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SUPPLEMENT TO "WHEN ARE LOCAL INCENTIVE CONSTRAINTS SUFFICIENT?"

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THIS SUPPLEMENT PROVIDES a more detailed study of conditions under which the basic method of proof used for the sufficiency results in the main paper can be applied, with an eye to understanding how much the method might potentially be further generalized and whether the results still hold when the method does not apply. We restrict ourselves to cardinal type spaces and no transfers, as in Section 3.1.

All of the proofs of sufficiency results in the main paper follow the general method of showing that the linear inequality corresponding to any desired incentive constraint can be obtained by adding up inequalities corresponding to local incentive constraints. We show here that for *finite* type spaces, whenever a set S of incentive constraints is sufficient, there exists a proof of sufficiency by adding up (Lemma S-1 below). Moreover, with minor exceptions, whenever an incentive constraint (u, v) is provable by adding up, there exists such a proof that uses only types along the line segment $[u, v]$ or types cardinally equivalent to them (Proposition S-1). The conclusion, then, is that for finite type spaces, there exist essentially no sufficiency results beyond those that can be proven using the method of Proposition 1.

However, for *infinite* type spaces, the conclusions are not as tight. We give an example (Proposition S-2) of a type space where local incentive constraints are sufficient, but sufficiency cannot be proven by adding up. In that example, we prove sufficiency by a combination of adding-up arguments and limiting arguments exploiting the compactness of the space $\Delta(X)$.

To begin the investigation, we must first be precise about what it means for an incentive constraint to be provable by adding up other constraints. Let T be a cardinal type space and let S be a set of incentive constraints. Let $\mathbf{1} \in \mathbb{R}^m$ denote the vector all of whose components are 1 and let e_p denote the pth unit vector for $p = 1, \dots, m$. For any mechanism f, we have

(S-1) $$\mathbf{1} \cdot f(u) = 1$$

for all $u \in T$ and

(S-2) $$e_p \cdot f(u) \geq 0$$

for $p = 1, \dots, m$ and all $u \in T$. If f satisfies S, then we also have

(S-3) $$u \cdot (f(u) - f(v)) \geq 0$$

for each $(u, v) \in S$.

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We say that an incentive constraint $(u^*, v^*) \in T \times T$ is *provable from S by adding up* if the inequality

$$u^* \cdot (f(u^*) - f(v^*)) \geq 0 \tag{S-4}$$

can be obtained as a finite linear combination of the equation (S-1) and inequalities (S-2) and (S-3), with nonnegative coefficients on the inequalities. That is, (u^*, v^*) is provable from S by adding up if there exist real numbers

- a_u for $u \in T$
- b_{pu} for $p = 1, \ldots, m$, $u \in T$
- c_{uv} for $(u, v) \in S$

such that all but finitely many of these numbers are zero, such that all the b_{pu} and c_{uv} are nonnegative, and such that adding up a_u times (S-1), b_{pu} times (S-2), and c_{uv} times (S-3) gives (S-4). (For notational convenience, we assume c_{uv} to be defined for all $u, v \in T$, with $c_{uv} = 0$ whenever $(u, v) \notin S$.)

We can write out the adding-up conditions explicitly by comparing coefficients of $f(u)$ for each $u \in T$. Assume $u^* \neq v^*$ (otherwise (S-4) just reads $0 = 0$, which is trivially provable by adding up). Then the adding-up condition says that for each u, we have

$$a_u \mathbf{1} + \sum_{p=1}^{m} b_{pu} e_p + \sum_{v \in T} c_{uv} u - \sum_{v \in T} c_{vu} v = \begin{cases} u^*, & \text{if } u = u^*, \\ -u^*, & \text{if } u = v^*, \\ 0, & \text{otherwise.} \end{cases} \tag{S-5}$$

Also, for the constant terms, the adding-up condition is simply

$$\sum_{u \in T} a_u = 0. \tag{S-6}$$

We say that the set S of incentive constraints *implies* the incentive constraint $(u^*, v^*) \in T \times T$ if every mechanism that satisfies S also satisfies (u^*, v^*).

The present question is, If S implies (u^*, v^*), must the constraint (u^*, v^*) necessarily be provable from S by adding up? When S is finite, the answer is affirmative; this is just a form of the theorem of the alternative.

LEMMA S-1: *If T is a cardinal type space and S is a finite set of incentive constraints that implies the incentive constraint (u^*, v^*), then (u^*, v^*) is provable from S by adding up.*

PROOF: We may as well assume that T consists only of u^*, v^*, and the types that appear in constraints of S. Thus, T is finite. A mechanism f satisfying S then consists simply of a choice of $m \cdot |T|$ real numbers—the components of the $|T|$ vectors $f(u)$ for $u \in T$—satisfying (S-1), (S-2), and also (S-3) for $(u, v) \in S$. The hypothesis is that any such numbers must also satisfy (S-4).

This can be recast as a linear programming statement: for any choice of $m \cdot |T|$ real numbers satisfying the nonnegativity constraints (S-2), the linear equations (S-1), and inequalities (S-3), the minimum value of the linear function $u^* \cdot (f(u^*) - f(v^*))$ is 0. (This minimum is attained, for example, by any mechanism such that $f(u)$ is constant across all u.) The duality theorem of linear programming then tells us that (S-4) is expressible as a linear combination of (S-1), (S-2), and (S-3) with nonnegative coefficients on the inequalities; that is, (u^*, v^*) is provable from S by adding up. *Q.E.D.*

To proceed further, it is helpful to have an alternative, cleaner definition of provability by adding up. Let $\Pi \subseteq \mathbb{R}^m$ be the hyperplane orthogonal to $\mathbf{1}$, as in Section 4. For any $u \in \mathbb{R}^m$, let $\overline{u}$ denote its orthogonal projection onto Π.

LEMMA S-2: *Assume $u^* \neq v^*$. Then (u^*, v^*) is provable from S by adding up if and only if there exist numbers $c_{uv} \geq 0$, finitely many of which are nonzero, such that the equation*

$$\sum_{v \in T} c_{uv}\overline{u} - \sum_{v \in T} c_{vu}\overline{v} = \begin{cases} \overline{u^*}, & \text{if } u = u^*, \\ -\overline{u^*}, & \text{if } u = v^*, \\ 0, & \text{otherwise}, \end{cases} \tag{S-7}$$

holds for each $u \in T$, and $c_{uv} = 0$ unless $(u, v) \in S$.

PROOF: First suppose that (u^*, v^*) is provable from S by adding up under the original definition. Let a_u, b_{pu}, and c_{uv} be the coefficients satisfying (S-5). By summing (S-5) over all choices of u, we get $\sum_u a_u \mathbf{1} + \sum_p \sum_u b_{pu} e_p = 0$. (On the left side, each c_{uv} occurs once multiplied by u and once multiplied by $-u$. On the right side, we get one u^*, one $-u^*$, and all zeroes otherwise.) From (S-6), this reduces to $\sum_p \sum_u b_{pu} e_p = 0$. Since the b_{pu} are nonnegative, they must all be zero. Once we know this, then, taking (S-5) and projecting orthogonally onto Π gives (S-7).

Conversely, suppose there are coefficients c_{uv} that satisfy (S-7). Put $b_{pu} = 0$ for all p and all u. Note that (S-7) implies that for each u, the expression

$$\begin{aligned} &\sum_v c_{uv} u - \sum_v c_{vu} v - u^*, \quad \text{if } u = u^*, \\ &\sum_v c_{uv} u - \sum_v c_{vu} v + u^*, \quad \text{if } u = v^*, \\ &\sum_v c_{uv} u - \sum_v c_{vu} v, \quad \text{otherwise}, \end{aligned}$$

must be some multiple of $\mathbf{1}$. Choose a_u so that this expression is equal to $-a_u \mathbf{1}$. Then it is immediate that (S-5) is satisfied for each u. Moreover, summing (S-5) across all $u \in T$, the c_{uv} terms cancel as in the previous paragraph and we are simply left with $\sum_u a_u \mathbf{1} = 0$; hence, with this choice of a_u, (S-6) is satisfied as well. Finally, $a_u \neq 0$ only when $u = u^*, v^*$ or when c_{uv} or c_{vu} is nonzero for some v; thus, only finitely many of the a_u are nonzero. Thus, the original definition of provability by adding up is satisfied. *Q.E.D.*

We need just a few more definitions. Say that two types u and v are *equivalent* if $v = \alpha u + \beta \mathbf{1}$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, and that a type is *indifferent* if it is equivalent to 0. For $u^*, v^* \in T$, let $T_{[u^*,v^*]}$ be the set of all types in T that are equivalent to some type on the segment $[u^*, v^*]$ and let

$$S_{[u^*,v^*]} = \{(u, v) \in S \mid u, v \in T_{[u^*,v^*]}\}.$$

We now arrive at the main result of this supplement.

PROPOSITION S-1: *Let T be a cardinal type space and let S be a set of incentive constraints such that (u^*, v^*) is provable from S by adding up. Assume that v^* is not equivalent to $-u^*$. Then (u^*, v^*) is provable from $S_{[u^*,v^*]}$ by adding up.*

This result says that if an incentive constraint (u^*, v^*) can be proved by adding up constraints in S, then it can be proved by adding up in a way that only uses types equivalent to convex combinations of u^* and v^*. Thus, the method used to prove Proposition 1 is (almost) the only possible adding-up argument.

The proof of Proposition S-1 is a bit long, but the main idea is straightforward. It consists of taking the coefficients c_{uv} that satisfy (S-7) and successively replacing them with zeroes, checking that (S-7) still holds at each step, until only constraints in $S_{[u^*,v^*]}$ have nonzero coefficients.

PROOF OF PROPOSITION S-1: We may assume that u^* is not indifferent, since otherwise the conclusion is immediate: (S-7) holds with all c_{uv} equal to 0. We also assume $u^* \neq v^*$; otherwise the conclusion is again trivial.

Let c_{uv} be the coefficients that satisfy (S-7), with $c_{uv} > 0$ only if $(u, v) \in S$. We may as well assume that S consists only of the (finitely many) incentive constraints (u, v) for which $c_{uv} > 0$, and T consists only of the types appearing in these constraints.

Now consider any fixed vector $w \in \Pi$ with the following properties:

(i) $w \cdot \overline{u^*} > 0$;
(ii) $w \cdot \overline{v^*} \geq 0$;
(iii) if $u \in T$ and $w \cdot \overline{u} = 0$, then $\overline{u} = 0$.

We claim that if $(u, v) \in S$ such that either

(a) $w \cdot \overline{u} > 0$ and $w \cdot \overline{v} < 0$ or
(b) $w \cdot \overline{u} > 0$ and $w \cdot \overline{v} = 0$ and $v \neq v^*$ or
(c) $w \cdot \overline{u} < 0$ and $w \cdot \overline{v} \geq 0$,

then $c_{uv} = 0$.

To prove the claim, consider any $u \in T$ such that $w \cdot \overline{u} < 0$. Take the dot product of w with (S-7). We get

$$\sum_{v \in T} c_{uv}(w \cdot \overline{u}) - \sum_{v \in T} c_{vu}(w \cdot \overline{v}) = 0$$

(note that $u \neq u^*, v^*$). Now sum over all u such that $w \cdot \overline{u} < 0$. For each incentive constraint $(u, v) \in S$ such that $w \cdot \overline{u} < 0$ and $w \cdot \overline{v} < 0$, the term $c_{uv}(w \cdot \overline{u})$ appears once with a $+$ sign and once with a $-$ sign, so they cancel out. The remaining terms give us

$$\sum_{w \cdot \overline{u} < 0; w \cdot \overline{v} \geq 0} c_{uv}(w \cdot \overline{u}) - \sum_{w \cdot \overline{u} < 0; w \cdot \overline{v} \geq 0} c_{vu}(w \cdot \overline{v}) = 0.$$

Since each c_{uv} is nonnegative, every term in the first sum is less than or equal to 0 and every term in the second sum is greater than or equal 0. Hence, every term must be equal to zero. This implies that whenever $w \cdot \overline{u} < 0$ and $w \cdot \overline{v} \geq 0$, $c_{uv} = 0$, and, moreover, when $w \cdot \overline{v} > 0$, we also have $c_{vu} = 0$.

This covers (a) and (c). For (b), when $w \cdot \overline{v} = 0$ and $v \neq v^*$, (S-7) for v gives $\sum_u c_{vu}\overline{v} - \sum_u c_{uv}\overline{u} = 0$. Taking the dot product with w gives $\sum_u c_{uv}(w \cdot \overline{u}) = 0$ (after canceling). We have already established that $c_{uv} = 0$ if $w \cdot \overline{u} < 0$, so all the terms on the left are nonnegative and hence they must all be zero. So $c_{uv} = 0$ whenever $w \cdot \overline{u} > 0$. This proves the claim.

Next, for each $u, v \in T$, define $c'_{uv} = c_{uv}$ if $w \cdot \overline{u} \geq 0$ and $w \cdot \overline{v} \geq 0$, and define $c'_{uv} = 0$ otherwise. We claim that we again have, for each u

$$\text{(S-8)} \qquad \sum_v c'_{uv}\overline{u} - \sum_v c'_{vu}\overline{v} = \begin{cases} \overline{u^*}, & \text{if } u = u^*, \\ -\overline{u^*}, & \text{if } u = v^*, \\ 0, & \text{otherwise.} \end{cases}$$

To prove this claim, proceed as follows: if u is such that $w \cdot \overline{u} < 0$, then (S-8) is trivial since both sides are zero. If $w \cdot \overline{u} > 0$, then the left side of (S-8) differs from the left side of (S-7) by the terms $c_{uv}\overline{u}$ and $-c_{vu}\overline{v}$ for $w \cdot \overline{v} < 0$. These are all zero, by cases (a) and (c) of the previous claim, respectively; thus (S-8) follows from (S-7). If $w \cdot \overline{u} = 0$ and $u \neq v^*$, then again all the left-hand-side terms of (S-8) are zero:

- All the $c'_{uv}\overline{u}$ are zero because $\overline{u} = 0$, by condition (iii) on w.
- $c'_{vu}\overline{v} = 0$ for $w \cdot \overline{v} > 0$ by (b) of the previous claim.
- $c'_{vu}\overline{v} = 0$ for $w \cdot \overline{v} = 0$ again by (iii) on w.
- $c'_{vu}\overline{v} = 0$ for $w \cdot \overline{v} < 0$ by definition of c'_{vu}.

So both sides of (S-8) are zero and it again holds.

Thus, (S-8) is verified for all u except possibly for $u = v^*$. But summing (S-8) over all $u \in T$ gives the identity $0 = 0$, so if it holds for all u except $u = v^*$, it must hold for $u = v^*$ as well.

At this point, we have shown the following: If we start with coefficients c_{uv} for which (S-7) holds, pick any $w \in \Pi$ satisfying (i)–(iii), and replace c_{uv} with 0 whenever $w \cdot \overline{u} < 0$ or $w \cdot \overline{v} < 0$, then (S-7) still holds.

If we find any finite set of vectors $w_1, \dots, w_q \in \Pi$, each satisfying conditions (i)–(iii), and for each w_k, we successively replace c_{uv} with 0 whenever $w_k \cdot \overline{u} < 0$ or $w_k \cdot \overline{v} < 0$, then the resulting coefficients still satisfy (S-7).

Now let $T_{[u^*,v^*]^+}$ consist of the types in $T_{[u^*,v^*]}$ together with all indifferent types (alternatively stated, all types that are equivalent to $\alpha u^* + \beta v^*$ for some $\alpha, \beta \geq 0$) and let $S_{[u^*,v^*]^+} = \{(u, v) \in S | u, v \in T_{[u^*,v^*]^+}\}$. We show that for any $u \in T$ that is not in $T_{[u^*,v^*]^+}$, there is some $w \in \Pi$ that satisfies (i)–(iii) with $w \cdot \overline{u} < 0$. If we consider each such w in turn and successively replace c_{uv}'s with 0 as in the previous paragraph, we will be left with coefficients $c_{uv} \geq 0$ that still satisfy (S-7) and such that $c_{uv} = 0$ unless $u, v \in T_{[u^*,v^*]^+}$. Therefore, we will have shown that (u, v) is provable from $S_{[u^*,v^*]^+}$ by adding up.

Thus, consider any $u \in T \setminus T_{[u^*,v^*]^+}$. We wish to show that there exists $w \in \Pi$ that satisfies (i)–(iii) with $w \cdot \overline{u} < 0$. The assumptions that v^* is not equivalent to $-u^*$ and u^* is not indifferent imply that there exists $w' \in \Pi$ with

$$w' \cdot \overline{u^*} > 0, \quad w' \cdot \overline{v^*} \geq 0,$$

and the latter inequality holds strictly unless $\overline{v^*} = 0$. The assumption $u \notin T_{[u^*,v^*]^+}$ implies that $\overline{u}$ is not a nonnegative combination of $\overline{u^*}$ and $\overline{v^*}$; hence there is some $w'' \in \Pi$ such that

$$w'' \cdot \overline{u^*} \geq 0, \quad w'' \cdot \overline{v^*} \geq 0, \quad w'' \cdot \overline{u} < 0.$$

Taking $w = w' + \kappa w''$ for large κ gives (i), (ii), and $w \cdot \overline{u} < 0$. Finally, by perturbing w slightly, we can ensure $w \cdot \overline{v} \neq 0$ for all $v \in T$, $\overline{v} \neq 0$, without breaking any of the strict inequalities; thus we get (iii) as well.

At this point, we have finished showing that (u^*, v^*) is provable from $S_{[u^*,v^*]^+}$ by adding up.

If v^* is indifferent, then $S_{[u^*,v^*]^+} = S_{[u^*,v^*]}$ and so we are done. Otherwise, we have to do just a little more work. Let c_{uv} now be the coefficients used to prove (u^*, v^*) from $S_{[u^*,v^*]^+}$ by adding up (i.e., the coefficients that satisfy (S-7)). Whenever $\overline{u} = 0$, we can replace c_{uv} with 0 without affecting the validity of (S-7) (since c_{uv} only ever appears as part of the product $c_{uv}\overline{u}$). So we may assume $c_{uv} = 0$ whenever u is indifferent.

Since u^* and v^* are both non-indifferent and v^* is not equivalent to $-u^*$, we can find $w \in \Pi$ such that $w \cdot \overline{u^*} > 0$ and $w \cdot \overline{v^*} > 0$. Thus, for any element of $T_{[u^*,v^*]^+}$ that is not indifferent, its projection has a positive dot product with w.

Now for any indifferent u, considering (S-7) and taking the dot product with w gives $-\sum_v c_{vu}(w \cdot \overline{v}) = 0$. Each term in the sum is nonnegative, so they must all be zero. Hence $c_{vu} = 0$ whenever $\overline{v}$ has a positive dot product with w, and the remaining $v \in T_{[u^*,v^*]^+}$ are indifferent, so $c_{vu} = 0$ for them too by assumption. Thus, if u is indifferent, then $c_{uv}, c_{vu} = 0$ for all v. But this means that (S-7) holds with c_{uv} zero unless $u, v \in T_{[u^*,v^*]}$, so in fact (u^*, v^*) is provable from $S_{[u^*,v^*]}$ by adding up. *Q.E.D.*

Proposition S-1 is stated as a description of the form of proofs by adding up. However, it also provides us with a tool to show when a particular constraint is *not* provable by adding up. In particular, we can apply it to give an example of an infinite type space and a set of local incentive constraints that are sufficient, but whose sufficiency cannot be proven by adding up, as promised at the beginning of this supplement. In fact, we give a stronger example: a type space such that *any* set of local incentive constraints is sufficient, yet there exist fairly large such sets whose sufficiency cannot be proven by adding up.

Let X have four elements and let w be some utility function on X that is not indifferent. Let T_{w^+} be the set of all cardinal types that are either indifferent or equivalent to w and let $T = \mathbb{R}^4 \setminus T_{w^+}$ be the set of cardinal types not in T_{w^+}. Say that two types $u, v \in T$ are T_{w^+}-*opposed* if $[u, v] \cap T_{w^+} \neq \emptyset$. Let S be any set of local incentive constraints such that if u and v are T_{w^+}-opposed, then $(u, v) \notin S$.

This requirement on S can be easily satisfied. Indeed, for each $u \in T$, start with any neighborhood N_u and let $d(u, T_{w^+}) > 0$ be the Euclidean distance from u to T_{w^+}. Then the set $N'_u = \{v \in N_u | d(u, v) < d(u, T_{w^+})\}$ is again an open neighborhood of u, not containing any types T_{w^+}-opposed to u. So $S = \{(u, v) | u \in N'_v \text{ or } v \in N'_u\}$ is a set of local incentive constraints meeting our requirement.

PROPOSITION S-2: *With T and S as above, S is sufficient. However, for any $u^*, v^* \in T$ that are T_{w^+}-opposed, with u^* not equivalent to $-v^*$, the constraint (u^*, v^*) is not provable from S by adding up.*

PROOF: First we show that S is sufficient. Let f be any mechanism that satisfies S. For any possible incentive constraint (u, v), if u and v are not T_{w^+}-opposed, then the entire line segment from u to v is contained in T. Therefore, the usual argument from Proposition 1 of the main paper shows that f satisfies (u, v).

So we need only deal with the case where u and v are T_{w^+}-opposed. In this case, notice that we can choose $u_k \in T$ arbitrarily close to $(u+v)/2$ such that u_k is not T_{w^+}-opposed to either u or v. (Any type T_{w^+}-opposed to u must lie on the hyperplane Π_{uw} generated by u, w, and $\mathbf{1}$. Similarly, any type T_{w^+}-opposed to v must lie on the hyperplane generated by v, w, and $\mathbf{1}$, which is again Π_{uw}. There are types in T arbitrarily close to $(u+v)/2$ that do not lie on this hyperplane.) For any such u_k, then, we have already shown that f satisfies the constraints (u, u_k), (v, u_k), and (u_k, v); that is,

(S-9) $$u \cdot (f(u) - f(u_k)) \geq 0,$$

(S-10) $$v \cdot (f(v) - f(u_k)) \geq 0,$$

(S-11) $$u_k \cdot (f(u_k) - f(v)) \geq 0.$$

So we can choose a sequence of types $u_1, u_2, \dots$ in T with $u_k \to (u+v)/2$ such that (S-9)–(S-11) are satisfied for each u_k. Moreover, because the image of f is contained in the compact set $\Delta(X)$, we may assume by passing to a subsequence that $f(u_k)$ converges to some limit f^*. Then, taking limits, we get

$$u \cdot (f(u) - f^*) \geq 0, \tag{S-12}$$

$$v \cdot (f(v) - f^*) \geq 0, \tag{S-13}$$

$$\frac{u+v}{2} \cdot (f^* - f(v)) \geq 0. \tag{S-14}$$

Adding (S-12), (S-13), and twice (S-14) gives

$$u \cdot (f(u) - f(v)) \geq 0,$$

so f satisfies the constraint (u, v). This shows that S is sufficient.

It remains to prove that if $u^*, v^* \in T$ are T_{w^+}-opposed and u^* is not equivalent to $-v^*$, then (u^*, v^*) is not provable from S by adding up. By Proposition S-1, if (u^*, v^*) were provable from S by adding up, then it would be provable from $S_{[u^*,v^*]}$ by adding up. So we just need to show that the latter is not the case.

For any $\alpha \in [0, 1]$, let $u_\alpha = (1-\alpha)u^* + \alpha v^*$. Let $\alpha^* \in (0, 1)$ be such that $u_{\alpha^*} \in T_{w^+}$. Notice that if u and v are equivalent to u_α and u_β respectively, and $(u, v) \in S$, then α and β are either both less than α^* or both greater than α^*: otherwise u and v are T_{w^+}-opposed.

Suppose that (u^*, v^*) is provable from $S_{[u^*,v^*]}$ by adding up. Let c_{uv} be the coefficients that satisfy (S-7). Let $T_<$ be the set of types in $T_{[u^*,v^*]}$ that are equivalent to some u_α for $\alpha < \alpha^*$. The observation of the previous paragraph implies that if $c_{uv} > 0$, and one of u or v is in $T_<$, then the other is as well.

Sum up (S-7) over all $u \in T_<$. The $c_{uv}\overline{u}$ terms on the left side appear in pairs of opposite sign, which cancel; thus we are left with $0 = \overline{u^*}$. Since $u^* \in T$ cannot be indifferent, we have a contradiction. *Q.E.D.*

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