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SUPPLEMENT TO "INTERTEMPORAL HEDGING AND TRADE IN REPEATED GAMES WITH RECURSIVE UTILITY"

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IN THIS SUPPLEMENT, we first extend our folk theorem to general paths of play. We then provide the missing proofs regarding our analysis of the prisoners' dilemma under IMI.

APPENDIX S-A: A FOLK THEOREM FOR GENERAL PATHS OF PLAY

The folk theorem in the main text (Theorem 1) restricted attention to situations in which on-path behavior is history-independent, and hence, can be identified with a sequence $(\alpha^0, \alpha^1, \dots) \in (\Delta(A))^\infty$. While under standard preferences such sequences can be used to attain any payoff, the same is not true under recursive utility. The general definition of a path of play, which allows for history dependence on path, is that of an infinite probability tree $\mu \in \Delta(D)$; see Mailath and Samuelson (2006, page 20). We begin by giving a formal definition of the set D as the inverse limit of finite trees, which is necessary for the statement of our folk theorem. The definition mimics well-known arguments from the work Mertens and Zamir (1985) and Epstein and Zin (1989). We provide it for the sake of completeness, given the relative complexity of the subject matter and some slight differences in setups.

S-A.1. *The Path of Play as an Infinite Probability Tree*

Given a finite set A of action profiles, we define the set D of infinite probability trees as follows.[1] Let $D^1 = A$, and for $t > 0$, $D^t = A \times \Delta(D^{t-1})$. Let $\pi_1 : D^2 \to D^1$ be the projection of $D^2 = A \times \Delta(A)$ on A, and for $t > 1$, define $\pi_t : D^{t+1} \to D^t$ by letting

$$\pi_t(a, \mu^t) = (a, \mu^t \circ \pi_{t-1}^{-1}) \in D^t \quad \forall (a, \mu^t) \in A \times \Delta(D^t) \equiv D^{t+1}.$$

Above, $\mu^t \circ \pi_{t-1}^{-1} \in \Delta(D^{t-1})$ is the image measure of $\mu^t \in \Delta(D^t)$ under the function $\pi_{t-1} : D^t \to D^{t-1}$. To understand the construction, think of each $\mu^t \in \Delta(D^t)$ as a t-stage compound lottery, with pure actions $a \in A$ as outcomes. Then $\mu^t \circ \pi_{t-1}^{-1}$ is the $(t-1)$-stage compound lottery obtained by removing the last stage of randomization from μ^t.

Say that a sequence $(d^t)_t \in \times_t D^t$ is consistent if $d^t = \pi_t(d^{t+1})$ for each t, and let $D \subset \times_t D^t$ be the set of all such sequences. In other words, D is the inverse limit of the sets D^t relative to the projections π_t. It is well known that D is a compact, metric space,

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[1]We adopt the measure-theoretic and topological assumptions described at the start of the Appendix in the main text.

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and by a version of the Kolmogorov consistency theorem, that D is homeomorphic to $A \times \Delta(D)$; see, for instance, Theorems 2.6 and 3.2 in Parthasarathy (1967).

To see how each strategy $\sigma \in \Sigma$ induces a path of play $\mu \in \Delta(D)$, note that σ^0 induces a mixed action $\mu^1 \in \Delta(A) = \Delta(D^1)$, while (σ^0, σ^1) induce a two-stage lottery $\mu^2 \in \Delta(A \times \Delta(A)) = \Delta(D^2)$. Proceeding inductively, we obtain a sequence $(\mu^t)_t \in \bigtimes_t \Delta(D^t)$ such that $\mu^t \circ \pi_{t-1}^{-1} = \mu^{t-1}$. By the Kolmogorov consistency theorem again, there is a unique $\mu \in \Delta(D)$ such that, for each t, its marginal on D^t is μ^t. The next lemma verifies that conversely each $\mu \in \Delta(D)$ is induced by some strategy $\sigma \in \Sigma$.

LEMMA S-1: *Each $\mu \in \Delta(D)$ is induced by some strategy $\sigma \in \Sigma$.*

PROOF: Fix $\mu \in \Delta(D)$. Being a compact metric space, D is also separable and complete, and thus, a standard Borel space. Since D is also uncountable, it is isomorphic to the unit interval by the Borel isomorphism theorem; see Srivastava (2008, Theorem 3.3.13). Thus, there is a bijection $\phi : D \to [0, 1]$ such that both ϕ and ϕ^{-1} are measurable in the respective Borel algebras. Let $\hat{\mu} = \mu \circ \phi^{-1}$ be the induced image measure on the unit interval and let $g : [0, 1] \to [0, 1]$ be such that $\hat{\mu}$ is the image of the Lebesgue measure μ_L on the unit interval under g, that is, $\hat{\mu} = \mu_L \circ g^{-1}$. For example, as shown in Billingsley (1995, page 189), g could be the quantile function (the inverse of the cdf) of $\hat{\mu}$. Then let $f : [0, 1] \to D$ be the function $\phi^{-1} \circ g$ and note that, by construction, $\mu = \mu_L \circ f^{-1}$. Since D is homeomorphic to $A \times \Delta(D)$, we can also think of f as a pair of functions, (f^0, f^1), where $f^0 : [0, 1] \to A$ and $f^1 : [0, 1] \to \Delta(D)$. Then, suppressing the private signals, which are not needed for this proof, we construct the strategy $(\sigma^t)_t$ inductively as follows. First, let $\sigma^0 = f^0$. Then, given $\omega^0 \in [0, 1]$ and the associated on-path history $h^1 = (\omega^0, f^0)$, construct $\sigma^1(h^1) : [0, 1] \to A$ by applying the same arguments to the distribution $f^1(\omega^0) \in \Delta(D)$, which can be interpreted as the path of play in the subgame given h^1, and so on. *Q.E.D.*

S-A.2. *SIR and the Folk Theorem*

To generalize our folk theorem to paths of play $\mu \in \Delta(D)$, we must first extend the definition of SIR to such paths. For each $\mu \in \Delta(D)$, let $\operatorname{supp}_0 \mu = \{\mu\}$; for $t > 0$, let $\operatorname{supp}_t \mu$ be the set of $\mu' \in \Delta(D)$ for which there exist $(a_k, \mu_k) \in D$, $k = 1, \dots, t$ such that:

1. (a_1, μ_1) is in the support of μ,
2. (a_k, μ_k) is in the support of μ_{k-1} for all $k = 2, \dots, t$,
3. $\mu_t = \mu'$.[2]

If a strategy σ induces a path of play μ, then $\operatorname{supp}_t \mu$ represents (up to a set of measure zero) the paths of play μ' that arise in some subgame of the game given some on-path history h^t. If $\operatorname{supp}_t \mu$ is a singleton for every t, we say that $\mu \in \Delta(D)$ is history-independent; else, μ is history-dependent.

As is now intuitive, say that $\mu \in \Delta(D)$ is ε-sequentially individually rational (ε-SIR) if $v(\mu') \geq \varepsilon$ for every $\mu' \in \cup_t \operatorname{supp}_t \mu$. Finally, define a family $\{\Gamma_\lambda\}_\lambda$ of repeated games as in Section 3 and let $\Delta_\lambda^{\mathrm{SIR}}(\varepsilon) \subset \Delta(D)$ be the set of all ε-SIR μ in the game Γ_λ.

THEOREM S-1—Folk theorem for general paths of play: *For every $\varepsilon > 0$, there exists $\underline{\lambda}$ such that for all $\lambda > \underline{\lambda}$, every $\mu \in \Delta_\lambda^{\mathrm{SIR}}(\varepsilon) \subset \Delta(D)$ can be supported in a SPE of the game Γ_λ.*

[2]It is important to distinguish the sets $\operatorname{supp}_t \mu$, which are subsets of $\Delta(D)$, from the support of μ when μ is viewed as a measure on D. The latter set, which we denote as $\operatorname{supp} \mu$, is a subset of $A \times \Delta(D)$, not $\Delta(D)$.

The proof requires a single, minor tweak to that in the main text. Suppose player i deviates. From Lemma B7, recall that in the last phase of the DPSPs we constructed, play eventually switches to the worst SIR "path" $\mathbf{w}^i_\lambda$ for player i. When we wanted to sustain history-independent paths of play, it sufficed that this worst path be history-independent as well. Presently, we also want to sustain *history-dependent* paths of play, so the worst path must be redefined with respect to the larger set $\Delta^{\text{SIR}}_\lambda(\varepsilon)$; otherwise, the worst may not be worse enough. With this change, the proof of Theorem S-1 proceeds as in the main text.[3] We only verify that the set $\Delta^{\text{SIR}}_\lambda(\varepsilon)$ is closed so that a worst path can be found (whenever the set is nonempty).

LEMMA S-2: *For each λ, the set $\Delta^{\text{SIR}}_\lambda(\varepsilon)$ of ε-SIR paths of play is a closed subset of $\Delta(D)$.*

PROOF: Suppose $\mu_n \to_n \mu$ and each μ_n is ε-SIR. By the continuity of UzE preferences, $v_\lambda(\mu_n) \geq \varepsilon$ for all n implies $v(\mu) \geq \varepsilon$. Take some $\mu' \in \operatorname{supp}_1 \mu$, which means that $(a', \mu') \in \operatorname{supp} \mu$ for some $a' \in A$. Since D is compact and metrizable, D is separable. Likewise, $A \times \Delta(D)$ is compact, metrizable, and separable. It follows (see Aliprantis and Border (1999, Theorem 16.15)) that the correspondence $\hat{\mu} \mapsto \operatorname{supp} \hat{\mu}$ is lower hemicontinuous. Thus, there is a sequence $(a'_n, \mu'_n) \to_n (a', \mu')$. By the continuity of UzE preferences again, $v_\lambda(\mu') \geq \varepsilon$. It follows by induction that μ is ε-SIR. *Q.E.D.*

APPENDIX S-B: THE PRISONERS' DILEMMA UNDER IMI

In this section, we formalize the analysis of the prisoners' dilemma in Section 6.1.

S-B.1. *The Set of Efficient Pure Paths*

We begin by describing the set of all efficient pure paths in the two cases depicted in Figure 2. We let $\mathcal{X} = \{(DC, DC, \ldots), (CD, CD, \ldots)\}$ be the set of "extreme" paths in which the utility of a single player is maximized.

Intratemporal Cooperation. This is the case in Figure 2(a) in which $(CC, CC, \ldots)$ is efficient, but the alternating paths $(CD, DC, \ldots)$ and $(DC, CD, \ldots)$ are not. Let $\mathcal{C}_1$ be the set of paths such that DC is played in at most one period while CC is played in all other periods. The subscript "1" is used to designate the fact that the action profile DC, if it occurs, favors player 1. Next, let $\mathcal{E}_1\mathcal{C}_1$ be the set of paths $\mathbf{a} \in A^\infty$ such that for some $T \geq 0$, depending on the path, $a^t = DC$ for all $t < T$ and ${}_T\mathbf{a} \in \mathcal{C}_1$. Here, the letter $\mathcal{E}$ is mnemonic for the fact that cooperation prevails *eventually*, that is, after some period. Define the sets $\mathcal{C}_2$ and $\mathcal{E}_2\mathcal{C}_2$ analogously. Finally, let $\mathcal{EC} := \mathcal{E}_1\mathcal{C}_1 \cup \mathcal{E}_2\mathcal{C}_2 \cup \mathcal{X}$.

Intertemporal Cooperation. This corresponds to the case in Figure 2(b) in which the alternating paths $(CD, DC, \ldots)$ and $(DC, CD, \ldots)$, but not $(CC, CC, \ldots)$, are efficient. Consider the pairs (DC, CD) and (CD, DC) in A^2 and interpret each such pair as a "simple trade" in which the players swap turns defecting. Let $\mathcal{A}$ be the set of all play paths in which the players make such simple trades in succession:

$$\mathcal{A} := \left\{\mathbf{a} \in A^\infty : a^{2t}, a^{2t+1} \in \{DC, CD\} \text{ and } a^{2t} \neq a^{2t+1}\ \forall t\right\}.$$

[3]The definition of a DPSP must of course be amended to account for history dependence. As in the case of SIR, this is easily done.

One can verify that the payoffs from the paths $\mathbf{a} \in \mathcal{A}$ are dispersed along a linear segment of the frontier perpendicular to the 45-degree line. Next, let $\mathcal{E}_1\mathcal{A}$ be the set of play paths $\mathbf{a} \in A^\infty$ such that for some $T \geq 0$, depending on the path, $a^t = DC$ for all $t < T$, and ${}_T\mathbf{a} \in \mathcal{A}$. Define $\mathcal{E}_2\mathcal{A}$ analogously and let $\mathcal{E}\mathcal{A} := \mathcal{E}_1\mathcal{A} \cup \mathcal{E}_2\mathcal{A} \cup \mathcal{X}$.

Mixed Cases. Though such a situation is not depicted in Figure 2, it is also possible that $(CD, DC, \dots)$, $(DC, CD, \dots)$, and $(CC, CC, \dots)$ are simultaneously efficient. If so, there are three subcases.

Case I. The payoffs from $(CD, DC, \dots)$, $(DC, CD, \dots)$, and $(CC, CC, \dots)$ all lie on face of the Pareto frontier orthogonal to the direction $\eta = (1, 1)$. To describe this case, let $X = CC$, $Y = (DC, CD)$, and $Z = (CD, DC)$. Let $\mathcal{CA}^{\text{I}} := \{X, Y, Z\}^\infty$ and identify $\mathcal{CA}^{\text{I}}$ with a subset of A^∞ in the obvious manner. Define $\mathcal{E}_1\mathcal{CA}^{\text{I}}$ to be the set of paths such that DC is played until some period $T \geq 0$ and ${}_T\mathbf{a} \in \mathcal{CA}^{\text{I}}$. Define $\mathcal{E}_2\mathcal{CA}^{\text{I}}$ analogously. Finally, let $\mathcal{ECA}^{\text{I}} := \mathcal{E}_1\mathcal{CA}^{\text{I}} \cup \mathcal{E}_2\mathcal{CA}^{\text{I}} \cup \mathcal{X}$.

Case II. The payoffs from $(CD, DC, \dots)$, $(DC, CD, \dots)$, and $(CC, CC, \dots)$ are all extreme points of the Pareto frontier. In particular, $v_\lambda(CD, DC, \dots)$ lies strictly above the linear segment connecting $v_\lambda(CD, CC, CC, \dots)$ and $v(CC, CC, \dots)$. To describe the case, let $X = CC$ and $Y = (DC, CD)$, and $\mathcal{CA}_1^{\text{II}} = \{X, Y\}^\infty$. Define $\mathcal{CA}_2^{\text{II}}$ analogously. Let $\mathcal{CA}^{\text{II}} := \mathcal{CA}_1^{\text{II}} \cup \mathcal{CA}_2^{\text{II}}$. Observe that $\mathcal{CA}^{\text{II}} \subset \mathcal{CA}^{\text{I}}$. Define $\mathcal{E}_1\mathcal{CA}^{\text{II}}$ to be the set of paths such that DC is played until some period $T \geq 0$ and ${}_T\mathbf{a} \in \mathcal{CA}^{\text{II}}$. Define $\mathcal{E}_2\mathcal{CA}^{\text{II}}$ analogously and let $\mathcal{ECA}^{\text{II}} := \mathcal{E}_1\mathcal{CA}^{\text{II}} \cup \mathcal{E}_2\mathcal{CA}^{\text{II}} \cup \mathcal{X}$.

Case III. $(CC, CC, \dots)$ is the unique pure path that maximizes the sum of the players utilities, while the alternating paths $(CD, DC, \dots)$ and $(DC, CD, \dots)$ are efficient but their payoffs are not extreme points of the Pareto frontier. In particular, $v(CD, DC, \dots)$ lies on the linear segment connecting $v(CD, CC, CC, \dots)$ and $v(CC, CC, \dots)$. To describe this case, let $\mathcal{CA}^{\text{III}}$ be the set of paths $\mathbf{a}$ such that if $a^t \in \{CD, DC\}$ and T is the smallest integer $k > t$ such that $a^k \neq CC$, then $a^T \in \{CD, DC\} \setminus \{a^t\}$. We note that $\mathcal{CA}^{\text{II}} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \subset \mathcal{CA}^{\text{III}}$ but $\mathcal{CA}^{\text{I}} \nsubseteq \mathcal{CA}^{\text{III}}$. Define $\mathcal{E}_1\mathcal{CA}^{\text{III}}$ to be the set of paths such that DC is played until some period $T \geq 0$ and ${}_T\mathbf{a} \in \mathcal{CA}^{\text{III}}$. Define $\mathcal{E}_2\mathcal{CA}^{\text{III}}$ analogously and let $\mathcal{ECA}^{\text{III}} := \mathcal{E}_1\mathcal{CA}^{\text{III}} \cup \mathcal{E}_2\mathcal{CA}^{\text{III}} \cup \mathcal{X}$.

Given λ, let $P_\lambda \subset A^\infty$ be the set of all efficient pure paths in Γ_λ. Our next result shows that P_λ falls into one of the cases described above. Furthermore, the mixed cases do not arise for any λ sufficiently high. To state the result, write $0.5CD + 0.5DC$ for the mixed action $\alpha \in \Delta(A)$ that randomizes between CD and DC with equal probability and, as in the Appendix to the main text, write $v_i(\alpha)$ for $v_i(\alpha^{\text{iid}})$.

THEOREM S-2: *Consider the prisoners' dilemma under IMI. For every λ, $P_\lambda \in \{\mathcal{EC}, \mathcal{EA}, \mathcal{ECA}^{\text{I}}, \mathcal{ECA}^{\text{II}}, \mathcal{ECA}^{\text{III}}\}$. In addition*:

1. $v(0.5CD + 0.5DC) \geq v(CC)$ *if and only if* $P_\lambda = \mathcal{EA}$ *for all* λ.
2. *if* $v(0.5CD + 0.5DC) < v(CC)$, *then there exist* $\underline{\lambda} < \overline{\lambda} < 1$ *such that*
   (a) $P_\lambda = \mathcal{EA}$ *for all* $\lambda < \underline{\lambda}$,
   (b) $P_{\underline{\lambda}} = \mathcal{ECA}^{\text{I}}$,
   (c) $P_\lambda = \mathcal{ECA}^{\text{II}}$ *for all* $\lambda \in (\underline{\lambda}, \overline{\lambda})$,
   (d) $P_{\overline{\lambda}} = \mathcal{ECA}^{\text{III}}$,
   (e) $P_\lambda = \mathcal{EC}$ *for all* $\lambda > \overline{\lambda}$.

In the second part of the theorem, it is allowed that $\overline{\lambda} < 0$ so that $P_\lambda = \mathcal{EC}$ for all $\lambda \in [0, 1)$.

TABLE I

THE PRISONERS' DILEMMA.

	C	D
C	c, c	b, d
D	d, b	$0, 0$

PROOF OF THEOREM S-2: For ease of notation, let $g_1, g_2 : A \to \mathbb{R}$ be as in Table I and let $\beta(b) := \beta_1(CD)$, $\beta(c) := \beta_1(CC)$, and $\beta(d) := \beta_1(DC)$. Write v_c for $c(1-\beta(c))^{-1}$ and define v_d and v_b similarly. Since $v_d > v_c > 0 > v_b$, IMI implies $\beta(d) < \beta(c) < \beta(b)$. Given λ, write $\beta_\lambda(d)$ for $\lambda + (1-\lambda)\beta(d)$, etc., and note that $\beta_\lambda(d) < \beta_\lambda(c) < \beta_\lambda(b)$ for all λ. Also, recall that given $\mathbf{a} \in A^\infty$, $s_\lambda(\mathbf{a}, \eta) = \eta \cdot v_\lambda(\mathbf{a})$ and $P_\lambda(\eta)$ is the set of pure play paths $\mathbf{a} \in A^\infty$ that maximize $s_\lambda(\cdot, \eta)$. Finally, let

$$\mathbf{a}^{\mathcal{A},1} := (DC, CD, DC, CD, \ldots) \quad \text{and} \quad \mathbf{a}^{\mathcal{A},2} := (CD, DC, CD, DC, \ldots).$$

LEMMA S-1: *For every $i \in I$, $\lambda \in [0, 1)$ and $\eta \in \mathbb{R}^2_{++}$, we have $\mathbf{a}^{\max,i} \notin P_\lambda(\eta)$.*

PROOF: Since $\beta_\lambda(d) < \beta_\lambda(b)$, there is $T > 0$ large enough such that $\frac{\eta_i}{\eta_j}[\frac{\beta_\lambda(d)}{\beta_\lambda(b)}]^T$ is almost zero and so $\mathbf{a}^{\max,i} \notin P_\lambda(\eta^T_\lambda)$. By Lemma E22, $\mathbf{a}^{\max,i} \notin P_\lambda(\eta)$. *Q.E.D.*

LEMMA S-2: *For every $\lambda \in [0, 1)$, $\eta \in \mathbb{R}^2_+$, and $\mathbf{a} \in P_\lambda(\eta)$, if $a^0 = CD$ and $a^1 = DC$, then $\mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta)$. Similarly, if $a^0 = DC$ and $a^1 = CD$, then $\mathbf{a}^{\mathcal{A},1} \in P_\lambda(\eta)$.*

PROOF: If $a^0 = CD$ and $a^1 = DC$, then $\eta^2_\lambda = (\eta_1\beta_\lambda(b)\beta_\lambda(d), \eta_2\beta_\lambda(d)\beta_\lambda(b))$. It follows that $P_\lambda(\eta) = P_\lambda(\eta^2_\lambda)$, and by Lemma E22, ${}_2\mathbf{a} \in P_\lambda(\eta)$. Thus, $s_\lambda(\mathbf{a}, \eta) = s_\lambda({}_2\mathbf{a}, \eta)$, from which we deduce that

$$s_\lambda({}_2\mathbf{a}, \eta) = \eta_1(1-\lambda)\frac{b + \beta_\lambda(b)d}{1 - \beta_\lambda(b)\beta_\lambda(d)} + \eta_2(1-\lambda)\frac{d + \beta_\lambda(d)b}{1 - \beta_\lambda(b)\beta_\lambda(d)}.$$

Thus, $s_\lambda({}_2\mathbf{a}, \eta) = s_\lambda(\mathbf{a}^{\mathcal{A},2}, \eta)$ and so $\mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta)$. *Q.E.D.*

LEMMA S-3: *For every $\lambda \in [0, 1)$, $\eta \in \mathbb{R}^2_+$, and $\mathbf{a} \in P_\lambda(\eta)$, if $\frac{\eta_1}{\eta_2} < 1$, then $v_{1\lambda}(\mathbf{a}) \leq v_{2\lambda}(\mathbf{a})$ and $a^0 \neq DC$; if $\frac{\eta_1}{\eta_2} > 1$, then $v_{1\lambda}(\mathbf{a}) \geq v_{2\lambda}(\mathbf{a})$ and $a^0 \neq CD$.*

PROOF: It is enough to consider the case when $\frac{\eta_1}{\eta_2} < 1$. Then, by the symmetry of the game, $v_{1\lambda}(\mathbf{a}) \leq v_{2\lambda}(\mathbf{a})$. Next, suppose $a^0 = DC$ and let $T \geq 1$ be the first period t such that $a^t \neq DC$. Such T exists because $v_{1\lambda}(\mathbf{a}) \leq v_{2\lambda}(\mathbf{a})$. Suppose $a^T = CC$ and consider the path $\hat{\mathbf{a}}$ such that $\hat{a}^t = DC$ for all $t < T$ and $\hat{a}^t = CC$ for all $t \geq T$. From Lemma E23, $\hat{\mathbf{a}} \in P_\lambda(\eta)$. But, by construction, $v_{1\lambda}(\hat{\mathbf{a}}) > v_{2\lambda}(\hat{\mathbf{a}})$, contradicting the first assertion of the lemma. If $a^T = CD$, then by Lemma S-2, $\mathbf{a}^{\mathcal{A},1} \in P_\lambda(\eta^{T-1}_\lambda(\mathbf{a}))$. Also,

$$\frac{\eta^{T-1}_{1\lambda}(\mathbf{a})}{\eta^{T-1}_{2\lambda}(\mathbf{a})} = \frac{[\beta_\lambda(d)]^{T-1}}{[\beta_\lambda(b)]^{T-1}}\frac{\eta_1}{\eta_2} \leq \frac{\eta_1}{\eta_2} < 1,$$

where the first inequality follows from $\beta_\lambda(d) < \beta_\lambda(b)$. But then $v_{1\lambda}(\mathbf{a}^{\mathcal{A},1}) > v_{2\lambda}(\mathbf{a}^{\mathcal{A},1})$, contradicting the first assertion in the lemma. Thus, $a^0 \neq DC$. *Q.E.D.*

Next, let $\mathbf{a}^C(0) := \mathbf{a}^C$ and for every $T \geq 1$, let $\mathbf{a}^C(T)$ be the path such that $a^t = CD$ for all $0 \leq t < T$ and ${}_T\mathbf{a} = \mathbf{a}^C$. Recall $\eta^{\text{sym}} = (1, 1)$. Define $P_\lambda^{\text{sym}} := P_\lambda(\eta^{\text{sym}})$. For simplicity, write $s_\lambda(\mathbf{a})$ instead of $s_\lambda(\mathbf{a}, \eta^{\text{sym}})$. The next lemma shows that there is no λ such that $\mathbf{a}^C(1) \in P_\lambda^{\text{sym}}$.

LEMMA S-4: $s_\lambda(\mathbf{a}^C(1)) < \max\{s_\lambda(\mathbf{a}^C), s_\lambda(\mathbf{a}^{\mathcal{A},2})\}$ *for all* $\lambda \in [0, 1)$.

PROOF: By construction, $s_\lambda(\mathbf{a}^C) > s_\lambda(\mathbf{a}^C(1))$ if and only if

$$v_c > \frac{b+d}{1-\beta(b)+1-\beta(d)}. \tag{S-1}$$

If (S-1) holds, the proof is complete. Suppose that (S-1) holds with equality. Then

$$s_\lambda\big(\mathbf{a}^C(1)\big) = \frac{2(b+d)}{1-\beta(b)+1-\beta(d)}. \tag{S-2}$$

Also, since $s_{\lambda'}(\mathbf{a}^{\mathcal{A},2})$ is decreasing in λ', we have

$$s_\lambda\big(\mathbf{a}^{\mathcal{A},2}\big) > \lim_{\lambda'\to 1} s_{\lambda'}\big(\mathbf{a}^{\mathcal{A},2}\big) = \frac{2(b+d)}{1-\beta(b)+1-\beta(d)}. \tag{S-3}$$

Combining (S-2) and (S-3) gives $s_\lambda(\mathbf{a}^{\mathcal{A},2}) > s_\lambda(\mathbf{a}^C(1))$. Finally, if the strict inequality in (S-1) is reversed, then

$$\begin{aligned} s_\lambda\big(\mathbf{a}^C(1)\big) &< (1-\lambda)(b+d) + \big(\beta_\lambda(b)+\beta_\lambda(d)\big)\frac{b+d}{1-\beta(b)+1-\beta(d)} \\ &= \frac{2(b+d)}{1-\beta(b)+1-\beta(d)} < s_\lambda\big(\mathbf{a}^{\mathcal{A},2}\big). \end{aligned}$$

The equality follows from direct simplification. The last inequality follows from (S-3). *Q.E.D.*

Say that λ is *irregular* if $P_\lambda \in \{\mathcal{ECA}^{\text{I}}, \mathcal{ECA}^{\text{II}}, \mathcal{ECA}^{\text{III}}\}$, that is, if $\mathbf{a}^{\mathcal{A},1}$, $\mathbf{a}^{\mathcal{A},2}$, and $\mathbf{a}^C$ are simultaneously efficient. The next lemma provides a characterization for irregular λ. It can be seen that irregular λ may not always exist and is bounded away from one.

LEMMA S-5: λ *is irregular if and only if*

$$1 \leq f(\lambda) := \frac{v_c - v_{1\lambda}\big(\mathbf{a}^{\mathcal{A},2}\big)}{v_{2\lambda}\big(\mathbf{a}^{\mathcal{A},2}\big) - v_c} \leq \sqrt{\frac{\beta_\lambda(b)}{\beta_\lambda(d)}}. \tag{S-4}$$

PROOF: Suppose $\mathbf{a}^{\mathcal{A},2}, \mathbf{a}^C \in P_\lambda(\eta)$ for some λ and $\eta \in \mathbb{R}^2_+$. If $f(\lambda) < 0$, then either $\mathbf{a}^C$ or $\mathbf{a}^{\mathcal{A},2}$ is strictly Pareto dominated; if $f(\lambda) \in [0, 1)$, then $\mathbf{a}^C$ is strictly dominated by some path in $\mathcal{A}$. Thus, $f(\lambda) \geq 1$. Turn to the second inequality. Since $\mathbf{a}^{\mathcal{A},2}, \mathbf{a}^C \in P_\lambda(\eta)$, Lemma E22 implies that $(CD, DC, \mathbf{a}^C) \in P_\lambda(\eta)$. By Lemma E22, the paths $(DC, \mathbf{a}^C)$ and $(DC, \mathbf{a}^{\mathcal{A},2}) = \mathbf{a}^{\mathcal{A},1}$ are efficient given the direction $(\eta_1\beta_\lambda(b), \eta_2\beta_\lambda(d))$. By the symmetry of the game, the paths $(CD, \mathbf{a}^C)$ and $(CD, DC, \mathbf{a}^{\mathcal{A},2}) = \mathbf{a}^{\mathcal{A},2}$ are efficient given

$\eta' := (\eta_2\beta_\lambda(d), \eta_1\beta_\lambda(b))$. Thus, $\mathbf{a}^{\mathcal{A},2}$ is efficient under both η and η'. By the convexity of the feasible set, we have $\frac{\eta_1'}{\eta_2'} \leq \frac{\eta_1}{\eta_2}$. Since $\mathbf{a}^{\mathcal{A},2}, \mathbf{a}^C \in P_\lambda(\eta)$, it must be that $\eta = (v_{2\lambda}(\mathbf{a}^{\mathcal{A},2}) - v_c, v_c - v_{1\lambda}(\mathbf{a}^{\mathcal{A},2}))$. Deduce that $\frac{\eta_1'}{\eta_2'} \leq \frac{\eta_1}{\eta_2}$ if and only if the second inequality in (S-4) holds. Identical arguments show that $\mathbf{a}^{\mathcal{A},2}, \mathbf{a}^C \in P_\lambda(\eta)$ whenever (S-4) holds. *Q.E.D.*

We first focus on regular λ. Given a path $\mathbf{a} \in A^\infty$ and some T, say that $(T, T+1)$ is an alternation for $\mathbf{a}$ if $a^T, a^{T+1} \in \{CD, DC\}$ and $a^T \neq a^{T+1}$.

LEMMA S-6: *Fix a regular λ. For every $\eta \in \mathbb{R}^2_{++}$ and $\mathbf{a} \in P_\lambda(\eta)$, if $(T, T+1)$ is an alternation for $\mathbf{a}$, then $a^t \neq CC$ for every t.*

PROOF: It is w.l.o.g. to assume that $a^T = CD$ and $a^{T+1} = DC$. By Lemma S-2, $\mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta^T_\lambda(\mathbf{a}))$. Assuming $T \geq 1$, we are going to show that $a^t \neq CC$ for every $t < T$. If not, let T' be the greatest integer $k < T$ such that $a^k = CC$. By Lemma E23, we know that $\mathbf{a}^C \in P_\lambda(\eta^{T'}_\lambda(\mathbf{a}))$. The latter is possible only if

$$v_c > v_{1\lambda}(\mathbf{a}^{\mathcal{A},2}). \tag{S-5}$$

Otherwise, we would have $v_c \leq v_{1\lambda}(\mathbf{a}^{\mathcal{A},2}) < v_{2\lambda}(\mathbf{a}^{\mathcal{A},2})$, and hence, $\mathbf{a}^C$ would be strictly Pareto dominated by $\mathbf{a}^{\mathcal{A},2}$. Next, observe that, by construction, $T' \leq T-1$. Suppose first that $T' = T-1$. Since $a^{T'} = CC$,

$$(\eta^T_{1\lambda}(\mathbf{a}), \eta^T_{2\lambda}(\mathbf{a})) = (\eta^{T'}_{1\lambda}(\mathbf{a})\beta_\lambda(c), \eta^{T'}_{2\lambda}(\mathbf{a})\beta_\lambda(c)).$$

Thus, $P_\lambda(\eta^T_\lambda(\mathbf{a})) = P_\lambda(\eta^{T'}_\lambda(\mathbf{a}))$. But then $\mathbf{a}^C, \mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta^T_\lambda(\mathbf{a}))$, contradicting the regularity of λ. Suppose now that $T' < T-1$. It is w.l.o.g. to assume that $a^t = CD$ for all $T' < t < T$. Else, there would be an alternation $(k, k+1)$ where $T' < k < T$ and we can use the latter alternation in place of $(T, T+1)$. The direction $\eta^{T'+2}_\lambda(\mathbf{a})$ satisfies

$$(\eta^{T'+2}_{1\lambda}(\mathbf{a}), \eta^{T'+2}_{2\lambda}(\mathbf{a})) = ((\eta^{T'}_{1\lambda}(\mathbf{a})\beta_\lambda(c)\beta_\lambda(b), \eta^{T'}_{2\lambda}(\mathbf{a})\beta_\lambda(c)\beta_\lambda(d)).$$

Since $\beta_\lambda(b) > \beta_\lambda(d)$, we have $\frac{\eta^{T'+2}_{1\lambda}(\mathbf{a})}{\eta^{T'+2}_{2\lambda}(\mathbf{a})} > \frac{\eta^{T'}_{1\lambda}(\mathbf{a})}{\eta^{T'}_{2\lambda}(\mathbf{a})}$. But then $\mathbf{a}^C \in P_\lambda(\eta^{T'}_\lambda(\mathbf{a}))$ implies that

$$v_{1\lambda}(\mathbf{a}') \geq v_c \quad \forall \mathbf{a}' \in P_\lambda(\eta^{T'+2}_\lambda(\mathbf{a})). \tag{S-6}$$

Let $\hat{\mathbf{a}} \in A^\infty$ be a path such that $\hat{a}^t = a^t = CD$ for all $T'+2 \leq t < T$ and ${}_T\hat{\mathbf{a}} = \mathbf{a}^{\mathcal{A},2}$. By Lemma E22, the fact that ${}_{T'+2}\mathbf{a} \in P_\lambda(\eta^{T'+2}_\lambda(\mathbf{a}))$ and $\mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta^T_\lambda(\mathbf{a}))$ implies that ${}_{T'+2}\hat{\mathbf{a}} \in P_\lambda(\eta^{T'+2}_\lambda(\mathbf{a}))$. We claim that $v_{1\lambda}({}_{T'+2}\hat{\mathbf{a}}) \leq v_{1\lambda}(\mathbf{a}^{\mathcal{A},2}) < v_c$. The first inequality follows since ${}_{T'+2}\hat{\mathbf{a}}$ begins with a repetitive play of CD, which hurts player 1, followed by the more desirable path $\mathbf{a}^{\mathcal{A},2}$. The second inequality follows from (S-5). Together, the inequalities contradict (S-6).

Next, we show that $a^t \neq CC$ for every $t > T+1$. Note that

$$(\eta^{T+2}_{1\lambda}(\mathbf{a}), \eta^{T+2}_{2\lambda}(\mathbf{a})) = (\eta^T_{1\lambda}(\mathbf{a})\beta_\lambda(b)\beta_\lambda(d), \eta^T_{2\lambda}(\mathbf{a})\beta_\lambda(d)\beta_\lambda(b)).$$

Thus, $P_\lambda(\eta^T_\lambda(\mathbf{a})) = P_\lambda(\eta^{T+2}_\lambda(\mathbf{a}))$. By way of contradiction, suppose first that $a^{T+2} = CC$. By Lemma E23, $\mathbf{a}^C \in P_\lambda(\eta^{T+2}_\lambda(\mathbf{a}))$. But then $\mathbf{a}^C, \mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta^T_\lambda(\mathbf{a}))$, contradicting the regularity of λ. Suppose now that $a^k = CC$ for some $k > T+2$. Let T' be the smallest such k. It is w.l.o.g. to assume that $a^t = DC$ for all $T+1 < t < T'$. Else, there would be an alternation $(k, k+1)$ where $T < k < T'$ and we can use the latter alternation in place of $(T, T+1)$. Since $(T, T+1)$ is an alternation, $\eta_\lambda^T(\mathbf{a})$ and $\eta_\lambda^{T+2}(\mathbf{a})$ determine the same direction and so $P_\lambda(\eta_\lambda^T(\mathbf{a})) = P_\lambda(\eta_\lambda^{T+2}(\mathbf{a}))$. Since $a^T = CD$, Lemma S-3 shows that $\eta_{1\lambda}^T(\mathbf{a}) \leq \eta_{2\lambda}^T(\mathbf{a})$. And, since $a^{T+2} = DC$, Lemma S-3 shows that $\eta_{1\lambda}^{T+2}(\mathbf{a}) \geq \eta_{2\lambda}^{T+2}(\mathbf{a})$. Conclude that both $\eta_\lambda^T(\mathbf{a})$ and $\eta_\lambda^{T+2}(\mathbf{a})$ determine the same direction as η^{sym}. To complete the proof, suppose first that $T' = T+3$. Hence, $a^{T+3} = CC$, and by Lemma E23, we know that $\mathbf{a}^C \in P_\lambda(\eta_\lambda^{T+3}(\mathbf{a}))$. Then, by Lemma E22, $(DC, \mathbf{a}^C) \in P_\lambda(\eta_\lambda^{T+2}(\mathbf{a}))$. But recall that $\eta_\lambda^{T+2}(\mathbf{a})$ and η^{sym} determine the same direction. Thus, $(DC, \mathbf{a}^C) \in P_\lambda^{\text{sym}}$, contradicting Lemma S-4. Alternatively, suppose that $T' > T+3$. Then $a^{T+2} = a^{T+3} = DC$, and hence,

$$\eta_\lambda^{T+3}(\mathbf{a}) = \big(\eta_{1\lambda}^{T+2}(\mathbf{a})\beta_\lambda(d),\, \eta_{2\lambda}^{T+2}(\mathbf{a})\beta_\lambda(b)\big).$$

Since $\eta_{1\lambda}^{T+2}(\mathbf{a}) = \eta_{2\lambda}^{T+2}(\mathbf{a})$, we may conclude that $\eta_{1\lambda}^{T+3}(\mathbf{a}) < \eta_{2\lambda}^{T+3}(\mathbf{a})$. But then Lemma S-3 shows that a^{T+3} cannot be DC, a contradiction. *Q.E.D.*

LEMMA S-7: *Fix a regular λ. For every $\eta \in \mathbb{R}_{++}^2$ such that $\eta_1 < \eta_2$ and every path $\mathbf{a} \in P_\lambda(\eta)$, if $a^0 = CC$, then $\mathbf{a} \in \mathcal{C}_2$.*

PROOF: If $\mathbf{a} = \mathbf{a}^C$, we are done. Suppose that $\mathbf{a} \neq \mathbf{a}^C$. We are going to show that ${}_1\mathbf{a} \in \mathcal{C}_2$, and hence, $\mathbf{a} \in \mathcal{C}_2$. Let T be the first period t such that $a^t \neq CC$. Since $a^0 = CC$, we know that $T > 0$. By the choice of T, we know that the direction $\eta_\lambda^t(\mathbf{a})$ is the same as η for every $0 < t \leq T$. Since $\frac{\eta_1}{\eta_2} < 1$, Lemma S-3 shows that $a^T \neq DC$. Thus, $a^T = CD$. Next, we are going to show that $a^{T+1} = CC$. By Lemma S-1, the constant path $(CD, CD, \ldots) \notin P_\lambda(\eta_\lambda^T(\mathbf{a}))$. Hence, there exists $t > T$ such that $a^t \neq CD$. Let T' be the smallest such t. By construction, $a^{T'-1} = CD$. Since $a^0 = CC$, Lemma S-6 implies that $a^{T'} \neq DC$. Else, $(T'-1, T')$ would be an alternation for a path that contains CC. Conclude that $a^{T'} = CC$. Next, observe that

$$\big(\eta_{1\lambda}^{T+1}(\mathbf{a}), \eta_{2\lambda}^{T+1}(\mathbf{a})\big) = \big(\eta_1\big[\beta_\lambda(c)\big]^T\beta_\lambda(b),\, \eta_2\big[\beta_\lambda(c)\big]^T\beta_\lambda(d)\big).$$

Since $\beta_\lambda(b) > \beta_\lambda(d)$, we have $\frac{\eta_{1\lambda}^{T+1}(\mathbf{a})}{\eta_{2\lambda}^{T+1}(\mathbf{a})} > \frac{\eta_1}{\eta_2}$. Since $a^0 = CC$, Lemma E23 shows that $\mathbf{a}^C \in P_\lambda(\eta)$. Combining the last two observations, conclude that

$$v_{1\lambda}(\mathbf{a}') \geq v_c \quad \forall \mathbf{a}' \in P_\lambda\big(\eta_\lambda^{T+1}(\mathbf{a})\big). \tag{S-7}$$

Recall that $a^{T'} = CC$. By Lemma E23, $\mathbf{a}^C \in P_\lambda(\eta_\lambda^{T'}(\mathbf{a}))$. Define the path $\hat{\mathbf{a}} \in A^\infty$ such that $\hat{a}^t = a^t = CD$ for $T+1 \leq t < T'$ and ${}_{T'}\hat{\mathbf{a}} = \mathbf{a}^C$. Lemma E22 implies that ${}_{T+1}\hat{\mathbf{a}} \in P_\lambda(\eta_\lambda^{T+1}(\mathbf{a}))$. If $T' > T+1$, then $v_{1\lambda}({}_{T+1}\hat{\mathbf{a}}) < v_c$, contradicting (S-7). Hence, $T' = T+1$, that is, $a^{T+1} = CC$. To summarize, we have shown that for every $\mathbf{a} \in P_\lambda(\eta)$ such that $a^t = CC$ for all $t < T$ and $a^T = CD$, we have $a^{T+1} = CC$.

Next, we are going to show that, in fact, ${}_{T+1}\mathbf{a} = \mathbf{a}^C$. If not, we can find $k > T+1$ such that $a^k \neq CC$. Let T'' be the smallest such k. By the choice of T'', we know that $\eta_\lambda^{T''}(\mathbf{a})$ and $\eta_\lambda^{T+1}(\mathbf{a})$ determine the same direction so that $P_\lambda(\eta_\lambda^{T+1}(\mathbf{a})) = P_\lambda(\eta_\lambda^{T''}(\mathbf{a}))$. By Lemma E22, ${}_{T''}\mathbf{a} \in P_\lambda(\eta_\lambda^{T''}(\mathbf{a}))$. Thus, ${}_{T''}\mathbf{a} \in P_\lambda(\eta_\lambda^{T+1}(\mathbf{a}))$. But then, by Lemma E22, $\tilde{\mathbf{a}} := (a^0, a^1, \ldots, a^T, {}_{T''}\mathbf{a}) \in P_\lambda(\eta)$. By construction, $\tilde{\mathbf{a}}$ is such that $\tilde{a}^t = CC$ for all $t < T$, $\tilde{a}^T = CD$, and $\tilde{a}^{T+1} \neq CC$, contradicting the first part of the proof. *Q.E.D.*

LEMMA S-8: *Fix a regular λ. For every $\eta \in \mathbb{R}_{++}^2$ and $\mathbf{a} \in P_\lambda(\eta)$, if $a^0 = CD$, and $a^1 = CC$, then ${}_2\mathbf{a} \notin \mathcal{C}_1$.*

PROOF: If ${}_2\mathbf{a} \in \mathcal{C}_1$, there is $T > 1$ such that $a^T = DC$ and ${}_{T+1}\mathbf{a} = \mathbf{a}^C$. By the choice of T, we know that $\eta_\lambda^T(\mathbf{a})$ and $\eta_\lambda^1(\mathbf{a})$ determine the same direction, so that $P_\lambda(\eta_\lambda^T(\mathbf{a})) = P_\lambda(\eta_\lambda^1(\mathbf{a}))$. But, by Lemma E22, ${}_T\mathbf{a} \in P_\lambda(\eta_\lambda^T(\mathbf{a}))$. Thus, ${}_T\mathbf{a} \in P_\lambda(\eta_\lambda^1(\mathbf{a}))$. But then, by Lemma E22, $\hat{\mathbf{a}} := (CD, {}_T\mathbf{a}) = (CD, DC, \mathbf{a}^C) \in P_\lambda(\eta)$. Thus, $\hat{\mathbf{a}}$ contains an alternation followed by a play of CC, contradicting Lemma S-6. *Q.E.D.*

LEMMA S-9: *Fix a regular λ. For every $\mathbf{a} \in P_\lambda^{\text{sym}}$, if $a^0 = CD$, then $\mathbf{a} \in \mathcal{A}$.*

PROOF: Let $\eta := \eta^{\text{sym}}$. Since $a^0 = CD$ and $\beta_\lambda(b) > \beta_\lambda(d)$,

$$\frac{\eta_{1\lambda}^1(\mathbf{a})}{\eta_{2\lambda}^1(\mathbf{a})} = \frac{\eta_1\beta_\lambda(b)}{\eta_2\beta_\lambda(d)} = \frac{\beta_\lambda(b)}{\beta_\lambda(d)} > 1.$$

Since ${}_1\mathbf{a} \in P_\lambda(\eta_\lambda^1(\mathbf{a}))$, we can apply Lemma S-3 to deduce that $a^1 \in \{CC, DC\}$. If $a^1 = CC$, it follows from Lemma E23 that $\mathbf{a}^C \in P_\lambda(\eta_\lambda^1(\mathbf{a}))$. But then, by Lemma E22, $(CD, \mathbf{a}^C) \in P_\lambda^{\text{sym}}$, contradicting Lemma S-4. Thus, $a^1 = DC$. Deduce that $\eta_\lambda^2(\mathbf{a})$ and η determine the same direction and, by Lemma S-2, that $\mathbf{a}^{\mathcal{A},2} \in P_\lambda(\eta)$. Since (0,1) is an alternation for the path $\mathbf{a}$, we know from Lemma S-6 that $a^t \neq CC$ for all $t > 1$. Thus, $a^2 \in \{CD, DC\}$. By Lemma E22, ${}_2\mathbf{a} \in P_\lambda(\eta_\lambda^2(\mathbf{a}))$. But, since $\eta_\lambda^2(\mathbf{a})$ and η determine the same direction, we have ${}_2\mathbf{a} \in P_\lambda(\eta)$. We also know that $a^2 \in \{CD, DC\}$. Thus, the same arguments that showed that $a^1 = DC$ now show that $a^3 \in \{CD, DC\} \setminus \{a^2\}$. Proceeding like this, conclude that $\mathbf{a} \in \mathcal{A}$. *Q.E.D.*

LEMMA S-10: *Fix a regular λ. For every $\eta \in \mathbb{R}_{++}^2$ and $\mathbf{a} \in P_\lambda(\eta)$, if $a^0 = CD$ and $a^1 = DC$, then $\mathbf{a} \in \mathcal{E}_2\mathcal{A}$.*

PROOF: Since $a^0 = CD$, it follows from Lemma S-3 that $\eta_1 \leq \eta_2$. Suppose $\eta_1 = \eta_2$. We know from Lemma S-9 that $\mathbf{a} \in \mathcal{A}$, and hence, that $\mathbf{a} \in \mathcal{E}_2\mathcal{A}$. Next, suppose $\eta_1 < \eta_2$. Since $a^0 = CD$ and $a^1 = DC$, η and η_λ^2 determine the same direction. Hence, $\eta_{1\lambda}^2 < \eta_{2\lambda}^2$. Since the path $\mathbf{a}$ has an alternation $(0, 1)$, we know from Lemma S-6 that $a^t \neq CC$ for all $t > T$. Hence, Lemma S-3 implies that $a^2 = CD$. Moreover, since $a^1 = DC$, Lemma S-3 shows that $\eta_{1\lambda}^1 \geq \eta_{2\lambda}^1$. If $\eta_{1\lambda}^1 = \eta_{2\lambda}^1$, we know from Lemma S-9 that $P_\lambda(\eta_\lambda^1) \subset \mathcal{A}$. Therefore, $\mathbf{a} \in \mathcal{E}_2\mathcal{A}$, as desired. Now suppose $\eta_{1\lambda}^1 > \eta_{2\lambda}^1$. Recall that $a^1 = DC$ and $a^2 = CD$. Thus, η_λ^1 and η_λ^3 determine the same direction. As a result, we have $\eta_{1\lambda}^3 > \eta_{2\lambda}^3$. Recall that $a^t \neq CC$ for all $t > T$. Hence, Lemma S-3 implies that $a^3 = DC$. Proceeding like this, we get $\eta_{1\lambda}^{2t} < \eta_{2\lambda}^{2t}$ and $\eta_{1\lambda}^{2t+1} > \eta_{2\lambda}^{2t+1}$ for all t. Lemma S-3 implies that $a^{2t} = CD$ and $a^{2t+1} = DC$ for all t. That is, $\mathbf{a} = \mathbf{a}^{\mathcal{A},2} \in \mathcal{E}_2\mathcal{A}$. *Q.E.D.*

Let $P_\lambda^{++} := \cup_{\eta \in \mathbb{R}_{++}^2} P_\lambda(\eta)$.

LEMMA S-11: *Fix a regular λ. Then $P_\lambda^{++} \subset \mathcal{E}_1\mathcal{C}_1 \cup \mathcal{E}_2\mathcal{C}_2 \cup \mathcal{E}_1\mathcal{A} \cup \mathcal{E}_2\mathcal{A}$.*

PROOF: First, we show that if $\mathbf{a} \in P_\lambda^{\text{sym}}$, then $\mathbf{a} \in \mathcal{A} \cup \{\mathbf{a}^C\}$. By Lemma E24, DD cannot be played along any efficient path. Hence, $a^0 \in \{CC, DC, CD\}$. If $a^0 \in \{CD, DC\}$, Lemma S-9 shows that $\mathbf{a} \in \mathcal{A}$. Alternatively, suppose $a^0 = CC$. By Lemma E23, the path $\mathbf{a}^C$ is efficient. Assume that $\mathbf{a} \neq \mathbf{a}^C$. Let T be the first period t such that $a^t \neq CC$. By construction, for any $t \leq T$, the direction η_λ^t is the same as η^{sym}. Thus, ${}_T\mathbf{a} \in P_\lambda(\eta)$. W.l.o.g., assume $a^T = CD$. The proof of Lemma S-9 shows that $a^{T+1} = DC$. Thus, $(T, T+1)$ is an alternation for $\mathbf{a}$. Since $a^0 = CC$, Lemma S-6 is contradicted. Conclude that $P_\lambda(\eta^{\text{sym}}) = \{\mathbf{a}^C\}$.

Next, take any $\mathbf{a} \in P_\lambda(\eta)$ where $0 < \eta_1 < \eta_2$. Since $\eta_1 < \eta_2$, Lemma S-3 shows that $v_{1\lambda}(\mathbf{a}) \leq v_{2\lambda}(\mathbf{a})$ and $a^0 \neq DC$. By Lemma E24, DD cannot be played along any efficient path. Hence, $a^0 \in \{CC, CD\}$. If $a^0 = CC$, Lemma S-7 shows that ${}_1\mathbf{a} \in \mathcal{C}_2$, and hence, $\mathbf{a} \in \mathcal{C}_2$. Alternatively, suppose $a^0 = CD$. By Lemma S-1, the constant path $(CD, CD, \dots)$ is not efficient. Let T be the first period t such that $a^t \neq CD$. Suppose $a^T = CC$. If $\eta^T_{1\lambda} < \eta^T_{2\lambda}$, then Lemma S-7 shows that ${}_T\mathbf{a} \in \mathcal{C}_2$. If $\eta^T_{1\lambda} = \eta^T_{2\lambda}$, we have already shown that ${}_T\mathbf{a} = \mathbf{a}^C$. If $\eta^T_{1\lambda} > \eta^T_{2\lambda}$, Lemma S-7 implies that ${}_T\mathbf{a} \in \mathcal{C}_1$. Moreover, Lemma S-8 implies that $a^t \neq DC$ for all $t > T$. Therefore, ${}_T\mathbf{a} = \mathbf{a}^C$. Finally, suppose $a^T = DC$. Lemma S-10 shows that ${}_{T-1}\mathbf{a} \in \mathcal{E}_2\mathcal{A}$, and hence, $\mathbf{a} \in \mathcal{E}_2\mathcal{A}$. *Q.E.D.*

LEMMA S-12: *Fix a regular λ. If $s_\lambda(\mathbf{a}^C) > s_\lambda(\mathbf{a}^{\mathcal{A},2})$, then $P^{++}_\lambda \supseteq \mathcal{E}_1\mathcal{C}_1 \cup \mathcal{E}_2\mathcal{C}_2$. Else, $P^{++}_\lambda \supseteq \mathcal{E}_1\mathcal{A} \cup \mathcal{E}_2\mathcal{A}$.*

PROOF: We prove that $\mathcal{E}_2\mathcal{C}_2 \subset P^{++}_\lambda$. Everything else follows from analogous arguments. Recall the paths $\mathbf{a}^C(T)$, $T \geq 0$, defined prior to Lemma S-4. Note that $\mathbf{a}^C(T) \in \mathcal{E}_2\mathcal{C}_2$ for every T. Let $\eta(0) := (1, 1)$ and, for every $T > 0$,

$$\eta(T) := \big(v_{2\lambda}\big(\mathbf{a}^C(T)\big) - v_{2\lambda}\big(\mathbf{a}^C(T-1)\big), v_{1\lambda}\big(\mathbf{a}^C(T-1)\big) - v_{1\lambda}\big(\mathbf{a}^C(T)\big)\big).$$

We claim that $\mathbf{a}^C(T) \in P_\lambda(\eta(T))$ for every $T \geq 0$. The proof is by induction. From Lemma S-11, we know that $\mathbf{a}^C(0) \in P_\lambda(\eta(0))$. Suppose $\mathbf{a}^C(T) \in P_\lambda(\eta(T))$ for some $T > 0$. We have to show that $\mathbf{a}^C(T+1) \in P_\lambda(\eta(T+1))$. From Lemma S-11, we know that $P_\lambda(\eta(T+1)) \subset \mathcal{E}\mathcal{C}$. It is therefore enough to show that

$$s_\lambda\big(\mathbf{a}^C(T+1), \eta(T+1)\big) \geq s_\lambda\big(\mathbf{a}, \eta(T+1)\big) \quad \forall \mathbf{a} \in \mathcal{E}\mathcal{C}. \tag{S-8}$$

By construction, $\frac{\eta_1(T+1)}{\eta_2(T+1)} < 1$ and, Lemma S-3, $v_{1\lambda}(\mathbf{a}) \leq v_{2\lambda}(\mathbf{a})$ for all $\mathbf{a} \in P_\lambda(\eta(T+1))$. Hence, it is enough to show that (S-8) holds for all paths $\mathbf{a} \in \mathcal{E}_2\mathcal{C}_2$. Begin with paths in the set $\{\mathbf{a}^C(T') : T' \geq 0\} \subset \mathcal{E}_2\mathcal{C}_2$. If $T' > T+1$, then (S-8) is equivalent to

$$\beta_\lambda(d) + \cdots + \big[\beta_\lambda(d)\big]^{T'-T-1} \leq \beta_\lambda(b) + \cdots + \big[\beta_\lambda(b)\big]^{T'-T-1},$$

which holds since $\beta_\lambda(d) < \beta_\lambda(b)$. If $T' = T$, then (S-8) holds since, by the definition of $\eta(T+1)$, we have

$$s_\lambda\big(\mathbf{a}^C(T+1), \eta(T+1)\big) = s_\lambda\big(\mathbf{a}^C(T), \eta(T+1)\big). \tag{S-9}$$

Finally, take $T' < T$. By the induction hypothesis, $\mathbf{a}(T) \in P_\lambda(\eta(T))$, and hence,

$$s_\lambda\big(\mathbf{a}^C(T), \eta(T)\big) \geq s_\lambda\big(\mathbf{a}^C(T'), \eta(T)\big).$$

The above inequality is equivalent to

$$\frac{\eta_2(T)}{\eta_1(T)} \geq \frac{v_{1\lambda}\big(\mathbf{a}^C(T')\big) - v_{1\lambda}\big(\mathbf{a}^C(T)\big)}{v_{2\lambda}\big(\mathbf{a}^C(T)\big) - v_{2\lambda}\big(\mathbf{a}^C(T')\big)}.$$

Also, by construction, $\frac{\eta_2(T+1)}{\eta_1(T+1)} > \frac{\eta_2(T)}{\eta_1(T)}$. Hence,

$$\frac{\eta_2(T+1)}{\eta_1(T+1)} > \frac{v_{1\lambda}\big(\mathbf{a}^C(T')\big) - v_{1\lambda}\big(\mathbf{a}^C(T)\big)}{v_{2\lambda}\big(\mathbf{a}^C(T)\big) - v_{2\lambda}\big(\mathbf{a}^C(T')\big)}. \tag{S-10}$$

Combining (S-9) and (S-10) yield $s_\lambda(\mathbf{a}^C(T+1), \eta(T+1)) \geq s_\lambda(\mathbf{a}^C(T'), \eta(T+1))$, as desired. Now, we show that (S-8) holds for every $\mathbf{a} \in \mathcal{E}_2\mathcal{C}_2 \setminus \{\mathbf{a}^C(T') : T' \geq 0\}$. For such a path $\mathbf{a}$, there are periods $T^* < T^{**}$ such that CD is played in all periods $t < T^*$, CD is played in period T^{**} as well, and CC is played in all other periods. Letting $\varrho := 1 - [\beta_\lambda(c)]^{T^{**}-T^*}$, observe that

$$v_\lambda(\mathbf{a}) = \varrho v_\lambda\big(\mathbf{a}^C\big(T^*\big)\big) + (1-\varrho) v_\lambda\big(\mathbf{a}^C\big(T^*+1\big)\big).$$

Conclude that (S-8) holds for all paths $\mathbf{a} \in \mathcal{E}_2\mathcal{C}_2$, and hence, that every path $\mathbf{a}^C(T')$, $T' \geq 0$, is efficient. It remains to show that every path $\mathbf{a} \in \mathcal{E}_2\mathcal{C}_2 \setminus \{\mathbf{a}^C(T') : T' \geq 0\}$ is efficient. But, as we just showed, $v(\mathbf{a})$ is a convex combination of $v(\mathbf{a}^C(T))$ and $v(\mathbf{a}^C(T+1))$ for some T. Since $\mathbf{a}^C(T), \mathbf{a}^C(T+1) \in P_\lambda(\eta(T+1))$, we see that $\mathbf{a} \in P_\lambda(\eta(T+1))$. *Q.E.D.*

The next three lemmas provide the characterization of the set of efficient paths when λ is irregular. Recall from Lemma S-5 that $f(\lambda) = \frac{v_c - v_{1\lambda}(\mathbf{a}^{\mathcal{A},2})}{v_{2\lambda}(\mathbf{a}^{\mathcal{A},2}) - v_c}$.

LEMMA S-13: *If $f(\lambda) = 1$, then $P_\lambda = \mathcal{E}\mathcal{C}\mathcal{A}^{\mathrm{I}}$.*

PROOF: We first prove that $P_\lambda^{\mathrm{sym}} = \mathcal{C}\mathcal{A}^{\mathrm{I}}$. Note that $f(\lambda) = 1$ implies $s_\lambda(\mathbf{a}^C) = s_\lambda(\mathbf{a}^{\mathcal{A},2})$, which means $\mathbf{a}^{\mathcal{A},1}$, $\mathbf{a}^{\mathcal{A},2}$, and $\mathbf{a}^C$, simultaneously maximize the sum of the players' utilities. By Lemma E22, any path $\mathbf{a} \in \mathcal{C}\mathcal{A}^{\mathrm{I}}$ maximizes the sum of the players' utilities and $\mathbf{a} \in P_\lambda^{\mathrm{sym}}$. Thus, $\mathcal{C}\mathcal{A}^{\mathrm{I}} \subset P_\lambda^{\mathrm{sym}}$. Next, we prove $P_\lambda^{\mathrm{sym}} \subset \mathcal{C}\mathcal{A}^{\mathrm{I}}$ by contradiction. Take $\mathbf{a} \in P_\lambda^{\mathrm{sym}}$ and $\mathbf{a} \notin \mathcal{C}\mathcal{A}^{\mathrm{I}}$. By Lemma E24, DD cannot be played along any efficient path. Hence, $a^t \in \{CC, DC, CD\}$ for all t. Since $\mathbf{a} \notin \mathcal{C}\mathcal{A}^{\mathrm{I}}$, either there exists T such that $a^T = CD$ and $a^{T+1} \neq DC$ or $a^T = DC$ and $a^{T+1} \neq CD$. Let T be the first period t such that $a^t \in \{CD, DC\}$ and $a^{t+1} \neq \{CD, DC\} \setminus \{a^t\}$. W.l.o.g., assume $a^T = CD$. We know that $\eta_{1\lambda}^T(\mathbf{a}) = \eta_{2\lambda}^T(\mathbf{a})$. Otherwise, there would be a $t < T$ such that this is the case. Since $a^T = CD$ and $\beta_\lambda(b) > \beta_\lambda(d)$,

$$\frac{\eta_{1\lambda}^{T+1}(\mathbf{a})}{\eta_{2\lambda}^{T+1}(\mathbf{a})} = \frac{\beta_\lambda(b)}{\beta_\lambda(d)} > 1.$$

Since ${}_{T+1}\mathbf{a} \in P_\lambda(\eta_\lambda^{T+1}(\mathbf{a}))$, we can apply Lemma S-3 to deduce that $a^{T+1} \in \{CC, DC\}$. If $a^{T+1} = CC$, it follows from Lemma E23 that $\mathbf{a}^C \in P_\lambda(\eta_\lambda^{T+1}(\mathbf{a}))$. But then, by Lemma E22, $(CD, \mathbf{a}^C) \in P_\lambda^{\mathrm{sym}}$, contradicting Lemma S-4. Thus, $a^{T+1} = DC$, a contradiction.

We next show $P_\lambda^{++} \subset \mathcal{E}\mathcal{C}\mathcal{A}^{\mathrm{I}}$. Take any $\mathbf{a} \in P_\lambda(\eta)$ where $0 < \eta_1 < \eta_2$. We are going to prove $\mathbf{a} \in \mathcal{E}_2\mathcal{C}\mathcal{A}^{\mathrm{I}}$. Since $\eta_1 < \eta_2$, Lemma S-3 shows that $v_{1\lambda}(\mathbf{a}) \leq v_{2\lambda}(\mathbf{a})$ and $a^0 \neq DC$. By Lemma E24, DD cannot be played along any efficient path. Hence, $a^0 \in \{CC, CD\}$. If $a^0 = CC$, then Lemma E23 implies that $\mathbf{a}^C \in P_\lambda(\eta)$. However, since $v(\mathbf{a}^C)$ is not an extreme point of the Pareto frontier, $\mathbf{a}^C$ cannot be efficient both for η^{sym} and η where $\eta_1 < \eta_2$. Thus, $a^0 = CD$. By Lemma S-1, the constant path $(CD, CD, \ldots)$ is not efficient. Let $T > 0$ be the first period t such that $a^t \neq CD$. If $a^T = CC$, then Lemma E23 implies that $\mathbf{a}^C \in P_\lambda(\eta^T)$. Since $v(\mathbf{a}^C)$ is not an extreme point of the frontier and $\mathbf{a}^C \in P_\lambda^{\mathrm{sym}}$, then $\eta_{1\lambda}^T = \eta_{2\lambda}^T$. By Lemma E22, ${}_T\mathbf{a} \in P_\lambda^{\mathrm{sym}}$. We have shown that $P_\lambda^{\mathrm{sym}} = \mathcal{C}\mathcal{A}^{\mathrm{I}}$. Thus, $\mathbf{a} \in \mathcal{E}_2\mathcal{C}\mathcal{A}^{\mathrm{I}}$. If $a^T = DC$, then $a^{T-1} = CD$ implies that η_λ^{T+1} determines the same direction as η_λ^{T-1}. If $\eta_{1\lambda}^{T-1} = \eta_{2\lambda}^{T-1}$, by Lemma E22, ${}_{T-1}\mathbf{a} \in P_\lambda^{\mathrm{sym}} = \mathcal{C}\mathcal{A}^{\mathrm{I}}$. Thus, $\mathbf{a} \in \mathcal{E}_2\mathcal{C}\mathcal{A}^{\mathrm{I}}$. If $\eta_{1\lambda}^{T-1} < \eta_{2\lambda}^{T-1}$, then $\eta_{1\lambda}^{T+1} < \eta_{2\lambda}^{T+1}$, which implies $a^{T+1} = CD$. Since $a^T = DC$, η_λ^{T+2} determines the same direction as η_λ^T. By Lemma S-3, $\eta_{1\lambda}^T \geq \eta_{2\lambda}^T$, and hence, $\eta_{1\lambda}^{T+2} \geq \eta_{2\lambda}^{T+2}$. If $\eta_{1\lambda}^{T+2} = \eta_{2\lambda}^{T+2}$, Lemma E22 implies ${}_T\mathbf{a} \in \mathcal{C}\mathcal{A}^{\mathrm{I}}$, and hence, $\mathbf{a} \in \mathcal{E}_2\mathcal{C}\mathcal{A}^{\mathrm{I}}$, as desired. If $\eta_{1\lambda}^{T+2} > \eta_{2\lambda}^{T+2}$, then $a^{T+2} = DC$. As a result, ${}_{T-1}\mathbf{a} = \mathbf{a}^{\mathcal{A},2}$ and $\mathbf{a} \in \mathcal{E}_2\mathcal{C}\mathcal{A}^{\mathrm{I}}$.

The proof for $\mathcal{ECA}^{\mathrm{I}} \subset P_\lambda^{++}$ follows from analogous arguments of the proof of Lemma S-12. *Q.E.D.*

LEMMA S-14: *If* $1 < f(\lambda) < \sqrt{\frac{\beta_\lambda(b)}{\beta_\lambda(d)}}$, *then* $P_\lambda = \mathcal{ECA}^{\mathrm{II}}$.

PROOF: We are going to prove $P_\lambda^{++} \subset \mathcal{ECA}^{\mathrm{II}}$. Take any $\mathbf{a} \in P_\lambda(\eta)$ where $0 < \eta_1 < \eta_2$. We will show $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{II}}$. Since $\eta_1 < \eta_2$, Lemma S-3 shows that $v_{1\lambda}(\mathbf{a}) \le v_{2\lambda}(\mathbf{a})$ and $a^0 \neq DC$. By Lemma E24, DD cannot be played along any efficient path. Hence, $a^0 \in \{CC, CD\}$. Suppose $a^0 = CC$. By using the assumption that $1 < f(\lambda) < \sqrt{\frac{\beta_\lambda(b)}{\beta_\lambda(d)}}$ and following analogous arguments as in the proof of Lemma S-7, we can show that ${}_1\mathbf{a} \in \mathcal{CA}_2^{\mathrm{II}}$. As a result, $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{II}}$. Alternatively, suppose $a^0 = CD$. By Lemma S-1, the constant path $(CD, CD, \ldots)$ is not efficient. Let T be the first period t such that $a^t \neq CD$. Suppose $a^T = CC$. If $\eta_{1\lambda}^T < \eta_{2\lambda}^T$, then it is the same as the previous case and we have ${}_{T+1}\mathbf{a} \in \mathcal{CA}_2^{\mathrm{II}}$ and $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{II}}$. Suppose $\eta_{1\lambda}^T = \eta_{2\lambda}^T$. Note that $f(\lambda) > 1$ implies $s_\lambda(\mathbf{a}^C) > s_\lambda(\mathbf{a}^{\mathcal{A},2})$, and hence, $P_\lambda^{\mathrm{sym}} = \{\mathbf{a}^C\}$. This implies that ${}_T\mathbf{a} = \mathbf{a}^C$ and $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{II}}$. If $\eta_{1\lambda}^T > \eta_{2\lambda}^T$, following analogous arguments as in the proof of Lemma S-7, we can obtain that ${}_{T+1}\mathbf{a} \in \mathcal{CA}_1^{\mathrm{II}}$. As a result, $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{II}}$. Finally, suppose $a^T = DC$. Analogous arguments as in the proof of Lemma S-10 show that ${}_T\mathbf{a} \in \mathcal{CA}_1^{\mathrm{II}}$, and hence, $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{II}}$.

The proof for $\mathcal{ECA}^{\mathrm{II}} \subset P_\lambda^{++}$ follows from analogous arguments of the proof of Lemma S-12. *Q.E.D.*

The last case is when $f(\lambda) = \sqrt{\frac{\beta_\lambda(b)}{\beta_\lambda(d)}}$. This case is similar to Case II except that $v_\lambda(\mathbf{a}^{\mathcal{A},2})$ is not an extreme point of the Pareto frontier. In particular, the proof of the next lemma is similar to that of Lemma S-14 and omitted.

LEMMA S-15: *If* $f(\lambda) = \sqrt{\frac{\beta_\lambda(b)}{\beta_\lambda(d)}}$, *then* $P_\lambda = \mathcal{ECA}^{\mathrm{III}}$.

The possible transitions of P_λ as $\lambda \nearrow 1$ follow from the preceding lemmas.

S-B.2. *Proof of Proposition 4*

Take $\lambda > \underline{\lambda}$ and $\mathbf{a} \in P_\lambda$ such that $v_\lambda(\mathbf{a}) \gg (1-\lambda)g_1(CD)$. We need to prove that $v_\lambda({}_t\mathbf{a}) \gg (1-\lambda)g_1(DC)$ for all t. The case when $P_\lambda = \mathcal{EC}$ was handled in the main text. If $P_\lambda = \mathcal{EA}$, then it is automatic that $v_{i\lambda}({}_t\mathbf{a}) \ge \min\{v_{1\lambda}(\mathbf{a}), v_{2\lambda}(\mathbf{a})\}$ for all i and t, and we are done.

Suppose $P_\lambda = \mathcal{ECA}^{\mathrm{III}}$. The other "mixed" cases follow from analogous arguments. W.l.o.g., assume $\mathbf{a} \in \mathcal{E}_2\mathcal{CA}^{\mathrm{III}}$. By definition of the set $\mathcal{E}_2\mathcal{CA}^{\mathrm{III}}$, there is $\hat{T} \ge 0$ such that ${}_{\hat{T}}\mathbf{a} \in \mathcal{CA}^{\mathrm{III}}$ and $v_{1\lambda}({}_t\mathbf{a}) \ge v_{1\lambda}(\mathbf{a})$ for all $t \le \hat{T}$. Thus, it suffices to prove that $v_{1\lambda}({}_t\mathbf{a}) \ge v_{1\lambda}(CD, CC, \ldots)$ for all $t \ge \hat{T}$. By the definition of $\mathcal{CA}^{\mathrm{III}}$, we have ${}_t\mathbf{a} \in \mathcal{CA}^{\mathrm{III}}$ for all $t \ge \hat{T}$. Also, for any $\hat{\mathbf{a}} \in \mathcal{CA}^{\mathrm{III}}$ and t, player 1's payoff from ${}_t\hat{\mathbf{a}}$ is lowest when $\hat{a}^t = CD$. It is thus w.l.o.g. to assume that $a^{\hat{T}} = CD$ and we only need to show that $v_{1\lambda}({}_{\hat{T}}\mathbf{a}) \ge v_{1\lambda}(CD, CC, CC, \ldots)$. If ${}_{\hat{T}}\mathbf{a} = (CD, CC, CC, \ldots)$, we are done. If not, let T be the smallest integer $k > \hat{T}$ such that $a^k = DC$. If ${}_{T+1}\mathbf{a} = (CC, CC, \ldots)$, then $v_{1\lambda}({}_{\hat{T}}\mathbf{a}) > v_{1\lambda}(CD, CC, CC, \ldots)$ since $v_{1\lambda}({}_T\mathbf{a}) = v_{1\lambda}(DC, CC, \ldots) > v_1(CC)$. If ${}_{T+1}\mathbf{a} \neq (CC, CC, \ldots)$, then by the definition of $\mathcal{CA}^{\mathrm{III}}$, there exists $T' > T$ such that $a^{T'} = CD$ and $a^t = CC$ for all $T < t < T'$. We are going to show that $v_{1\lambda}({}_T\mathbf{a}) > v_1(CC)$. By assumption, $a^T = DC$. Let η

be such that $\mathbf{a} \in P_\lambda(\eta)$. Then Lemma S-3 implies that $\eta^T_{1\lambda}(\mathbf{a}) > \eta^T_{2\lambda}(\mathbf{a})$. Since $a^{T'} = CD$, we have

$$\frac{\eta^{T'+1}_{1\lambda}(\mathbf{a})}{\eta^{T'+1}_{2\lambda}(\mathbf{a})} = \frac{\eta^{T}_{1\lambda}(\mathbf{a})\beta_\lambda(b)\beta_\lambda(d)}{\eta^{T}_{2\lambda}(\mathbf{a})\beta_\lambda(b)\beta_\lambda(d)} = \frac{\eta^{T}_{1\lambda}(\mathbf{a})}{\eta^{T}_{2\lambda}(\mathbf{a})} > 1.$$

Then Lemma S-3 implies that $v_{1\lambda}({}_{T'+1}\mathbf{a}) \geq v_1(CC)$. Also, since $(DC, CD, DC, CD, \dots)$ is efficient, we have $v_{1\lambda}(DC, CD, DC, CD, \dots) > v_1(CC)$, and hence,

$$v_{1\lambda}(DC, CD, CC, CC, \dots) > v_1(CC).$$

Then, by the assumption that $a^T = DC$, $a^{T'} = CD$, and $a^t = CC$ for all $T < t < T'$, we have

$$v_{1\lambda}({}_T\mathbf{a}) = v_{1\lambda}(DC, CC, \dots, CC, CD, {}_{T'+1}\mathbf{a}) \geq v_{1\lambda}(DC, CD, CC, CC, \dots) > v_1(CC).$$

This further implies $v_{1\lambda}({}_{\hat{T}}\mathbf{a}) > v_1(CD, CC, \dots)$, as desired.

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