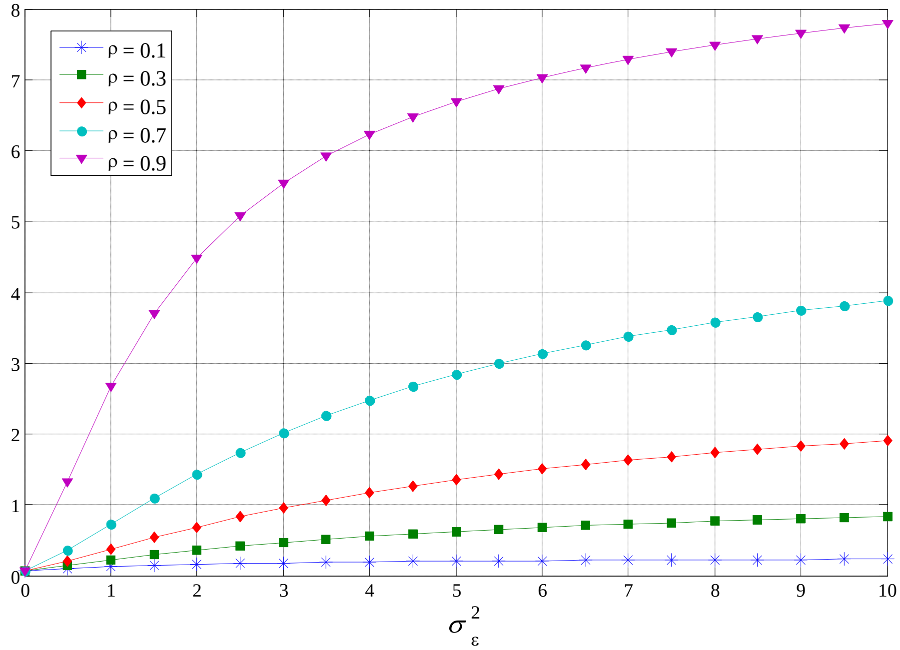

Figure S.4b. $E[DWL_n]/n$ as a function of σ_ε^2 for different values of ρ (with parameters $\beta = \lambda = 1$, $\overline{\theta} = 30$, $\alpha = 50$, $\sigma_\theta^2 = 1$, and $n = 10$).

The result of Figure S.4a generalizes and $E[DWL_n]/n$ decreases as the market gets large and it is found that the rate of decrease is slow for low σ_θ^2. The result is driven by the decrease in d_n with n (which overwhelms the effects that when averaging over predictions $E\left[\left(\alpha - \tilde{t}_n\right)^2\right]$ may increase with n and that $E\left[\left(t_i - \tilde{t}_n\right)^2\right]$ does increase with n). Typically, the speed of convergence of the deadweight loss to zero (in terms of the constant of convergence) is slower when ρ is larger. That is, the limit as n tends to infinity of $n\left(ETS_n^o - ETS_n\right)$ is increasing with ρ. (This is so since $\left(ETS_n^o - ETS_n\right)$ is typically increasing in ρ for any n and the limit of $n\left(ETS_n^o - ETS_n\right)$ as n tends to infinity is well defined.)

The simulations suggest also that the (per capita) deadweight loss at the full information equilibrium $\left(ETS_n^o - ETS_n^f\right)/n$ (or deadweight loss due to standard market power) also decreases with n. This is driven by the fact that it can be checked analytically that c_n^f increases in n and therefore d_n^f decreases in n. However, the (per capita) deadweight loss due to the private-information-induced market power (that is, $\left(ETS_n^f - ETS_n\right)/n$) may increase with n for n low. This is due to the fact that the deadweight loss due to standard market power falls more sharply with n at the beginning than the one due to the private information induced market power.[3]

S.3.3 Comparison with Cournot

It is worth to compare the relative efficiency of the Cournot market (ETS^C) in relation to the supply function market (denoted now by ETS^{SF}). A typical pattern for n not too large is for $ETS^{SF} - ETS^C$ to be positive for ρ close to zero and negative for ρ close to 1, being zero at the point for which the supply function equilibrium calls for a vertical supply and both equilibria coincide. Furthermore, when signals are perfect ($\sigma_\varepsilon^2 = 0$) or close to perfect we have that $ETS^{SF} - ETS^C > 0$. (See Figure S.5.) For σ_ε^2 or ρ small sellers at the supply function market act with full information and have less market power. In the (Bayesian) Cournot equilibrium sellers do not act with full information (see Section S.2).[4] For larger ρ and $\sigma_\varepsilon^2 > 0$ supply functions slope downwards and sellers in the supply function market have more market power and this may dominate the information effect.

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3 For example, this happens from $n = 2$ to $n = 4$ when $\beta = \lambda = 1$, $\overline{\theta} = 30$, $\alpha = 50$, $\sigma_\theta^2 = 1$, $\sigma_\varepsilon^2 = 10$ and $\rho = .9$.

4 However, with constant marginal costs the Bayesian Cournot equilibrium does replicate the full information outcome under some regularity conditions (see Palfrey (1985) and Vives (1988)).

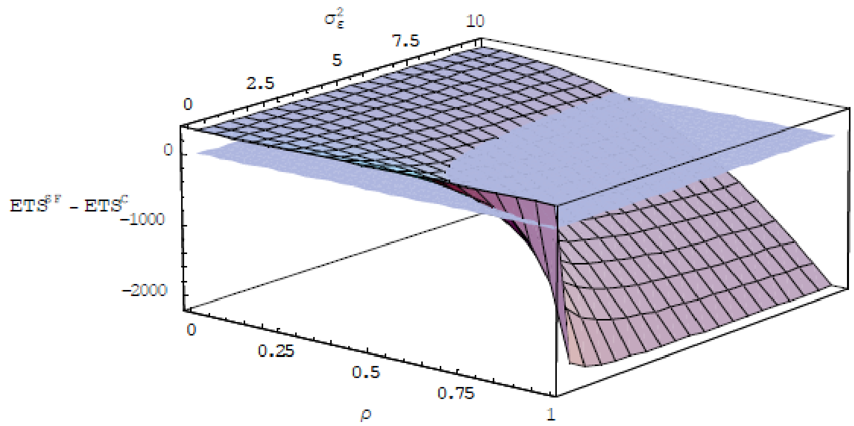
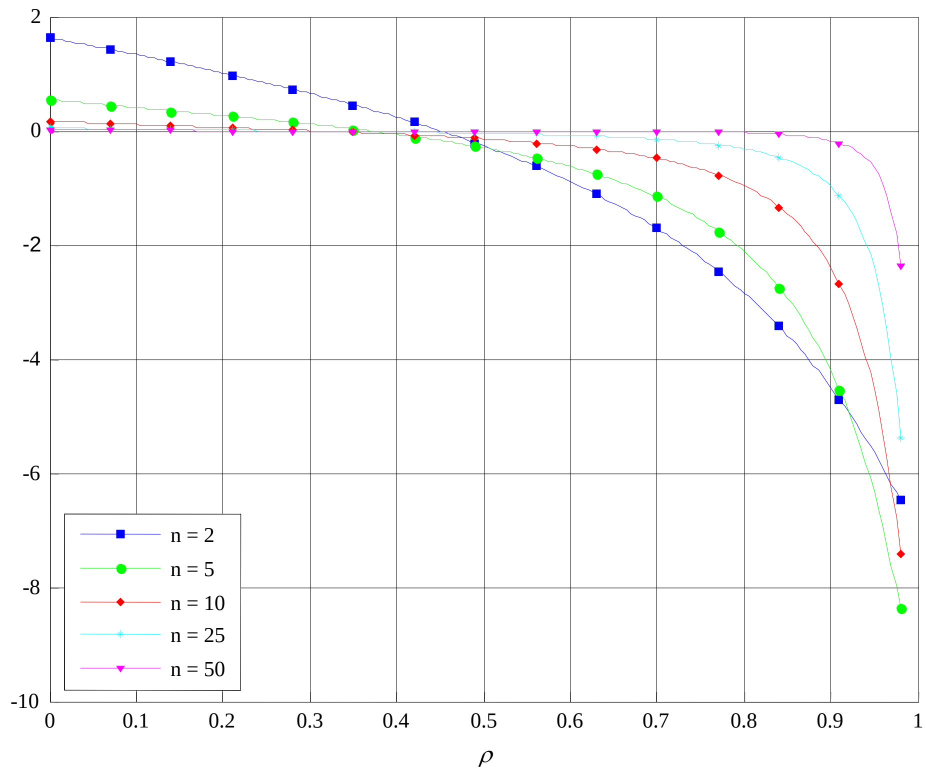


Figure S.5. Efficiency differential between supply function and Cournot equilibria $ETS^{SF} - ETS^{C}$ as a function of ρ and σ_ε^2 (with parameters $\beta = \lambda = 1$, and $\sigma_\theta^2 = 1$, $n = 4$, $\alpha = 200$, $\bar{\theta} = 20$).

It is worth comparing the relative efficiency of the Cournot market (ETS_n^C) in relation to the supply function market (ETS_n^{SF}) in per capita terms. For large n we have $\left(ETS_n^{SF} - ETS_n^{C}\right)/n > 0$ (except for ρ close to 1). (See Figure S.6a.)

$\left(ETS_n^{SF} - ETS_n^{C}\right)/n$



<u>Figure S.6a</u>. Efficiency differential $\left(ETS_n^{SF} - ETS_n^{C}\right)/n$ between supply function and Cournot equilibria as a function of ρ for different values of n (with parameters $\beta = \lambda = 1$, $\bar{\theta} = 30$, $\alpha = 50$, and $\sigma_\varepsilon^2 = \sigma_\theta^2 = 1$).

The typical pattern of $\left(ETS_n^{SF} - ETS_n^{C}\right)/n$ as a function of σ_ε^2 is similar to the one for ρ whenever $c < 0$ obtains for the parameters under consideration, being positive for σ_ε^2 small and negative for σ_ε^2 large, and zero at the point for which the supply function equilibrium calls for a vertical supply. See Figure S.6b.

$\left(ETS_n^{SF} - ETS_n^{C}\right)/n$

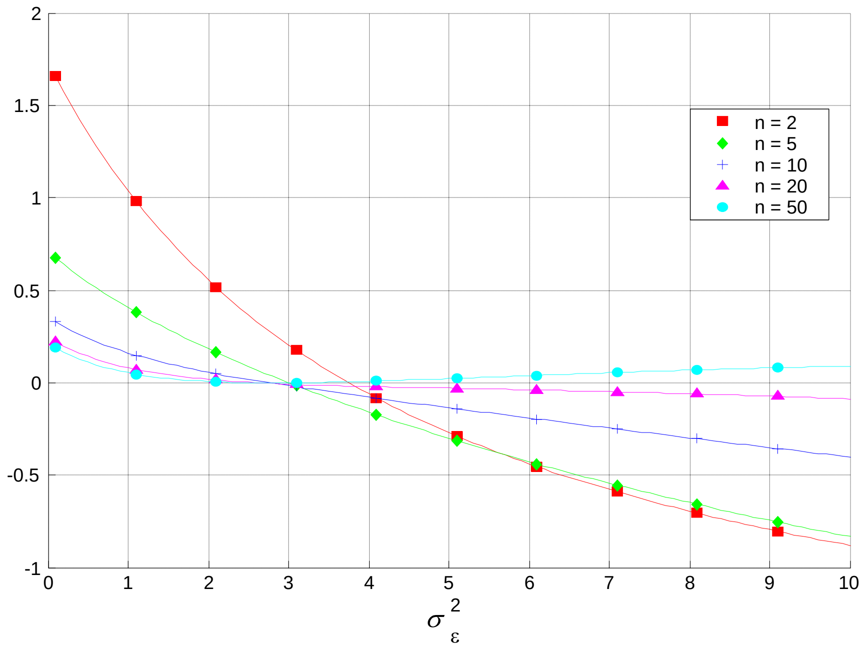


Figure S.6b. Efficiency differential $\left(ETS_n^{SF} - ETS_n^{C}\right)/n$ between supply function and Cournot equilibria as a function of σ_ε^2 for different values of n (with parameters $\beta = \lambda = 1$, $\bar{\theta} = 30$, $\alpha = 50$, $\sigma_\theta^2 = 5$, and $\rho = .5$).

Finally, and as an illustration of the theoretical result, for a given ρ we check that $\left(ETS_n^{SF} - ETS_n^{C}\right)/n > 0$ for large n. See Figure S.7.

$\left(ETS_n^{SF} - ETS_n^{C}\right)/n$

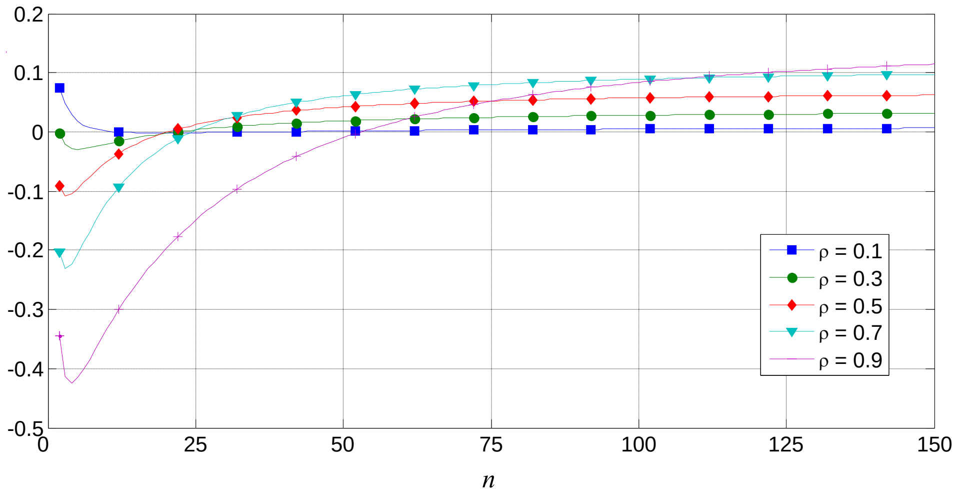


Figure S.7. Efficiency differential $\left(ETS_n^{SF} - ETS_n^{C}\right)/n$ between supply function and Cournot equilibria as a function of n (with parameters $\beta = \lambda = 1$, $\overline{\theta} = 15$, $\alpha = 20$, $\sigma_\theta^2 = 1$, and $\sigma_\varepsilon^2 = 5$).

S.4. Information acquisition

This section studies the incentives to acquire information in the supply function market.

Do sellers have incentives to gather information? If the sellers receive the private signals for free, in the course of their activity for example, then in a privately revealing equilibrium each seller has an incentive to rely on its private signal even though the price provides also information. Indeed, for seller i the signal s_i still helps in estimating θ_i even though p reveals $\tilde{s}$.

Consider the model of Section 2 and suppose now that private signals have to be purchased at a cost, increasing and convex in the precision $\tau_\varepsilon \equiv 1/\sigma_\varepsilon^2$ of the signal, according to a smooth function $H(\cdot)$ that satisfies $H(0) = 0$, $H' > 0$ for $\tau_\varepsilon > 0$, and $H'' \geq 0$. There are, thus, nonincreasing returns to information acquisition. A strategy for seller i is a pair $\left(\tau_{\varepsilon_i}, X_i(.,.)\right)$ determining the precision purchased and the supply function strategy. Note that we consider the case where each seller does not observe the precision purchased by other sellers and look therefore at a simultaneous move game where each seller chooses its precision and the supply function.

We analyze the symmetric Nash equilibria of the game. Note that in order for $\left(\tau_{\varepsilon_i}, X_i(\cdot,\cdot)\right)_{i=1,\dots n}$ to be a pure equilibrium of the game, $\left(X_i(\cdot,\cdot)\right)_{i=1,\dots n}$ needs to be the equilibrium of a game for a given precision tuple $\left(\tau_{\varepsilon_i}\right)_{i=1,\dots n}$.

Since we are interested in studying a symmetric equilibrium, we assume that any seller other than i, $j \neq i$, has the same precision, $1/\sigma_\varepsilon^2$, and the same coefficients, denoted by (b,a,c), for the candidate equilibrium supply function $X\left(s_j, p\right) = b - as_j + cp,\ j \neq i$. Seller i has precision $1/\sigma_{\varepsilon_i}^2$. Provided that $1+\beta(n-1)c > 0$ and exactly as in Section 2 we obtain an optimal supply function for seller i for given supply functions of the rivals:

$$X_i\left(s_i, p\right) = \left(p - E\left[\theta_i \middle| s_i, p\right]\right) \Big/ (d+\lambda)$$

with $d = \left(\beta^{-1} + (n-1)c\right)^{-1}$. Now, as in the proof of Proposition 1, from the point of view of seller i the price is informationally equivalent to $h_i \equiv \beta b(n-1) - \alpha + \left(1+\beta(n-1)c\right)p + \beta x_i = \beta a \sum_{j \neq i} s_j$, and therefore

$$\begin{pmatrix} \theta_i \\ s_i \\ h_i \end{pmatrix} \sim N\left(\begin{pmatrix} \bar{\theta} \\ \bar{\theta} \\ \beta a(n-1)\bar{\theta} \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \sigma_\theta^2 & (n-1)\rho\sigma_\theta^2 \\ \sigma_\theta^2 & \sigma_\theta^2 + \sigma_{\varepsilon_i}^2 & (n-1)\rho\sigma_\theta^2 \\ (n-1)\rho\sigma_\theta^2 & (n-1)\rho\sigma_\theta^2 & (n-1)\left(\left(\sigma_\theta^2 + \sigma_\varepsilon^2\right) + (n-2)\rho\sigma_\theta^2\right) \end{pmatrix} \right)$$

and using the projection theorem for normal random variables we obtain

$$E\left(\theta_i \mid s_i, h_i\right) = \bar{\theta} + \frac{\sigma_\theta^2\left(\sigma_\theta^2 + (n-2)\rho\sigma_\theta^2 + \sigma_\varepsilon^2\right) - (n-1)\left(\rho\sigma_\theta^2\right)^2}{\Delta_i}\left(s_i - \bar{\theta}\right) + \frac{\rho\sigma_\theta^2\sigma_{\varepsilon_i}^2}{\Delta_i}\left(h_i - \beta a(n-1)\bar{\theta}\right)$$

where $\Delta_i \equiv \left(\sigma_\theta^2 + \sigma_{\varepsilon_i}^2\right)\left(\left(1+(n-2)\rho\right)\sigma_\theta^2 + \sigma_\varepsilon^2\right) - (n-1)\sigma_\theta^4\rho^2$.

Using the expression for the supply function of i and for h_i and identifying coefficients with the candidate strategy $X_i(s_i, p) = b_i - a_i s_i + c_i p$ we obtain

$$a_i = \frac{\left(\sigma_\theta^2\left(\sigma_\theta^2 + (n-2)\rho\sigma_\theta^2 + \sigma_\varepsilon^2\right) - (n-1)\left(\rho\sigma_\theta^2\right)^2\right)a}{(d+\lambda)a\Delta_i + \rho\sigma_\theta^2\sigma_{\varepsilon_i}^2},$$

$$b_i = -\frac{a\left((1-\rho)\sigma_\theta^2 + \sigma_\varepsilon^2\right)\sigma_{\varepsilon_i}^2\overline{\theta} + \rho\sigma_\theta^2\sigma_{\varepsilon_i}^2\left((n-1)b - \alpha\beta^{-1}\right)}{(d+\lambda)a\Delta_i + \rho\sigma_\theta^2\sigma_{\varepsilon_i}^2}, \text{ and}$$

$$c_i = \frac{a\Delta_i - \rho\sigma_\theta^2\sigma_{\varepsilon_i}^2 d}{(d+\lambda)a\Delta_i + \rho\sigma_\theta^2\sigma_{\varepsilon_i}^2}.$$

Given $\left(\tau_\varepsilon, X(\cdot,\cdot)\right)_{j\neq i}$ with $X\left(s_j, p\right) = b - as_j + cp$ we have obtained the optimal supply strategy for seller i when he has precision $\tau_{\varepsilon_i} \equiv 1/\sigma_{\varepsilon_i}^2$. From the form of the optimal supply it follows that his expected profits from trading are given by

$$E\left[\pi_i\right] = \left(d + \frac{\lambda}{2}\right)E\left[\left(X_i\left(s_i, p\right)\right)^2\right].$$

Note that $E\left[\pi_i\right]$ is a function of $\left(b, a, c, \sigma_\varepsilon^2, \sigma_{\varepsilon_i}^2\right)$ since $d = \left(\beta^{-1} + (n-1)c\right)^{-1}$. After some lengthy manipulations it can be checked that $E\left[\pi_i\right] =$

$$\begin{aligned}
&\left(2\left(1+c\beta(n-1)\right)\left(2\beta+\lambda\left(1+c\beta(n-1)\right)\right)\right)^{-1}\times\\
&\left(\left(\sigma_\theta^4(1-\rho)\left(\rho(n-1)+1\right)+\sigma_\theta^2\sigma_\varepsilon^2+\sigma_{\varepsilon_i}^2\left(\sigma_\theta^2\left(\rho(n-2)+1\right)+\sigma_\varepsilon^2\right)\right)^{-1}\times\right.\\
&\left(\sigma_\theta^2\left(\sigma_\theta^2(1-\rho)\left(\rho(n-1)+1\right)+\sigma_\varepsilon^2\right)\left(\sigma_\theta^2\left(a\beta\rho(n-1)-\left(1+c\beta(n-1)\right)\right)^2\right.\right.\\
&\left.+a^2\beta^2(n-1)\left(\sigma_\theta^2(1-\rho)\left(\rho(n-1)+1\right)+\sigma_\varepsilon^2\right)\right)\\
&\left.+\sigma_{\varepsilon_i}^2(n-1)\left(\sigma_\theta^2\rho\left(1+c\beta(n-1)\right)-a\beta\left(\sigma_\theta^2\left(\rho(n-2)+1\right)+\sigma_\varepsilon^2\right)\right)^2\right)\\
&\left.+\left(\left(\alpha-\overline{\theta}\right)-\left(b+\overline{\theta}(c-a)\right)\beta(n-1)\right)^2\right).
\end{aligned}$$

When optimizing, seller i chooses $\sigma_{\varepsilon_i}^2$ and takes as given $\left(b,a,c,\sigma_{\varepsilon}^2\right)$. The marginal benefit of acquiring precision $\tau_{\varepsilon_i} \equiv 1/\sigma_{\varepsilon_i}^2$, $\partial E\left[\pi_i\right]/\partial\tau_{\varepsilon_i}$, evaluated at a symmetric solution $\tau_{\varepsilon_i} = \tau_{\varepsilon}$ can be seen to be equal to

$$\psi\left(\tau_{\varepsilon}\right) \equiv \frac{1}{2\left(2d+\lambda\right)} \frac{\left(\tau_{\varepsilon}\left(1-\rho\right)\left(1+\rho\left(n-1\right)\right)+\tau_{\theta}\right)^2}{\left(\tau_{\varepsilon}\left(1-\rho\right)+\tau_{\theta}\right)^2\left(\tau_{\varepsilon}\left(1+\rho\left(n-1\right)\right)+\tau_{\theta}\right)^2},$$

which is decreasing in τ_{ε} for a given c or d.[5] Interior symmetric equilibria are characterized by the solution of $\psi\left(\tau_{\varepsilon}\right)-H'\left(\tau_{\varepsilon}\right)=0$ with c given by the largest solution to the quadratic equation $g\left(c;M\right)=0$ for a given τ_{ε}. We know from Proposition 1 that there is a solution that fulfils $1+\beta nc>0$ provided that $-\left(n-1\right)^{-1}<\rho<1$. From Claim A.2 we know that $c\to\hat{c}$ as $\sigma_{\varepsilon}^2\to\infty$ or $\tau_{\varepsilon}\to 0$ (and $\hat{c}$ is decreasing in ρ and independent of τ_{θ}). Let $d\left(0+\right)\equiv\lim_{\tau_{\varepsilon}\to 0} d\left(\tau_{\varepsilon}\right)$. Then $\psi\left(0\right)\equiv\lim_{\tau_{\varepsilon}\to 0}\psi\left(\tau_{\varepsilon}\right)=\left(2\left(2d\left(0+\right)+\lambda\right)\tau_{\theta}^2\right)^{-1}>0$, and $\psi\left(\tau_{\varepsilon}\right)\to 0$ as $\tau_{\varepsilon}\to\infty$. If $H'\left(0\right)<\psi\left(0\right)$ for $\rho<1$ there is an interior solution $\tau_{\varepsilon}^*>0$ to the equation

$$\phi\left(\tau_{\varepsilon}\right)\equiv\psi\left(\tau_{\varepsilon}\right)-H'\left(\tau_{\varepsilon}\right)=0$$

since $\phi\left(0\right)>0$, $\phi\left(\infty\right)<0$ and $\phi\left(\cdot\right)$ is continuous.

If $H'\left(0\right)\geq\psi\left(0\right)$ then there can not be any information acquisition in a symmetric equilibrium and in fact there is no equilibrium (with $H'\left(0\right)$ not too high). We have that $\tau_{\varepsilon}^*=0$ at a candidate equilibrium but this can not be an overall equilibrium since if other sellers do not purchase information then the price contains no additional information for a seller and it will pay a single seller to get information (with $H'\left(0\right)$ not too high). This is akin to the Grossman-Stiglitz (1980) paradox on the impossibility of an informationally

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5 It can be checked that $E\left[\pi_i\right]$ is strictly concave in τ_{ε_i} if $d>0$ (which is always the case in equilibrium).

efficient market. As parameters β, λ, ρ and n move in such a way that $\psi(0) \downarrow H'(0)$, then $\tau_\varepsilon^* \to 0$ and the linear supply function equilibrium collapses.

The following proposition characterizes the equilibrium in the information acquisition game.[6]

Proposition S.3. Let $-(n-1)^{-1} < \rho < 1$, $\psi(0) \equiv \left(2\left(2d(0+)+\lambda\right)\tau_\theta^2\right)^{-1}$ and $d(0+) = \left(\beta^{-1} + (n-1)\hat{c}\right)^{-1}$. There is a symmetric equilibrium in the game with costly information acquisition provided that $H'(0) < \psi(0)$. At equilibrium sellers buy a positive precision of information $\tau_\varepsilon^* > 0$.

Remark S.3: Existence obtains in particular if $H'(0) = 0$ or the prior is diffuse enough (τ_θ small) even if the number of sellers is large and/or ρ close to 1. If $H'(0) > 0$, for any $-(n-1)^{-1} < \rho < 1$ and n we will have $\tau_\varepsilon^* > 0$ if $H'(0)$ or τ_θ are small enough. As $\rho \to 1$ we have that $\psi(0) \to \left(2\left(2\beta n + \lambda\right)\tau_\theta^2\right)^{-1}$ since $\hat{c} \to -1/\beta n$ (see Claim A.2) and $d(0+) \to \beta n$. Therefore for ρ close to 1 and a large number of sellers we will need a very diffuse prior in order to have positive precision acquisition (at the unique equilibrium). If $H'(0) > 0$ then for ρ close to 1 and n large enough there is no purchase of information. However, the situations changes, as we will see, in the natural case of a large market where the number of buyers and sellers grow together (in the replica economy).

The intuition for the result is as follows. The marginal benefit of acquiring precision τ_ε is declining with the level of precision acquired and is positive for $\tau_\varepsilon = 0$ with a finite

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6 Jackson (1991) shows the possibility of fully revealing prices in a common value environment with costly information acquisition with a finite number of agents and under some specific parametric assumptions.

number of sellers even if parameters are very correlated ($\psi(0) > 0$). This is so since even with high correlation a seller by purchasing a signal will improve the information on his random cost parameter despite learning the signals of the other sellers through the price. When the number of sellers is large the improvement will be small but if the seller can purchase a little bit of precision at a small cost he will do it. Furthermore, the more diffuse is the prior the higher the marginal value of information.

Let us consider the replica economy of Section 4 (with $\rho \in [0,1)$). Replacing β by β/n we obtain that $\psi_n(0) \equiv \left(2\left(2d_n(0+)+\lambda\right)\tau_\theta^2\right)^{-1}$ where $d_n(0+) = \left(n\beta^{-1} + (n-1)\hat{c}_n\right)^{-1}$. Proposition S.3 applies with $\psi_n(0)$ instead of $\psi(0)$. Now, for any n as $\rho \to 1$ we have that $\hat{c} \to -1/\beta$, $d_n(0+) \to \beta$ and $\psi_n(0) \to \left(2\left(2\beta+\lambda\right)\tau_\theta^2\right)^{-1}$. For ρ close to 1 for any number of sellers (size of replica) we need the same degree of diffusion of the prior in order to have positive precision acquisition.

What happens in a large market (that is, a market where the limit $n \to \infty$ is taken first)? When the number of buyers and sellers grow together we can consider the large market limit case as $n \to \infty$. It is possible to show then that whenever the marginal cost of acquiring precision at zero is positive, there is an upper bound on the degree of correlation of costs (strictly less than 1 and increasing in the diffuseness of the prior) below which there is an information acquisition equilibrium. (See Vives (2011).)

Remark S.4: We could also consider the case where information acquisition is observable in a first stage of the game and then at a second stage sellers compete in supply functions for given precisions. In this case we add a strategic effect of information acquisition and the characterization is much more involved. However, it is possible at least to check that the results are similar for ρ close to 1. (For example, we also have that $\psi_n(0) \to \left(2\left(2\beta+\lambda\right)\tau_\theta^2\right)^{-1}$ as $\rho \to 1$.)

Summary: If the signals are costly to acquire and agents face a convex cost of acquiring precision then it is possible to show that each seller will have an incentive to purchase some precision for any correlation of the costs parameters which is not perfect provided that at symmetric solutions the marginal cost of acquiring precision for zero precision is less than the marginal benefit. This will happen, for example, whenever the marginal cost of acquiring precision is zero for zero precision or when the prior diffuse enough (since then the marginal benefit of acquiring precision at zero precision is large) even if the number of sellers is large and/or ρ close to 1. The result is obtained in the case where each seller does not observe the precision purchased by other sellers and considers therefore a simultaneous move game where each seller chooses his precision and the supply function. In the case of a large market where the number of buyers and sellers grow together whenever the marginal cost of acquiring precision at zero is positive, there is an upper bound on the degree of correlation of costs (increasing in the diffuseness of the prior) below which there is an information acquisition equilibrium.

S.5 Demand uncertainty

This section deals with the case where demand is uncertain and characterizes the SFE providing the analysis and full results for Section 5.2.

Let $P(y) = \alpha + u - \beta y$ with $u \sim N(0, \sigma_u^2)$ independent of the other random variables in the model. The analysis of the equilibrium proceeds as in Section 2.1 positing candidate linear strategies $X(s_j, p) = b - as_j + cp$, $j \neq i$. Now the intercept of residual demand $p = I_i - dx_i$ for seller i is given by

$$I_i \equiv d\left((\alpha + u)\beta^{-1} - (n-1)b + a\sum\nolimits_{j \neq i} s_j\right) \text{ with } d \equiv \left(\beta^{-1} + (n-1)c\right)^{-1},$$

where I_i is informationally equivalent to $h_i \equiv u + \beta a \sum\nolimits_{j \neq i} s_j$.

S.5.1 Characterization of the equilibrium

The following proposition provides a full statement of the results presented in Section 5.2 (including Proposition 8).

Proposition S.5. Let $\lambda > 0$. For any $\rho \in \left[-(n-1)^{-1},\ 1\right]$, $\sigma_u^2 > 0$, and $\sigma_\varepsilon^2 \geq 0$ there exists a SFE. It is given by $X(s_i, p) = \left(p - E\left[\theta_i \middle| s_i, p\right]\right) \big/ (d + \lambda)$ with $0 < d < \beta n$.

(i) Let $\sigma_\varepsilon^2 > 0$, then when $\rho \in (0,1]$, or $\rho \in \left(-(n-1)^{-1}, 0\right)$, d is a root of the sixth degree polynomial

$$\begin{aligned}\Gamma(d) &= \rho \Upsilon(d)\left(\Lambda(d)(1-\rho) + dn(d + \lambda + n\beta)\right)^2 \sigma_u^2 \\ &+ \beta^2 \sigma_\theta^2 (1-\rho)\left(\rho(n-1)+1\right)\left(n\rho - M(1-\rho)\right)(n-1)^{-1}(d - n\beta)^2 \Lambda(d)\end{aligned}$$

where

$$\Lambda(d) = -(M+n)d^2 - \left((\lambda - n\beta)M + n\left(\beta(n-2)+\lambda\right)\right)d + n\beta\lambda(M+1),$$

$$\Upsilon(d) = -d^2 - \left(\beta(n-2)+\lambda\right)d + \beta\lambda, \text{ and } M \equiv \frac{\rho\sigma_\varepsilon^2 n}{(1-\rho)\left(\sigma_\varepsilon^2 + (1+(n-1)\rho)\sigma_\theta^2\right)},$$

that belongs to the interval (Υ_2, Λ_2) when $\rho \in (0,1]$, or to (Λ_2, Υ_2) when $\rho \in \left(-(n-1)^{-1}, 0\right)$, with $\Upsilon_2 \equiv$ largest root of $\Upsilon(d)$, $\Lambda_2 \equiv$ largest root of $\Lambda(d)$. The root is unique if $\rho \in (0,1]$ or $\rho \in \left(-(n-1)^{-1}, 0\right)$ and $n > 3$. When $\rho = -(n-1)^{-1}$ and $\sigma_\varepsilon^2 > 0$, d is a root (unique if $n > 3$) in $(0, \Upsilon_2)$ of the sixth degree polynomial $\dfrac{\Gamma(d)}{\left((n-1)\rho+1\right)\left(n\rho - M(1-\rho)\right)}$. When $\rho\sigma_\varepsilon^2 = 0$, $d = \Lambda_2 = \Upsilon_2 = d^f$.

(ii) As $\sigma_u^2 \to 0$, $d \to \Lambda_2$, and as $\sigma_u^2 \to \infty$, $d \to \Upsilon_2 = d^f$. In the cases with a unique equilibrium: When $\sigma_\varepsilon^2 > 0$, d increases in ρ for $\rho \in \left(-(n-1)^{-1},\ 1\right)$, and when $\rho\sigma_\varepsilon^2 > 0$ (resp. $\rho\sigma_\varepsilon^2 < 0$) d is decreasing (resp. increasing) with σ_u^2.

(iii) If $\rho\sigma_\varepsilon^2 > 0$, then $\partial d / \partial \sigma_\varepsilon^2 > 0$ if $\sigma_\varepsilon^2 \leq \sigma_\theta^2$.

Corollary: We have that $c=\left(d^{-1}-\beta^{-1}\right)/(n-1)$ with $c>-M\left((1+M)\beta n\right)^{-1}$ (and $\lambda^{-1}>c$ when $\rho\geq 0$)

$$a=\frac{(n\beta-d)(n\rho-M(1-\rho))}{(n-1)\rho((1-\rho)\Lambda(d)+dn(d+\lambda+n\beta))}>0,$$

$$b=\left(-\frac{(n\beta-d)}{d(n-1)(d+\lambda+n\beta)}+a\right)\bar{\theta}-\frac{\alpha}{d\beta(n-1)(d+\lambda+n\beta)}\Upsilon(d)$$

and when $\rho\sigma_\varepsilon^2>0$, both a and c increase in σ_u^2, and c is non-monotone in σ_ε^2.

Remark S.5: Simulations suggest that the equilibrium is unique also when $\rho<0$ and $n=2,3$.

Remark S.6: The largest root of $\Upsilon(d)$ is $\Upsilon_2=d^f$, the full information solution when $\sigma_u^2=0$ (as in Proposition 1 when $\rho\sigma_\varepsilon^2=0$). In equilibrium therefore $d>d^f$ when $\rho\sigma_\varepsilon^2>0$ and $d<d^f$ when $\rho\sigma_\varepsilon^2<0$.

Remark S.7: The largest root Λ_2 of $\Lambda(d)$ is the solution for d in Proposition 1 (where $\sigma_u^2=0$). Therefore as $\sigma_u^2\to 0$, the equilibrium d tends to the value of d when $\sigma_u^2=0$.

Remark S.8: Conditions for the noise independence property (equilibrium independent of σ_u^2). (i) When $\rho\sigma_\varepsilon^2=0$ ($M=0$), then $\Lambda(d)=n\Upsilon(d)$ and $d=\Upsilon_2=\Lambda_2=d^f$. (ii) When $\sigma_\varepsilon^2\to\infty$ and $\sigma_u^2>0$ then again $d\to d^f$ (yielding $X(p)=c^f\left(p-\bar{\theta}\right)$). When $\rho=0$, $d=d^f$. If $\rho\in(0,1]$ or $-(n-1)^{-1}<\rho<0$, then $(n\rho-M(1-\rho))\to 0$ as $\sigma_\varepsilon^2\to\infty$ and $\Gamma(d)=0$ if and only if $\Upsilon(d)=0$. If $\rho=-(n-1)^{-1}$ then $\frac{\Gamma(d)}{((n-1)\rho+1)(n\rho-M(1-\rho))\sigma_\varepsilon^2}\to\frac{\beta^2(n-1)(2d+\lambda)^2}{\sigma_\theta^2}\Upsilon(d)n\sigma_u^2$ as $\sigma_\varepsilon^2\to\infty$ and

$d \to \Upsilon_2 = d^f$ (recall that the S.O.C. is $2d + \lambda > 0$). It is easily checked that this limit is also the equilibrium when $\sigma_\varepsilon^2 = \infty$ ($\tau_\varepsilon = 0$).

Claim S.1. The coefficient of h_i in $E(\theta_i \mid s_i, h_i)$ increases in ρ if $\sigma_\varepsilon^2 > 0$.

Remark S.9: When $\lambda = 0$ we need $d > 0$ to fulfill the S.O.C. $2d > 0$. We have that $\Upsilon_2 = 0$ and $\Lambda_2 = \frac{-n\beta(n-2-M)}{M+n}$ and since $M + n > 0$ for a SFE to exist we need $n - 2 - M < 0$ proving Remark 7 in Section 5.2.

Claim S.2: With inelastic noisy demand u, $\rho = 1$, $n > 2$, and $\lambda = 0$, there is a unique SFE: $d = \frac{n\sigma_\theta^2}{\left(2\sigma_\varepsilon^2 + n\sigma_\theta^2\right)}\left(\frac{n\sigma_\varepsilon^2}{(n-1)(n-2)\sigma_u^2}\right)^{1/2}$ and $\left(nc\right)^{-1} = \frac{\sigma_\theta^2}{2\sigma_\varepsilon^2 + n\sigma_\theta^2}\left(\frac{(n-1)n\sigma_\varepsilon^2}{(n-2)\sigma_u^2}\right)^{1/2}$.

S.5.2 Proof of results

Proof of Proposition S.5 (i): Existence and uniqueness of a SFE. Given linear strategies of rivals $X\left(s_j, p\right) = b - as_j + cp$, $j \neq 1$, seller i faces a residual inverse demand

$$p = \alpha + u - \beta\left(n-1\right)\left(b + cp\right) + \beta a \sum\nolimits_{j \neq i} s_j - \beta x_i .$$

Provided $1 + \beta\left(n-1\right)c > 0$ it follows that $p = I_i - dx_i$ where

$$I_i \equiv d\left(\left(\alpha + u\right)\beta^{-1} - \left(n-1\right)b + a\sum\nolimits_{j \neq i} s_j\right) \text{ and } d \equiv \left(\beta^{-1} + \left(n-1\right)c\right)^{-1}.$$

Note that $\left(s_i, I_i\right)$ is informationally equivalent to $\left(s_i, h_i\right)$, where $h_i = u + \beta a \sum\nolimits_{j \neq i} s_j$ and therefore $E\left(\theta_i \mid s_i, I_i\right) = E\left(\theta_i \mid s_i, h_i\right)$ and

$$\operatorname{var}\left[s_i, h_i\right] = \begin{pmatrix} \sigma_\theta^2 + \sigma_\varepsilon^2 & \beta a(n-1)\rho\sigma_\theta^2 \\ \beta a(n-1)\rho\sigma_\theta^2 & \sigma_u^2 + \left(\beta a\right)^2\left(\left(n-1\right)\left(\sigma_\theta^2 + \sigma_\varepsilon^2\right) + \left(n-1\right)\left(n-2\right)\rho\sigma_\theta^2\right) \end{pmatrix}.$$

From the Gaussian updating formulæ it follows that

$$E\left(\theta_i \mid s_i, h_i\right) = \bar{\theta} + \frac{\left(\sigma_u^2 + a^2\beta^2\left(n-1\right)\left(\sigma_\theta^2\left(1-\rho\right)\left(\left(n-1\right)\rho+1\right)+\sigma_\varepsilon^2\right)\right)\sigma_\theta^2}{\Delta}\left(s_i - \bar{\theta}\right)$$
$$+\frac{\left(n-1\right)\beta\rho a\sigma_\theta^2\sigma_\varepsilon^2}{\Delta}\left(h_i - \beta a\left(n-1\right)\bar{\theta}\right),$$

where $\Delta \equiv \sigma_u^2\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)+a^2\beta^2\left(n-1\right)\left(\sigma_\varepsilon^2+\sigma_\theta^2\left(1-\rho\right)\right)\left(\sigma_\theta^2\left(\left(n-1\right)\rho+1\right)+\sigma_\varepsilon^2\right)$.

From the first order condition of the optimization problem for seller i, we obtain the optimal supply $X\left(s_i, p\right) = \left(p - E\left[\theta_i \middle| s_i, p\right]\right) / \left(d+\lambda\right)$ with second order condition $2d+\lambda > 0$. Taking into account the expression of the price and the symmetry of the equilibrium we obtain $h_i = p\left(1+\beta nc\right)-\alpha+\beta nb-\beta as_i$. Substituting in the expression for $E\left(\theta_i \mid s_i, h_i\right)$ and in the optimal supply for seller *i*, noting that $X\left(s_i, p\right) = b - as_i + cp$ and identifying coefficients, we have the following system of equations for a, b and c where $d = \left(\beta^{-1}+(n-1)c\right)^{-1}$:

$$a = \frac{\sigma_\theta^2\left(a^2\beta^2\left(n-1\right)\left(1-\rho\right)\left(\sigma_\theta^2\left(\rho\left(n-1\right)+1\right)+\sigma_\varepsilon^2\right)+\sigma_u^2\right)}{\Delta\left(d+\lambda\right)}$$
$$b = \left(d+\lambda\right)^{-1}\times$$
$$\left(\frac{\sigma_\theta^2\left(a^2\beta^2\left(n-1\right)\left(1-\rho\right)\left(\sigma_\theta^2\left(\rho\left(n-1\right)+1\right)+\sigma_\varepsilon^2\right)+\sigma_u^2\right)\bar{\theta}+\left(n-1\right)\rho\beta a\sigma_\theta^2\sigma_\varepsilon^2\left(\alpha-n\beta b+n\beta a\bar{\theta}\right)}{\Delta}-\bar{\theta}\right)$$
$$c = \left(1-\frac{\left(n-1\right)\rho\beta a\sigma_\theta^2\sigma_\varepsilon^2\left(n\beta c+1\right)}{\Delta}\right)\left(d+\lambda\right)^{-1}$$

From the expression for c it follows that

$$a^2 = \frac{\sigma_u^2\left(\left(d+\lambda\right)^{-1}-c\right)}{\left(n-1\right)\rho\beta\sigma_\varepsilon^2\left(n\beta c+1\right)-\beta^2\left(n-1\right)\left(1-\rho\right)\left(\sigma_\theta^2\left(\rho\left(n-1\right)+1\right)+\sigma_\varepsilon^2\right)\left(\left(d+\lambda\right)^{-1}-c\right)},$$

and using this expression in the first equation for a, we obtain

$$a = \left(1+n\beta c\right)\left(1+\left(n-1\right)\beta c\right)\sigma_\theta^2\left(f\left(c\right)\right)^{-1}$$

where

$$f(c) \equiv c^2\lambda\beta^2(n-1)^2\left(\sigma_\theta^2(1-\rho)+\sigma_\varepsilon^2\right)$$
$$+c\beta(n-1)\Big(2\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)(\beta+\lambda)+\sigma_\theta^2\rho\big(\beta(n-2)-\lambda\big)\Big)$$
$$+\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)(2\beta+\lambda)+\sigma_\theta^2\beta\rho(n-1).$$

Using the previous two expressions for a^2 and a in the first equation for c and after tedious computations, using $d=\left(\beta^{-1}+(n-1)c\right)^{-1}$, when $\rho \in (0,1]$, or $\rho \in \left(-(n-1)^{-1},0\right)$ we obtain the equilibrium d as the solution to $\Gamma(d)=0$ as defined in the proposition.

Let Λ_1 and Λ_2 (Υ_1 and Υ_2, respectively) denote the smallest and largest roots of the quadratic equations $\Lambda(d)$ ($\Upsilon(d)$, respectively). The discriminant of the quadratic equations is always positive for $\lambda>0$. We have that

$\Upsilon_2=\beta-\frac{1}{2}\lambda-\frac{1}{2}n\beta+\frac{1}{2}\sqrt{\beta^2(n-2)^2+\lambda^2+2n\beta\lambda}$ and

$$\Lambda_2=\frac{-\lambda(M+n)+n\beta(M-n+2)+\sqrt{n^2(M-n+2)^2\beta^2+2n\lambda(M+n)^2\beta+\lambda^2(M+n)^2}}{2(M+n)}.$$

It follows that $\Upsilon_2=\Lambda_2$ for $\rho\sigma_\varepsilon^2=0$ since then $M=0$. Let $\sigma_\varepsilon^2>0$, then $\Upsilon_1<\Lambda_1<-(\lambda/2)<0<\Upsilon_2<\Lambda_2$ for $\rho\in(0,1]$, and $\Lambda_1<\Upsilon_1<-(\lambda/2)<0<\Lambda_2<\Upsilon_2$ for $\rho\in\left(-(n-1)^{-1},0\right)$. In addition, $\Gamma(d)>0$ ($\Gamma(d)<0$) for all $d\in\left(\frac{-\lambda}{2},\Upsilon_2\right]$ and $\Gamma(d)<0$ ($\Gamma(d)>0$) for all $d\geq\Lambda_2$ for $\rho\in(0,1]$ ($\rho\in\left(-(n-1)^{-1},0\right)$). Therefore, we conclude that a root of $\Gamma(d)$ that satisfies $d>-\lambda/2$ (i.e. the S.O.C. $2d+\lambda>0$) exists and belongs to the interval (Υ_2,Λ_2) if $\rho\in(0,1]$ or to the interval (Λ_2,Υ_2) if $\rho\in\left(-(n-1)^{-1},0\right)$.

I show now that for $\rho \in (0,1]$, or $\rho \in \left(-(n-1)^{-1}, 0\right)$ and $n > 3$, d is the unique root of the sixth degree polynomial $\Gamma(d) = 0$. To obtain uniqueness of the equilibrium we show that $\Gamma'(d) < 0$ (resp. $\Gamma'(d) > 0$) for any root of $\Gamma(d)$ belonging to the interval (Υ_2, Λ_2) for $\rho \in (0,1]$ (resp. to the interval (Λ_2, Υ_2) for $\rho \in \left(-(n-1)^{-1}, 0\right)$ and $n > 3$). This fact guarantees the uniqueness of the equilibrium. Note that $\Upsilon(d) < 0$ and $\Lambda(d) > 0$ for $d \in (\Upsilon_2, \Lambda_2)$ for $\rho > 0$, and $\Upsilon(d) > 0$ and $\Lambda(d) < 0$ for $d \in (\Lambda_2, \Upsilon_2)$ for $\rho < 0$.

We have that

$$\Gamma'(d) = \rho\sigma_u^2 \Big(\Upsilon'(d)\big(\Lambda(d)(1-\rho) + dn(d+\lambda+n\beta)\big)^2 +$$
$$2\Upsilon(d)\big(\Lambda(d)(1-\rho) + dn(d+\lambda+n\beta)\big)\big(\Lambda'(d)(1-\rho) + n(2d+\lambda+n\beta)\big)\Big)$$
$$+\frac{\beta^2\sigma_\theta^2(1-\rho)(\rho(n-1)+1)(\rho(M+n)-M)}{(n-1)}\Big(2(d-n\beta)\Lambda(d) + (d-n\beta)^2\Lambda'(d)\Big).$$

Using the fact that in a zero of $\Gamma(d)$

$$\rho\sigma_u^2 = -\frac{\beta^2\sigma_\theta^2(1-\rho)(\rho(n-1)+1)(\rho(M+n)-M)}{(n-1)} \frac{(d-n\beta)^2\Lambda(d)}{\Upsilon(d)\big(\Lambda(d)(1-\rho)+dn(d+\lambda+n\beta)\big)^2},$$

we obtain

$$\Gamma'(d) = \frac{\beta^2\sigma_\theta^2(1-\rho)(\rho(n-1)+1)(\rho(M+n)-M)}{(n-1)} \times$$
$$\Bigg(-\frac{(d-n\beta)^2\Lambda(d)}{\Upsilon(d)}\Upsilon'(d) - 2\big(\Lambda'(d)(1-\rho) + n(2d+\lambda+n\beta)\big)\frac{(d-n\beta)^2\Lambda(d)}{\Lambda(d)(1-\rho)+dn(d+\lambda+n\beta)}$$
$$+2(d-n\beta)\Lambda(d) + (d-n\beta)^2\Lambda'(d)\Bigg).$$

If $\rho \in (0,1]$ then $\rho(M+n) - M > 0$. Hence, the first factor of the previous product is positive. Concerning to the second factor, note that adding the first and the fourth terms, we obtain

$$-\frac{\left(d-n\beta\right)^2 \Lambda(d)}{\Upsilon(d)}\Upsilon'(d)+\left(d-n\beta\right)^2 \Lambda'(d)=\frac{\left(d-n\beta\right)^2}{\Upsilon(d)}\left(M\beta\left(n-1\right)\left(2d\lambda+\lambda^2+2d^2+n\beta\lambda\right)\right).$$

Since $d>\Upsilon_2$, we know that $\Upsilon(d)<0,$ which implies that the previous expression is negative. On the other hand, adding the second and the third terms,

$$-2\left(\Lambda'(d)(1-\rho)+n\left(2d+\lambda+n\beta\right)\right)\frac{\left(d-n\beta\right)^2 \Lambda(d)}{\Lambda(d)(1-\rho)+dn\left(d+\lambda+n\beta\right)}+2\left(d-n\beta\right)\Lambda(d)=$$
$$\frac{2\left(d-n\beta\right)\Lambda(d)}{\Lambda(d)(1-\rho)+dn\left(d+\lambda+n\beta\right)}\times$$
$$\left(\left(d\left(-d+2n\beta\right)\left(-M+M\rho+n\rho\right)+n\beta\left(-\rho+n\rho+1\right)\lambda+n^2\beta^2\left(\left(1-\rho\right)\left(M+2\right)+n\rho\right)\right)\right).$$

Since $d<\Lambda_2,$ we know that $\Lambda(d)>0$ and $d<n\beta,$ which implies that the previous expression is negative. Combining all these results it follows that $\Gamma'(d)<0$ at any zero of $\Gamma(d)$ belonging to the interval $\left(\Upsilon_2,\Lambda_2\right)$ if $\rho\in\left(0,1\right]$.

If $\rho\in\left(-\frac{1}{n-1},0\right)$ then $\rho\left(M+n\right)-M<0.$ Hence, in the equilibrium expression of $\Gamma'(d)$, the first factor is negative. Concerning the second factor, adding the first and the third terms, we obtain

$$-\frac{\left(d-n\beta\right)^2 \Lambda(d)}{\Upsilon(d)}\Upsilon'(d)+2\left(d-n\beta\right)\Lambda(d)=$$
$$\frac{\left(d-n\beta\right)\Lambda(d)}{\Upsilon(d)}\left(-\beta\left(n-2\right)\left(\lambda+n\beta\right)-d\left(\lambda+\beta\left(3n-2\right)\right)\right),$$

which is negative since $\Lambda(d)<0,$ $\Upsilon(d)>0$ and $d<n\beta$ whenever $d\in(\Lambda_2,\Upsilon_2).$ Adding the second and the fourth terms, we obtain

$$-\frac{\left(d-n\beta\right)^2 \Lambda(d)}{\Lambda(d)(1-\rho)+dn\left(d+\lambda+n\beta\right)} 2\left(\Lambda'(d)(1-\rho)+n\left(2d+\lambda+n\beta\right)\right)+\left(d-n\beta\right)^2 \Lambda'(d)=$$
$$\frac{\left(d-n\beta\right)^2}{\Lambda(d)(1-\rho)+dn\left(d+\lambda+n\beta\right)}\times g(d),$$

where

$$g(d)\equiv\left(2\left(M+n\right)d+\left(M\lambda+n\lambda+n\beta\left(-M+n-2\right)\right)\right)\frac{M-\rho\left(M+n\right)}{M+n}\Lambda(d)$$
$$-\frac{1}{M+n}n^2\beta\left(M+1\right)\left(2n\beta\left(M-n+2\right)d+\lambda\left(\lambda\left(M+n\right)+n\beta\left(3M+n+2\right)\right)\right).$$

Straightforward computations yield $\Lambda(d)(1-\rho)+dn\left(d+\lambda+n\beta\right)>0$ whenever $d\in(\Lambda_2,\Upsilon_2)$. Hence, if we show that $g(d)<0$ whenever $d\in(\Lambda_2,\Upsilon_2)$, then we can conclude that the second factor of $\Gamma'(d)$ is negative, and hence, $\Gamma'(d)>0.$

Note that the first term of $g(d)$ is negative since $M-\rho\left(M+n\right)>0$ whenever $\rho\in\left(-\frac{1}{n-1},0\right)$ and $\Lambda(d)<0$ whenever $d\in(\Lambda_2,\Upsilon_2)$. Now, if $n>3$ then $d<\Upsilon_2\le\frac{\lambda\left(\lambda\left(M+n\right)+n\beta\left(3M+n+2\right)\right)}{2n\beta\left(-M+n-2\right)}$. In this case the second term of $g(d)$ is negative, and hence, $g(d)<0$ whenever $d\in\left(\Lambda_2,\Upsilon_2\right)$.

If $\rho=0$ or $\rho=-\left(n-1\right)^{-1}$ then $\Gamma\left(d\right)\equiv 0$. If $\rho=-\left(n-1\right)^{-1}$ (and $\sigma_\varepsilon^2>0$) it can be checked directly that d is a root in $\left(0,\Upsilon_2\right)$ of the polynomial:

$$\Omega\left(d\right)\equiv\frac{\Gamma\left(d\right)}{\left(\left(n-1\right)\rho+1\right)\left(n\rho-M\left(1-\rho\right)\right)}=$$
$$\frac{\left(\sigma_\theta^2 d^2+\left(\sigma_\theta^2\left(\lambda-n\beta\right)-2\beta\sigma_\varepsilon^2\left(n-1\right)\right)d-\beta\lambda\left(\sigma_\varepsilon^2\left(n-1\right)+n\sigma_\theta^2\right)\right)^2}{\sigma_\theta^2\sigma_\varepsilon^2\left(n-1\right)^2}\Upsilon\left(d\right)n\sigma_u^2$$
$$+\beta^2\sigma_\theta^2\left(d-n\beta\right)^2\frac{n}{\left(n-1\right)^2}\Lambda\left(d\right).$$

We have that that $\Lambda_2 = 0$ and $\Lambda_1 < \Upsilon_1 < \frac{-\lambda}{2} < 0 = \Lambda_2 < \Upsilon_2$. In addition, $\Omega(d) > 0$ for all $d \in (\frac{-\lambda}{2}, 0]$ and $\Omega(d) < 0$ for all $d \geq \Upsilon_2$. Therefore, a root of $\Omega(d)$ that satisfies $d > \frac{-\lambda}{2}$ exists and belongs to the interval $(0, \Upsilon_2)$. The root can be shown to be unique if $n > 3$ with a development parallel to the case $\rho \in \left(-\frac{1}{n-1}, 0\right)$ (in this case $\Omega(d)$ is decreasing around the equilibrium value).

When $\rho\sigma_\varepsilon^2 = 0$ we have that $M = 0$ and from the equations for the coefficients a, b, c it follows that $c = (d+\lambda)^{-1}$ and $d = \Upsilon_2$ (note also that when $M = 0$, $\Lambda(d) = n\Upsilon(d)$).

Note that we know from Proposition 1 and 2 that as ρ ranges from $-(n-1)^{-1}$ to 1, M ranges from -1 to ∞ and Λ_2 ranges from 0 to βn. Therefore, when $\sigma_u^2 > 0$, $0 < d < \beta n$ for $\rho \in \left[-(n-1)^{-1}, 1\right]$. ♦

<u>Proof of Proposition S.5 (ii)</u>: I show that (ii.a) in the cases where the equilibrium is unique and $\rho\sigma_\varepsilon^2 > 0$ (resp. $\rho\sigma_\varepsilon^2 < 0$) d is decreasing (resp. increasing) with σ_u^2; (ii.b) as $\sigma_u^2 \to 0$, $d \to \Lambda_2$; (ii.c) as $\sigma_u^2 \to \infty$, $d \to \Upsilon_2$; and (ii.d) when $\sigma_\varepsilon^2 > 0$, $\rho \in \left(-(n-1)^{-1}, 1\right)$, $n > 3$ if $\rho < 0$, then d is increasing with ρ.

(ii.a) Let $\sigma_\varepsilon^2 > 0$, the result follows since in the cases where the equilibrium is unique and $\rho \in (0,1]$ (resp. $\rho \in \left(-\frac{1}{n-1}, 0\right)$ and $n > 3$), $\Gamma(d)$ is decreasing (resp. increasing) around the equilibrium value, and in the relevant range for d: $\operatorname{sgn}\left\{\partial\Gamma(d)/\partial\sigma_u^2\right\} = \operatorname{sgn}\left\{\rho\Upsilon(d)\right\} < 0$ (since it can be checked that if $\rho > -(n-1)^{-1}$, $\Lambda(d)(1-\rho) + dn(d+\lambda+n\beta) > 0$ for $d \in [\Lambda_2, \Upsilon_2]$). In the case $\rho = -(n-1)^{-1}$ and $n > 3$, $\Omega(d)$ is decreasing around the equilibrium value, and

$\text{sgn}\{\partial\Omega(d)/\partial\sigma_u^2\} = \text{sgn}\{\Upsilon(d)\} > 0$ (since it can be checked that $\sigma_\theta^2 d^2 + \left(\sigma_\theta^2(\lambda - n\beta) - 2\beta\sigma_\varepsilon^2(n-1)\right)d - \beta\lambda\left(\sigma_\varepsilon^2(n-1) + n\sigma_\theta^2\right) < 0$ for $d \in [\Lambda_2, \Upsilon_2]$). It follows that the same result as in the case $\rho \in \left(-\frac{1}{n-1}, 0\right)$ holds.

(ii. b) We have that when $\rho \in \left(-(n-1)^{-1}, 1\right)$ as $\sigma_u^2 \to 0$, $\Gamma(d) \to$ non-zero constant$\times(d - n\beta)^2 \Lambda(d)$ and we know that $d < \Lambda_2 < n\beta$ for $\rho > 0$ and $d < \Upsilon_2 < n\beta$ for $\rho < 0$. Therefore, in the limit $d = \Lambda_2 > 0$. When $\rho = 1$, as $\sigma_u^2 \to 0$, $\Gamma(d) \to$ non-zero constant$\times(d - n\beta)^2\left(-d^2 - (\lambda - n\beta)d + n\beta\lambda\right)$ with unique root fulfilling the S.O.C. $d = n\beta$, and therefore, $d \to n\beta$ (recall that $\Lambda_2 \to n\beta$ as $\rho \to 1$ from Proposition 1). When $\rho = -(n-1)^{-1}$, then $\Omega(d) \to \beta^2\sigma_\theta^2 n(n-1)^{-2}(d - n\beta)^2 \Lambda(d)$ as $\sigma_u^2 \to 0$. Since $d < \Lambda_2 < n\beta$ for $\sigma_u^2 > 0$, it follows that $d \to \Lambda_2 = 0$.

(ii. c) As $\sigma_u^2 \to \infty$, $\sigma_u^{-2}\Gamma(d) \to \rho\Upsilon(d)\left(\Lambda(d)(1-\rho) + dn(d + \lambda + n\beta)\right)^2$ if $\rho > -(n-1)^{-1}$, and

$$\frac{\Omega(d)}{\sigma_u^2} \to \frac{\left(\sigma_\theta^2 d^2 + \left(\sigma_\theta^2(\lambda - n\beta) - 2\beta\sigma_\varepsilon^2(n-1)\right)d - \beta\lambda\left(\sigma_\varepsilon^2(n-1) + n\sigma_\theta^2\right)\right)^2}{\sigma_\theta^2\sigma_\varepsilon^2(n-1)^2}\Upsilon(d)n$$

if $\rho = -(n-1)^{-1}$. In both cases it follows that $d \to \Upsilon_2$ as $\sigma_u^2 \to \infty$ (since for $d \in [\Lambda_2, \Upsilon_2]$, $\left(\Lambda(d)(1-\rho) + dn(d + \lambda + n\beta)\right)^2 > 0$ and $\left(\sigma_\theta^2 d^2 + \left(\sigma_\theta^2(\lambda - n\beta) - 2\beta\sigma_\varepsilon^2(n-1)\right)d - \beta\lambda\left(\sigma_\varepsilon^2(n-1) + n\sigma_\theta^2\right)\right)^2 > 0$).

(ii.d) I show that whenever $\sigma_\varepsilon^2 > 0$, $\rho \in \left(-(n-1)^{-1}, 1\right)$, $n > 3$ if $\rho < 0$, then d is increasing with ρ. Let $\rho > 0$. Since in equilibrium $\Gamma'(d) < 0$ we know that $\text{sgn}\{\partial d/\partial\rho\} = \text{sgn}\{\partial\Gamma/\partial\rho\}$. It is possible to show that when $\Gamma(d) = 0$, we have that

$$\frac{\partial\Gamma}{\partial\rho} = \frac{\sigma_u^2\Upsilon(d)\left(\Lambda(d)(1-\rho) + dn(d + \lambda + n\beta)\right)}{\Lambda(d)\left(\rho(n-1) + 1\right)(1-\rho)}W(d)$$

where

$$W(d) = -\left(\rho(M+n)-M\right)n\left((1-\rho)(M+2)+n\rho\right)\left(\Upsilon(d)\right)^2$$
$$-\left(\rho(M+n)-M\right)\beta\left((\rho-1)^2(n-1)^2 M + n\left(n-2-2\rho(n-1)\right)\right)(2d+\lambda)\Upsilon(d)$$
$$-M\beta^2(1-\rho)(n-1)\left(\rho(n-1)+1\right)\left((1-\rho)(n-1)M+n\right)(2d+\lambda)^2.$$

Since $\Upsilon(d)<0$, $\Lambda(d)>0$, and $\Lambda(d)(1-\rho)+dn(d+\lambda+n\beta)>0$ for $d\in(\Upsilon_2,\Lambda_2)$, $\rho(n-1)+1>0$, $\text{sgn}\{\partial\Gamma/\partial\rho\}=-\text{sgn}\{W\}$. An elaborate analysis shows that $W(d)<0$. Note that $\left(\rho(M+n)-M\right)>0$ since $M\in\left[0,\frac{\rho n}{1-\rho}\right)$ for $\rho\geq 0$, $\left((1-\rho)(M+2)+n\rho\right)>0$, and $\left((1-\rho)(n-1)M+n\right)>0$. We distinguish two cases. If $(\rho-1)^2(n-1)^2 M+n\left(n-2-2\rho(n-1)\right)\leq 0$, then the three terms of $W(d)$ are negative. If $(\rho-1)^2(n-1)^2 M+n\left(n-2-2\rho(n-1)\right)>0$ note that

$$-\left(\rho(M+n)-M\right)\beta\left((\rho-1)^2(n-1)^2 M + n\left(n-2-2\rho(n-1)\right)\right)(2d+\lambda)\Upsilon(d)$$
$$-M\beta^2(1-\rho)(n-1)\left(\rho(n-1)+1\right)\left((1-\rho)(n-1)M+n\right)(2d+\lambda)^2 > W(d),$$

and, taking into account that $\Upsilon(d)=\left(\Lambda(d)+M(d+\lambda)(d-n\beta)\right)n^{-1}$:

$$-\left(\rho(M+n)-M\right)\beta\left((\rho-1)^2(n-1)^2 M + n\left(n-2-2\rho(n-1)\right)\right)(2d+\lambda)n^{-1}\Lambda(d)$$
$$+M\beta(2d+\lambda)K(d)>W(d)$$

where

$K(d)=$
$$-\left(\rho(M+n)-M\right)\left((\rho-1)^2(n-1)^2 M + n\left(n-2-2\rho(n-1)\right)\right)\frac{(d+\lambda)(d-n\beta)}{n}$$
$$-\beta(1-\rho)(n-1)\left(\rho(n-1)+1\right)\left((1-\rho)(n-1)M+n\right)(2d+\lambda).$$

Since $\Lambda(d) > 0$ and $(\rho-1)^2(n-1)^2 M + n\big(n-2-2\rho(n-1)\big) > 0$, we obtain $W(d) < M\beta(2d+\lambda)K(d)$. Since $\Upsilon(d) < 0$, we have that $-\big(\beta(n-2)+\lambda\big)d + \beta\lambda < d^2$. Using this inequality in the expression of $K(d)$, it follows that

$$K(d) < \frac{\beta(n-1)(2d+\lambda)}{n} \times$$
$$\Big((\rho-1)^3(n-1)^2 M^2 + n(\rho-1)\big(2n-3\rho(n-1)-3\big)M - n^2\big(\rho^2(n-1)+1\big)\Big).$$

It is easy to see that the last factor is negative whenever $(\rho-1)^2(n-1)^2 M + n\big(n-2-2\rho(n-1)\big) > 0$. This implies that in this case we also obtain $W(d) < 0$.

Let $\rho < 0$. Then in equilibrium $\Gamma'(d) > 0$ and therefore $\operatorname{sgn}\{\partial d/\partial\rho\} = \operatorname{sgn}\{-\partial\Gamma/\partial\rho\}$. Since $\Upsilon(d) > 0$ and $\Lambda(d) < 0$ for $d \in (\Lambda_2, \Upsilon_2)$, $\operatorname{sgn}\{\partial\Gamma/\partial\rho\} = -\operatorname{sgn}\{W\}$. We have that $W(d) < 0$ for $d \in (\Lambda_2, \Upsilon_2)$ since the three terms of $W(d)$ are negative. Note that since $M \in \left(\frac{\rho n}{1-\rho}, 0\right]$ for $\rho \le 0$ and $\big(\rho(n-1)+1\big) > 0$, we have that $\big((1-\rho)(M+2)+n\rho\big) > 0$, $\big(\rho(M+n)-M\big) > 0$, $\big((1-\rho)(n-1)M+n\big) > 0$. Furthermore, $\Upsilon(d) > 0$ and $\Big((\rho-1)^2(n-1)^2 M + n\big(n-2-2\rho(n-1)\big)\Big) > 0$ (since it is increasing in M and it is nonnegative when $M = \frac{\rho n}{1-\rho}$ and $\rho \in \big(-(n-1)^{-1},\ 0\big)$. It follows that $\partial\Gamma/\partial\rho < 0$ and $\partial d/\partial\rho > 0$. ♦

<u>Proof of Claim S.1</u>: The coefficient of h_i in $E(\theta_i \mid s_i, h_i)$ is $\nu \equiv \frac{(n-1)\beta\rho a\sigma_\theta^2\sigma_\varepsilon^2}{\Delta}$ (where $\Delta > 0$). We show it increases in ρ if $\sigma_\varepsilon^2 > 0$. Tedious algebra using the expressions for a and c in (i) and the definitions of d and Δ shows that $\nu = -\frac{\Upsilon(d)}{\beta(n\beta-d)}$. It follows that

$\frac{\partial v}{\partial \rho} = \frac{\partial v}{\partial d}\frac{\partial d}{\partial \rho}$ and $\frac{\partial v}{\partial d} = -\frac{\varphi(d)}{\beta(d-n\beta)^2}$, where $\varphi(d) \equiv d^2 - 2n\beta d - n\beta^2(n-2) - \lambda\beta(n-1) < 0$ in the relevant range (since $\varphi(d)$ is decreasing in d, as $d - \beta n < 0$, and $\varphi(0) < 0$) and therefore $\frac{\partial v}{\partial d} > 0$ and $\frac{\partial v}{\partial \rho} > 0$ since we know from (ii.d) that $\partial d/\partial \rho > 0$. ♦

<u>Proof of Proposition S.5 (iii)</u>: Let $\rho\sigma_\varepsilon^2 > 0$, then I show that $\partial d/\partial \sigma_\varepsilon^2 > 0$ if $\sigma_\varepsilon^2 \le \sigma_\theta^2$ (and note from Claim S.1 that $\operatorname{sgn}\{\partial v/\partial \sigma_\varepsilon^2\} = \operatorname{sgn}\{\partial d/\partial \sigma_\varepsilon^2\}$). From the expression for M and the fact that in equilibrium $\Gamma'(d) < 0$ we have that

$$\operatorname{sgn}\{\partial d/\partial \sigma_\varepsilon^2\} = \operatorname{sgn}\{\partial d/\partial M\} = \operatorname{sgn}\{\partial \Gamma/\partial M\}.$$

Furthermore,

$$\frac{\partial \Gamma}{\partial M} = \rho\sigma_u^2 \Upsilon(d)\big(\Lambda(d)(1-\rho) + dn(d+\lambda+n\beta)\big)\frac{l(d)}{(\rho(M+n)-M)\Lambda(d)},$$

where

$$\begin{aligned} l(d) &= n\beta(1-\rho)(2d+\lambda)\big(\rho(n-1)+1\big)\Lambda(d) \\ &-dn\big(\rho(M+n)-M\big)(d+\lambda+n\beta)(d+\lambda)(n\beta-d) \end{aligned}$$

and $\operatorname{sgn}\{\partial \Gamma/\partial M\} = -\operatorname{sgn}\{l(d)\}$ since $\Upsilon(d) < 0$ and $\Lambda(d) > 0$ whenever $d \in (\Upsilon_2, \Lambda_2)$. We have that $l(d)$ is a polynomial of degree 4 with coefficient $n\big(\rho(M+n)-M\big) > 0$ and therefore tends to ∞ when $d \to \pm\infty$. Furthermore,

$l(-\lambda) = -n^2\beta^2\lambda^2(1-\rho)(n-1)\big(\rho(n-1)+1\big) < 0$,

$l(0) = n^2\beta^2\lambda^2(1-\rho)\big(\rho(n-1)+1\big)(M+1) > 0$, and

$l(\Lambda_2) = -\big(\rho(M+n)-M\big)dn(d+\lambda+n\beta)(d+\lambda)(n\beta-d) < 0$ There is a root in $(-\infty, -\lambda)$, another in $(-\lambda, 0)$, and another in (Λ_2, ∞).

In addition, $\operatorname{sgn}\{l(\Upsilon_2)\} = \operatorname{sgn}\{(1-\rho)(n\rho(n-1)+2)M - n\rho\}$. Therefore, if $M \le n\rho(1-\rho)^{-1}(\rho(n-1)+2)^{-1}$ (or equivalently $\sigma_\varepsilon^2 \le \sigma_\theta^2$), we have $l(\Upsilon_2) \le 0$ and $l(d) < 0$ for $d \in (\Upsilon_2, \Lambda_2)$. It follows that $\operatorname{sgn}\{\partial\Gamma/\partial M\} = -\operatorname{sgn}\{l(d)\} > 0$ when $\sigma_\varepsilon^2 \le \sigma_\theta^2$. ♦

Proof of Claim S.2: Consider noise of the form βu in the model with inverse demand $P(y) = \alpha + \beta u - \beta y$, $\rho = 1$ and $\lambda = 0$. Note that $y = \beta^{-1}(\alpha + \beta u - p) \to u$ as $\beta \to \infty$. Consider the expression for $\Gamma(d)$ in the proof of Proposition S.5 and let $\rho = 1$, $\lambda = 0$. Note that $(1-\rho)M = \dfrac{n\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + n\sigma_\theta^2}$ and that now $\operatorname{var}[\beta u] = \beta^2\sigma_u^2$. Then it can be checked that $\Gamma(d)/\beta^5 \to \Xi(d)$ as $\beta \to \infty$ where $\Xi(d) = d\left(-d^2(n-2)(2\sigma_\varepsilon^2 + n\sigma_\theta^2)\sigma_u^2 + n^3\sigma_\theta^4\sigma_\varepsilon^2(n-1)^{-1}\right)$. The positive solution to $\Xi(d) = 0$ fulfilling the S.O.C. is the desired d and since $d = ((n-1)c)^{-1}$ with inelastic demand we obtain the result for $(nc)^{-1}$. ♦

S.6 Public signal

This section provides the proof of Proposition 9 and states and proves Claim S.3.

Proof of Proposition 9: If sellers other than i use the strategy $X(s_j, p) = b - as_j - er + cp$, for seller i from the market clearing condition the price is informationally equivalent to $h_i \equiv \beta(b-er)(n-1) - \alpha + (1+\beta(n-1)c)p + \beta x_i = \beta a \sum_{j\ne i} s_j$. We have that $E[\theta_i | s_i, r, p] = E[\theta_i | s_i, r, h_i]$,

$$\begin{pmatrix} \theta_i \\ s_i \\ r \\ h_i \end{pmatrix} \sim N\left[\begin{pmatrix} \bar\theta \\ \bar\theta \\ \bar\theta \\ (n-1)\bar\theta \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \sigma_\theta^2 & \operatorname{var}[\tilde\theta] & (n-1)\rho\sigma_\theta^2 \\ \sigma_\theta^2 & \sigma_\theta^2 + \sigma_\varepsilon^2 & \operatorname{var}[\tilde\theta] & (n-1)\rho\sigma_\theta^2 \\ \operatorname{var}[\tilde\theta] & \operatorname{var}[\tilde\theta] & \operatorname{var}[\tilde\theta] + \sigma_\delta^2 & (n-1)\operatorname{var}[\tilde\theta] \\ (n-1)\rho\sigma_\theta^2 & (n-1)\rho\sigma_\theta^2 & (n-1)\operatorname{var}[\tilde\theta] & (n-1)\left(\sigma_\theta^2 + \sigma_\varepsilon^2 + (n-2)\rho\sigma_\theta^2\right) \end{pmatrix}\right]$$

and we obtain

$$E\left[\theta_i \left| s_i, r, h_i \right.\right] = \frac{n\sigma_\delta^2\sigma_\varepsilon^2}{n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)}\overline{\theta}$$

$$+\frac{n^2\sigma_\delta^2\sigma_\theta^2\left(\left(1-\rho\right)\left(1+\left(n-1\right)\rho\right)\sigma_\theta^2+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^4\left(1-\rho\right)\left(n-1\right)\left(1+\left(n-1\right)\rho\right)}{\left(n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)\right)n\left(\sigma_\theta^2\left(1-\rho\right)+\sigma_\varepsilon^2\right)}s_i$$

$$+\frac{\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)}{n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)}r$$

$$+\frac{\sigma_\varepsilon^2\sigma_\theta^2\left(n^2\sigma_\delta^2\rho-\sigma_\theta^2\left(1-\rho\right)\left(1+\left(n-1\right)\rho\right)\right)}{\left(n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)\right)n\left(\sigma_\theta^2\left(1-\rho\right)+\sigma_\varepsilon^2\right)\beta a}h_i.$$

From the F.O.C., $p-E\left[\theta_i \left| s_i, r, h_i \right.\right]=\left(d+\lambda\right)x_i$, $d=\left(\beta^{-1}+\left(n-1\right)c\right)^{-1}$, $x_i=b-as_i-er+cp$, and the expression for h_i, we can obtain a system of equations to identify the coefficients of the linear strategy:

$$a=\frac{\left(1-\rho\right)\sigma_\theta^2}{\left(\sigma_\varepsilon^2+\left(1-\rho\right)\sigma_\theta^2\right)}\left(d+\lambda\right)^{-1}$$

$$b=\frac{1}{1+Q}\left(\frac{\alpha}{\beta n}Q-\frac{n\sigma_\delta^2\sigma_\varepsilon^2}{n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)}\left(d+\lambda\right)^{-1}\overline{\theta}\right)$$

$$e=\frac{\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)}{n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)}\frac{1}{\left(d+\lambda\right)\left(1+Q\right)}$$

where

$$Q=\frac{\sigma_\varepsilon^2\left(n^2\sigma_\delta^2\rho-\sigma_\theta^2\left(1-\rho\right)\left(1+\left(n-1\right)\rho\right)\right)}{\left(1-\rho\right)\left(n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)\right)}, \text{ and}$$

$c=\left(\left(d+\lambda\right)^{-1}-\dfrac{Q\left(1+\beta nc\right)}{\beta n}\right)$ yields a quadratic equation in c exactly as in Proposition 1 replacing M by $Q: g\left(c;Q\right)=0$. Its largest solution fulfills the S.O.C.. Comparative static properties follow immediately from the expression of Q. Recall that $-\left(n-1\right)^{-1}<\rho<1$ and let $\sigma_\varepsilon^2>0$. We obtain,

$$\frac{\partial Q}{\partial\sigma_\delta^2}=\frac{n\sigma_\varepsilon^2\sigma_\theta^2\left(\sigma_\varepsilon^2+\left(1-\rho\right)\sigma_\theta^2\right)\left(1+\left(n-1\right)\rho\right)^2}{\left(1-\rho\right)\left(n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)\right)^2}>0,$$

$$\frac{\partial Q}{\partial \rho}=\frac{n\sigma_\delta^2\sigma_\varepsilon^2\left(\left(n^2\sigma_\delta^2+\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)^2\right)\sigma_\varepsilon^2+n^2\sigma_\delta^2\sigma_\theta^2\left(1+\left(n-1\right)\rho^2\right)\right)}{\left(1-\rho\right)^2\left(n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)\right)^2}>0\text{, and}$$

$$\frac{\partial Q}{\partial \sigma_\theta^2}=-\frac{n\sigma_\delta^2\sigma_\varepsilon^2\left(1+\left(n-1\right)\rho\right)\left(n^2\sigma_\delta^2\rho+\sigma_\varepsilon^2\left(1+\left(n-1\right)\rho\right)\right)}{\left(1-\rho\right)\left(n\sigma_\delta^2\left(\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)+\sigma_\varepsilon^2\right)+\sigma_\varepsilon^2\sigma_\theta^2\left(1+\left(n-1\right)\rho\right)\right)^2}.$$

Therefore,

$$\operatorname{sgn}\left\{\frac{\partial Q}{\partial \sigma_\theta^2}\right\}=-\operatorname{sgn}\left\{n^2\sigma_\delta^2\rho+\sigma_\varepsilon^2\left(1+\left(n-1\right)\rho\right)\right\}$$

and $n^2\sigma_\delta^2\rho+\sigma_\varepsilon^2\left(1+\left(n-1\right)\rho\right)\geq 0$ if and only if $\rho\geq-\left(\left(n-1\right)+n^2\left(\sigma_\delta^2/\sigma_\varepsilon^2\right)\right)^{-1}$.

Furthermore, from the expression for Q, it follows that

$\operatorname{sgn}\left\{\partial Q/\partial\sigma_\varepsilon^2\right\}=\operatorname{sgn}\left\{Q\right\}=\operatorname{sgn}\left\{\left(\mathrm{n}^2\rho\sigma_\delta^2-\sigma_\theta^2\left(1-\rho\right)\left(1+\left(n-1\right)\rho\right)\right)\right\}$. ♦

<u>Claim S.3</u>: When $\sigma_\varepsilon^2>0$ the absolute value of the weight on h_i in $E\left[\theta_i\left|s_i,r,h_i\right.\right]$ increases in $|\rho|$ and σ_ε^2.

<u>Proof</u>: The coefficient of h_i in $E\left[\theta_i\left|s_i,r,h_i\right.\right]$ equals $\left(d+\lambda\right)Q/\beta n$ and is therefore increasing in Q since d is increasing in Q. If follows that when $\sigma_\varepsilon^2>0$ the coefficient is increasing in ρ since then Q is increasing in ρ and when $Q>0$ ($Q<0$) it is increasing (decreasing) in σ_ε^2 since then Q is increasing (decreasing) in σ_ε^2. We conclude that the absolute value of the weight on h_i in $E\left[\theta_i\left|s_i,h_i\right.\right]$ increases in $|\rho|$ and σ_ε^2. ♦

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