

SUPPLEMENT TO “INFORMATION AGGREGATION IN DYNAMIC  
MARKETS WITH STRATEGIC TRADERS”

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THIS SUPPLEMENT PROVIDES the proof of Theorem 6.

PROOF OF THEOREM 6:

STEP 0: Suppose that for some  $\varepsilon > 0$ , the statement is not true. Then one can find an increasing sequence  $m_1, m_2$ , etc., such that for any  $j$ , in equilibrium  $(S_{m_j}^*, Y_{m_j}^*)$ ,  $E[|y_{K_{m_j}} - X(\omega)|] \geq \varepsilon$ . Consider this sequence for the rest of the proof. For notational convenience, without loss of generality, we assume that  $m_1 = 1, m_2 = 2$ , etc.; that is, we have a sequence of games  $\Gamma_{K_m}$  and corresponding equilibria  $(S_m^*, Y_m^*)$  in which  $E[|y_{K_m} - X(\omega)|] \geq \varepsilon > 0$  for every  $m$ .

STEP 1: See Step 1 in the proof of Theorem 5.

STEP 2: See Step 2 in the proof of Theorem 5.<sup>1</sup>

STEP 3: In this step, we state and prove an auxiliary lemma on the convergence of prices over time.

Intuitively, this lemma says that as time approaches the end of the trading interval, the expected difference between the price of the security at that time and its average price in the subsequent periods converges to zero (for a sufficiently fine time grid, i.e., sufficiently large  $K_m$ ).

LEMMA S.1: *For any  $\delta > 0$ , there exists time  $t' < 1$  such that, for any  $t \in (t', 1)$ , there exists  $m'$  such that, for any  $m > m'$ , in equilibrium  $(S_m^*, Y_m^*)$  of game  $\Gamma_{K_m}$ , for  $k = \lfloor tK_m \rfloor$ , we have*

$$E \left[ \left| \frac{y_{k+1} + \dots + y_{K_m}}{K_m - k} - y_k \right| \right] < \delta.$$

<sup>1</sup>Two adjustments are needed in these steps. First, every instance of “the” strategic trader in Steps 1 and 2 needs to be replaced with “each” strategic trader, since we now have multiple strategic traders. Second, the definition of “ $a$ ” in Step 2, “Take the smallest  $a$  such that  $V^a - V^\infty < (\phi/4)^2$ ,” needs to be modified to say “Take the smallest  $a \geq 1/\phi$  such that  $V^a - V^\infty < (\phi/4)^2$ .” This ensures that for a small  $\phi$ ,  $\frac{1}{2^a}$  is close to zero and thus  $\frac{k(\phi, m)}{K_m}$  is close to 1 for a sufficiently large  $m$ . This fact is used at the end of Step 5 of the current proof (and is not needed in the proof of Theorem 5).

PROOF: Take any  $\delta > 0$ . Recall the sequence  $V^l$  from Step 1 and pick  $l$  such that  $V^l - V^\infty < \frac{\delta^2}{4}$ . Set  $t' = 1 - \frac{1}{2l}$  and take any  $t > t'$ . Let us show that the statement of the lemma holds for this  $t$ .

Let  $\gamma^* = \max_{\omega \in \Omega} \{|X(\omega)|\}$ . Take any  $t'' \in (t, 1)$  such that  $\frac{1-t''}{1-t} 2\gamma^* < \frac{\delta}{2}$ . Take a large  $m$  and let  $k = \lfloor tK_m \rfloor$  and  $k'' = \lfloor t''K_m \rfloor$  (assume that  $m$  is sufficiently large so that  $K_m > k'' > k$ ).

Now,

$$\begin{aligned} \mathbb{E} \left[ \left| \frac{y_{k+1} + \cdots + y_{K_m}}{K_m - k} - y_k \right| \right] &= \frac{1}{K_m - k} \mathbb{E} \left[ \left| \sum_{j=k+1}^{K_m} (y_j - y_k) \right| \right] \\ &\leq \frac{1}{K_m - k} \mathbb{E} \left[ \left| \sum_{j=k+1}^{k''} (y_j - y_k) \right| \right] \\ &\quad + \frac{1}{K_m - k} \mathbb{E} \left[ \left| \sum_{j=k''+1}^{K_m} (y_j - y_k) \right| \right]. \end{aligned}$$

For the second term in the sum, we have

$$\frac{1}{K_m - k} \mathbb{E} \left[ \left| \sum_{j=k''+1}^{K_m} (y_j - y_k) \right| \right] \leq \frac{K_m - k''}{K_m - k} 2\gamma^* = \frac{K_m - \lfloor t''K_m \rfloor}{K_m - \lfloor tK_m \rfloor} 2\gamma^*.$$

For the first term,

$$\begin{aligned} &\frac{1}{K_m - k} \mathbb{E} \left[ \left| \sum_{j=k+1}^{k''} (y_j - y_k) \right| \right] \\ &\leq \frac{1}{K_m - k} \sum_{j=k+1}^{k''} \mathbb{E}[|y_j - y_k|] \\ &\leq \frac{1}{K_m - k} \sum_{j=k+1}^{k''} \sqrt{\mathbb{E}[(y_j - y_k)^2]} \\ &\leq \frac{k'' - k}{K_m - k} \sqrt{\mathbb{E}[(y_{k''} - y_k)^2]} \\ &= \frac{\lfloor t''K_m \rfloor - \lfloor tK_m \rfloor}{K_m - \lfloor tK_m \rfloor} \sqrt{\mathbb{E}[(y_{\lfloor t''K_m \rfloor} - y_{\lfloor tK_m \rfloor})^2]}. \end{aligned}$$

Now, for a large  $m$ , the bound for the second term,  $\frac{K_m - \lfloor t''K_m \rfloor}{K_m - \lfloor tK_m \rfloor} 2\gamma^*$ , is close to  $\frac{1-t''}{1-t} 2\gamma^*$ , which by construction is less than  $\frac{\delta}{2}$ . The bound for the first

term is close to  $\frac{t''-t}{1-t} \sqrt{\mathbb{E}[(y_{\lfloor tK_m \rfloor} - y_{\lfloor tK_m \rfloor})^2]}$ , which for a sufficiently large  $m$  is less than or close to  $\frac{t''-t}{1-t} \sqrt{V^l - V^\infty} < \frac{t''-t}{1-t} \frac{\delta}{2}$ . Thus, for a sufficiently large  $m$ ,  $\mathbb{E}[|\frac{y_{k+1} + \dots + y_{K_m}}{K_m - k} - y_k|] < \delta$ . *Q.E.D.*

STEP 4: In this step, we identify potential arbitrage opportunities and show that they occur with a non-vanishing positive probability as time approaches the end of the trading interval.

The formal statement is as follows.

LEMMA S.2: *There exist time  $t' < 1$  and positive numbers  $\varepsilon^*$  and  $\delta^*$  such that, for any time  $t \in (t', 1)$ , there exists  $m'$  such that, for any  $m > m'$ , for  $k = \lfloor tK_m \rfloor$ , with probability greater  $\varepsilon^*$ , for at least one player  $i$ , the following statement holds: history  $h_{i,k}$  observed by player  $i$  up to period  $k$  is such that  $|\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k})| > \delta^*$ , that is, the belief of player  $i$  about the value of the traded security differs by at least  $\delta^*$  from its expected average price in the future periods if player  $i$  does not trade.*

PROOF: The proof consists of several parts.

*Part 1.* For any  $m$  and  $k < K_m$ , let  $Q(k, m)$  be the random variable denoting the beliefs of a market-maker about the true state  $\omega$  after period  $k$  in game  $\Gamma_{K_m}$ . As in Step 3 of the proof of Theorem 5, there exists  $\varepsilon_1 > 0$  such that, for any  $k$  and  $m$ , with probability at least  $\varepsilon_1$ , the realization of  $Q(k, m)$  is such that  $\text{Var}_{Q(k,m)}(X(\omega)) > \varepsilon_1$ . Let us call such realizations ‘‘arbitrageable.’’

Since set  $\Omega$  is finite and security  $X$  is separable, we can find  $\delta > 0$  such that, for any arbitrageable realization  $q$ , we can find strategic trader  $i$  and two elements of his partition,  $\pi_1$  and  $\pi_2$ , such that the probability of each of these elements under  $q$  is greater than  $\delta$  and also  $|\mathbb{E}_q[X(\omega)|\pi_1] - \mathbb{E}_q[X(\omega)|\pi_2]| > \delta$ .<sup>2</sup>

Clearly, it must be the case that  $|\mathbb{E}_{Q(k,m)}[X(\omega)|\pi_1] - \mathbb{E}_{Q(k,m)}[X(\omega)]| > \delta/2$ , or  $|\mathbb{E}_{Q(k,m)}[X(\omega)|\pi_2] - \mathbb{E}_{Q(k,m)}[X(\omega)]| > \delta/2$ , or both. Let  $\delta_1 = \delta/2$ .

Conclusion of Part 1: for some  $\varepsilon_1 > 0$  and  $\delta_1 > 0$ , for any  $m$  and  $k < K_m$ , with probability greater than  $\varepsilon_1$ , the realization of  $Q(k, m)$  is such that for at least one strategic trader  $i$  and at least one element of his partition  $\pi$ , the probability of  $\pi$  under  $Q(k, m)$  is greater than  $\delta_1$  and

$$|\mathbb{E}_{Q(k,m)}[X(\omega)|\pi] - \mathbb{E}_{Q(k,m)}[X(\omega)]| > \delta_1.$$

<sup>2</sup>Otherwise, take a sequence of  $\delta$  converging to zero, and the corresponding sequence of arbitrageable realizations  $q$  for which this statement is false. The sequence of realizations has a converging subsequence; denote the limit of this subsequence by  $q^*$ . On one hand, by continuity,  $q^*$  also has to be arbitrageable, that is,  $\text{Var}_{q^*}(X(\omega)) > 0$ . On the other hand, also by continuity, for any trader  $i$  and any two elements of his information partition,  $\pi_1$  and  $\pi_2$ , such that  $q^*(\pi_1) > 0$  and  $q^*(\pi_2) > 0$ , it has to be the case that  $\mathbb{E}_{q^*}[X(\omega)|\pi_1] = \mathbb{E}_{q^*}[X(\omega)|\pi_2]$ . The combination of these two statements contradicts the assumption that security  $X$  is separable.

*Part 2.* Take any arbitrageable realization  $q$  of  $Q(k, m)$  and the corresponding player  $i$  and element  $\pi$  of his information partition identified in Part 1, and assume for concreteness that  $E_q[X(\omega)|\pi] > E_q[X(\omega)]$  (the opposite case is completely analogous). Take any  $m$  and  $k < K_m$ , and let  $H(\pi, q)$  be the set of possible histories  $h_{i,k}$  (from player  $i$ 's point of view) in which player  $i$ 's initial signal about the security is  $\pi$  and the beliefs of outside observer about  $\omega$  at time  $t_k$  are equal to  $q$ . By construction,  $E[X(\omega)|h_{i,k} \in H(\pi, q)] = E_q[X(\omega)|\pi] > E_q[X(\omega)] + \delta_1$ . Since  $X$  is bounded, this implies that, for some  $\phi > 0$  that depends only on  $\delta_1$  and the highest possible value of  $X$ , conditional on  $h_{i,k} \in H(\pi, q)$ , with probability at least  $\phi$ , history  $h_{i,k}$  is such that  $E[X(\omega)|h_{i,k}] > E_q[X(\omega)] + \delta_1/2$ . Note that by definition,  $\bar{x}(h_{i,k}) = E[X(\omega)|h_{i,k}]$ . Also, by construction in Part 1, we know that conditional on  $Q(k, m)$  being equal to  $q$ , the probability of player  $i$ 's signal being equal to  $\pi$  is greater than  $\delta_1$ . Let  $\varepsilon_2 = \varepsilon_1 \delta_1 \phi$  and  $\delta_2 = \delta_1/2$ . Finally, recall that in equilibrium, the pricing rule is such that  $y_k = E_{Q(k,m)}[X(\omega)]$  (and is uniquely determined by  $h_{i,k}$ ). Combining all of the above, we get the following:

Conclusion of Part 2: for some  $\varepsilon_2 > 0$  and  $\delta_2 > 0$ , for any  $m$  and  $k < K_m$ , with probability greater than  $\varepsilon_2$ , for at least one player  $i$ , history  $h_{i,k}$  is such that

$$|\bar{x}(h_{i,k}) - y_k| > \delta_2.$$

*Part 3.* We know from Lemma S.1 from Step 3 that as time  $t$  gets close to the end of the trading interval, for a sufficiently fine time grid/large  $K_m$ , the expectation of the difference between  $y_k$  and the average price of the security in subsequent periods, that is,  $\frac{y_{k+1} + \dots + y_{K_m}}{K_m}$ , approaches zero. Take any player  $i$ . The expectation of the difference,  $E[|\frac{y_{k+1} + \dots + y_{K_m}}{K_m} - y_k|]$ , can be rewritten as  $E[E[|\frac{y_{k+1} + \dots + y_{K_m}}{K_m} - y_k||h_{i,k}]] = E[E[|\bar{y}(h_{i,k}) - y_k||h_{i,k}]]$ , where the outer expectation is taken over all possible histories  $h_{i,k}$  observed by player  $i$  up to time  $t_k$ . Since this expression becomes arbitrarily small as time  $t$  gets close to 1 and the grid becomes sufficiently fine, and the inner expectation is always nonnegative, it has to be the case that, for any positive number, the probability that the inner expectation exceeds that number also becomes arbitrarily small for  $t$  close to 1 and a sufficiently fine time grid. In particular, we can pick  $t'$  in such a way that for any  $i$ , any  $t > t'$ , and any sufficiently large  $m$ , the probability that the inner expectation exceeds  $\delta_2/2$  is less than  $\frac{\varepsilon_2}{2^n}$  (where  $n$  is the number of strategic traders). Combining this with the conclusion of Part 2, and setting  $\delta_3 = \delta_2/2$  and  $\varepsilon_3 = \varepsilon_2/2$ , we get the following:

Conclusion of Part 3: for some  $\varepsilon_3 > 0$ ,  $\delta_3 > 0$ , and  $t' < 1$ , for any  $t \in (t', 1)$ , there exists  $m'$  such that, for any  $m > m'$ , for  $k = \lfloor tK_m \rfloor$ , with probability greater than  $\varepsilon_3$ , for at least one player  $i$ , the realization of history  $h_{i,k}$  is such that

$$|\bar{x}(h_{i,k}) - \bar{y}(h_{i,k})| > \delta_3.$$

*Part 4.* This is the last part of the proof of Lemma S.2, and this is the part of the proof of Theorem 6 that relies on the assumption that the original sequence of equilibria is *not* infinitely destructive. From this assumption, we know that for any player  $i$  and any  $\phi > 0$ , for some  $D \geq 1$ , for any  $m$  and  $k < K_m$ , there is a less than  $\phi$  probability that history  $h_{i,k}$  is such that  $|\bar{x}(h_{i,k}) - \bar{y}(h_{i,k})| > D|\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k})|$ . Let  $\phi = \frac{\varepsilon_3}{2n}$ , and take the corresponding  $D$  (the highest of the ones for different players). Combining the above with the conclusion of Part 3, and setting  $\varepsilon^* = \varepsilon_3/2$  and letting  $\delta^* = \delta_3/D$ , we get the following:

Conclusion of Part 4: for some  $\varepsilon^* > 0$ ,  $\delta^* > 0$ , and  $t' < 1$ , for any  $t \in (t', 1)$ , there exists  $m'$  such that, for any  $m > m'$ , for  $k = \lfloor tK_m \rfloor$ , with probability greater than  $\varepsilon^*$ , for at least one player  $i$ , the realization of history  $h_{i,k}$  is such that

$$|\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k})| > \delta^*,$$

concluding the proof of the lemma. *Q.E.D.*

STEP 5: In this step, we show how trader  $i$  can take advantage of an arbitrage opportunity of the type identified in the previous step. (Note: This step is very similar to Step 4 of the proof of Theorem 5, although some details are adjusted for the environment with multiple strategic traders.)

Consider now the following trading strategy for trader  $i$  after a history  $h_{i,k}$  such that  $|\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k})| > \delta^*$ . Assume that  $\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k}) > \delta^*$  (the opposite case is completely analogous, except that the trader would be selling securities instead of buying). In every period between  $k + 1$  and  $K_m$ , buy  $c\Delta$  units of the security, where  $c$  is a constant to be determined below and  $\Delta = \frac{\sigma}{\sqrt{K_m(K_m-k)}}$ . Note that if  $c = 1$ , then over the  $K_m - k$  trading periods remaining after history  $h_{i,k}$ , the trader will end up buying a total of  $\sigma\sqrt{\frac{K_m-k}{K_m}}$  units, which is equal to one standard deviation of the total demand from noise traders over that period.

We now need to choose the constant  $c$ . Let  $\gamma(v_{k+1}, v_{k+2}, \dots, v_{K_m})$  be the expected average price of the security during periods  $k + 1$  through  $K_m$ , from the point of view of player  $i$ , following history  $h_{i,k}$ , conditional on player  $i$  himself not trading and on the realized demand from noise traders in each period  $k' > k$  being equal to  $v_{k'}$  (the expectation is thus taken over the initial state  $\omega$  and possible randomizations of other traders). By definition,

$$\bar{y}(0, h_{i,k}) = \int_{\mathbb{R}^{K_m-k}} \gamma(u_{k+1}, u_{k+2}, \dots, u_{K_m}) f(\bar{u}) d\bar{u},$$

where

$$f(\bar{u}) = \left( \sqrt{\frac{K_m}{2\pi\sigma^2}} \right)^{K_m-k} e^{-(K_m/(2\sigma^2))(u_{k+1}^2 + \dots + u_{K_m}^2)}$$

is the density of the multivariate normal distribution of noise traders' demands in periods  $k + 1$  and later.

Now, let  $\bar{y}(z, h_{i,k})$  denote the expected average price of the security following history  $h_{i,k}$  when trader  $i$  buys  $z$  units in every period  $k' > k$ . Crucially, from the point of view of other strategic traders and the Bayesian market makers, having demand  $u$  from noise traders and demand  $z$  from trader  $i$  is indistinguishable from having demand  $u + z$  from noise traders and demand zero from trader  $i$ , and thus has an identical effect on subsequent behavior of those other traders and on the prices. Therefore,

$$\begin{aligned} \bar{y}(z, h_{i,k}) &= \int_{\mathbb{R}^{K_m-k}} \gamma(u_{k+1} + z, \dots, u_{K_m} + z) \\ &\quad \times \left( \sqrt{\frac{K_m}{2\pi\sigma^2}} \right)^{K_m-k} e^{-(K_m/(2\sigma^2))(u_{k+1}^2 + \dots + u_{K_m}^2)} d\bar{u}. \end{aligned}$$

Now, following exactly the same calculations as those in Step 4 of the proof of Theorem 5, we can pick  $c$  in such a way that  $|\bar{y}(c\Delta, h_{i,k}) - \bar{y}(0, h_{i,k})| \leq \frac{\delta^*}{4}$ . Since by construction,  $\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k}) > \delta^*$ , we have  $\bar{x}(h_{i,k}) - \bar{y}(c\Delta, h_{i,k}) > \frac{3\delta^*}{4}$ . This implies that following history  $h_{i,k}$ , by buying  $c\Delta$  units in each period  $k + 1, \dots, K_m$ , trader  $i$  can in expectation obtain continuation profit

$$(\bar{x}(h_{i,k}) - \bar{y}(c\Delta, h_{i,k}))(K_m - k)c\Delta > \frac{3\delta^*}{4}c\sigma\sqrt{\frac{K_m - k}{K_m}}.$$

To finish the proof, take  $\varepsilon^*$ ,  $\delta^*$ , and  $t'$  as in the statement of Lemma S.2 in Step 4. Take any  $t \in (t', 1)$  and any sufficiently large  $m$ , and let  $k = \lfloor tK_m \rfloor$ . By Lemma S.2, with probability greater than  $\varepsilon^*$ , for at least one strategic trader  $i$ , history  $h_{i,k}$  is such that  $|\bar{x}(h_{i,k}) - \bar{y}(0, h_{i,k})| > \delta^*$ , and thus by the above calculation, his expected continuation payoff is greater than  $\frac{3\delta^*}{4}c\sigma\sqrt{\frac{K_m - k}{K_m}}$ .

This implies that for the chosen  $\varepsilon^*$ ,  $\delta^*$ ,  $t$ , and  $m$  (and thus  $k$ ), for at least one strategic trader  $i$ , with probability greater than  $\frac{\varepsilon^*}{n}$ , the realized history is such that his expected continuation payoff is greater than  $\frac{3\delta^*}{4}c\sigma\sqrt{\frac{K_m - k}{K_m}}$ . Since following any other history, his continuation payoff is at least zero, the expected continuation payoff of trader  $i$  (over all possible histories) in periods  $k + 1, \dots, K_m$  is greater than  $\lambda\sigma\sqrt{1 - \frac{k}{K_m}}$ , where  $\lambda = \frac{\varepsilon^*}{n}\frac{3\delta^*}{4}c$ . Since the expected continuation payoffs of market-makers after any period are by construction equal to zero, and those of other strategic traders are nonnegative, it therefore has to be the case that the expected losses of noise traders following period  $k$  are at least  $\lambda\sigma\sqrt{1 - \frac{k}{K_m}}$ .

This contradicts the finding in Step 2 that for any  $\varphi > 0$ , for a sufficiently large  $m$ , the expected loss of noise traders following period  $k(\varphi, m) \approx K_m(1 - \frac{1}{2^a})$  is less than  $\varphi\sigma\sqrt{\frac{1}{2^a}}$ . *Q.E.D.*

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