Econometrica Supplementary Material

SUPPLEMENT TO "NETWORKS, BARRIERS, AND TRADE"

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CONTENTS

APPENDIX A: DATA APPENDIX

TO CONDUCT the counterfactual exercises in Section 7, we use the World Input–Output Database (Timmer, Dietzenbacher, Los, Stehrer, and De Vries (2015)). We use the 2013 release of the data for the final year, which has no-missing data, that is, 2008. We use the 2013 release because it has more detailed information on the factor usage by industry. We aggregate the 35 industries in the database to get 30 industries to eliminate missing values, and zero domestic production shares, from the data. In Table A.I, we list our aggregation scheme, as well as the elasticity of substitution, based on Caliendo and Parro (2015) and taken from Costinot and Rodriguez-Clare (2014) associated with each industry. We calibrate the model to match the input–output tables and the socioeconomic accounts tables in terms of expenditure shares in steady state (before the shock).

For the growth accounting exercise in Section 7.1, we use both the 2013 and the 2016 release of the WIOD data. When we combine this data, we are able to cover a larger number of years. We compute our growth accounting decompositions for each release of the data separately, and then paste the resulting decompositions together starting with the year of overlap. To construct the consumer price index and the GDP deflator for each country, we use the final consumption weights or GDP weights of each country in each year to sum up the log price changes of each good. To arrive at the price of each good, we use the gross output prices from the socioeconomic accounts tables, which are reported at the (country of origin, industry) level into US dollars using the contemporaneous exchange rate, and then take log differences. This means that we assume that the log-change in the price of each good at the (origin, destination, industry of supply, industry of use) level is the same as (origin, industry of supply) level. If there are differential (changing) transportation costs over time, then this assumption is violated.

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TABLE A.I

THE SECTORS IN THE 2013 RELEASE OF THE WIOD DATA, AND THE AGGREGATED SECTORS IN OUR DATA.

	WIOD Sector	Aggregated Sector	Trade Elasticity
1	Agriculture, Hunting, Forestry, and Fishing	1	8.11
2	Mining and Quarrying	2	15.72
3	Food, Beverages, and Tobacco	3	2.55
4	Textiles and Textile Products	4	5.56
5	Leather, Leather and Footwear	4	5.56
6	Wood and Products of Wood and Cork	5	10.83
7	Pulp, Paper, Paper , Printing, and Publishing	6	9.07
8	Coke, Refined Petroleum, and Nuclear Fuel	7	51.08
9	Chemicals and Chemical Products	8	4.75
10	Rubber and Plastics	8	4.75
11	Other Nonmetallic Mineral	9	2.76
12	Basic Metals and Fabricated Metal	10	7.99
13	Machinery, Enc	11	1.52
14	Electrical and Optical Equipment	12	10.6
15	Transport Equipment	13	0.37
16	Manufacturing, Enc; Recycling	14	5
17	Electricity, Gas, and Water Supply	15	5
18	Construction	16	5
19	Sale, Maintenance and Repair of Motor Vehicles...	17	5
20	Wholesale Trade and Commission Trade, ...	17	5
21	Retail Trade, Except of Motor Vehicles and...	18	5
22	Hotels and Restaurants	19	5
23	Inland Transport	20	5
24	Water Transport	21	5
25	Air Transport	22	5
26	Other Supporting and Auxiliary Transport...	23	5
27	Post and Telecommunications	24	5
28	Financial Intermediation	25	5
29	Real Estate Activities	26	5
30	Renting of M&Req and Other Business Activities	27	5
31	Public Admin/Defence; Compulsory Social Security	28	5
32	Education	29	5
33	Health and Social Work	30	5
34	Other Community, Social and Personal Services	30	5
35	Private Households with Employed Persons	30	5

To arrive at the contemporaneous exchange rate, we use the measures of nominal GDP in the socioeconomic accounts for each year (reported in local currency) to nominal GDP in the world input–output database (reported in US dollars).

APPENDIX B: PROOFS

Throughout the proofs, let χ_c be the share of total world income accruing to country c.

PROOF OF THEOREM 1: Nominal GDP is equal to

$$P_{Y_c} Y_c = \sum_{i \in N_c} (1 - 1/\mu_i) p_i y_i + \sum_{f \in F_c} w_f L_f .$$

Hence,

$$\begin{aligned}
\mathrm{d}\log P_{Y_c} + \mathrm{d}\log Y_c &= \sum_{i\in N_c}(1-1/\mu_i)\lambda_i^{Y_c}\,\mathrm{d}\log\big((1-1/\mu_i)\lambda_i^{Y_c}\big) \\
&\quad + \sum_{f\in F_c}\Lambda_f^{Y_c}(\mathrm{d}\log w_f + \mathrm{d}\log L_f), \\
\mathrm{d}\log Y_c &= \sum_{i\in N_c}(1-1/\mu_i)\lambda_i^{Y_c}\,\mathrm{d}\log\big((1-1/\mu_i)\lambda_i^{Y_c}\big) \\
&\quad + \sum_{f\in F_c}\Lambda_f^{Y_c}(\mathrm{d}\log w_f + \mathrm{d}\log L_f) - \mathrm{d}\log P^{Y_c}.
\end{aligned}$$

The price of domestic goods is given by

$$\mathrm{d}\log p_i = \mathrm{d}\log \mu_i - \mathrm{d}\log A_i + \sum_{j\in N_c}\tilde{\Omega}_{ij}\,\mathrm{d}\log p_j + \sum_{j\notin N_c}\tilde{\Omega}_{ij}\,\mathrm{d}\log p_j,$$

which implies that

$$\mathrm{d}\log p = \big(I-\tilde{\Omega}^c\big)^{-1}\big(\mathrm{d}\log\mu_i - \mathrm{d}\log A_i + \tilde{\Omega}^F(\mathrm{d}\log\Lambda - \mathrm{d}\log L) + \tilde{\Omega}^M\,\mathrm{d}\log p^M\big),$$

where $\tilde{\Omega}^c$ is the cost-based domestic IO table, $\tilde{\Omega}^F$ are cost-based factor shares, and $\tilde{\Omega}^M$ are cost-based intermediate import shares, and $\mathrm{d}\log p^M$ represents the change in the price of imported intermediate goods. Use the fact that

$$\begin{aligned}
\mathrm{d}\log P_{Y_c} &= \sum_{i\in N_c}\Omega_{Y_c,i}\,\mathrm{d}\log p_i - \sum_{i\in N-N_c}\Lambda_i^{Y_c}\,\mathrm{d}\log p_i \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}(\mathrm{d}\log\mu_i - \mathrm{d}\log A_i) + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}(\mathrm{d}\log\Lambda_f - \mathrm{d}\log L_f) \\
&\quad + \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c}\,\mathrm{d}\log p_i - \sum_{i\in N-N_c}\Lambda_i^{Y_c}\,\mathrm{d}\log p_i.
\end{aligned}$$

For an imported intermediate,

$$\mathrm{d}\log p_i = \mathrm{d}\log\Lambda_i^{Y_c} - \mathrm{d}\log q_i + \mathrm{d}\log \mathrm{GDP}.$$

Substitute this back to get

$$\begin{aligned}
\mathrm{d}\log Y_c = {} & \sum_{i\in N_c}(1-1/\mu_i)\lambda_i^{Y_c}\,\mathrm{d}\log\big((1-1/\mu_i)\lambda_i^{Y_c}\big) + \sum_{f\in F_c}\Lambda_f^{Y_c}(\mathrm{d}\log w_f + \mathrm{d}\log L_f) \\
& - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}(\mathrm{d}\log\mu_i - \mathrm{d}\log A_i) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}(\mathrm{d}\log\Lambda_f - \mathrm{d}\log L_f) \\
& - \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c}\,\mathrm{d}\log p_i + \sum_{i\in N-N_c}\Lambda_i^{Y_c}\,\mathrm{d}\log p_i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{f\in F_c^*} \Lambda_f^{Y_c} \, \mathrm{d}\log\Lambda_f - \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} (\mathrm{d}\log\mu_i - \mathrm{d}\log A_i) \\
&\quad - \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} (\mathrm{d}\log\Lambda_f - \mathrm{d}\log L_f) \\
&\quad - \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big)\big(\mathrm{d}\log\Lambda_i^{Y_c} - \mathrm{d}\log q_i + \mathrm{d}\log \mathrm{GDP}\big) \\
&= \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} \, \mathrm{d}\log A_i + \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \, \mathrm{d}\log L_f + \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big) \, \mathrm{d}\log q_i \\
&\quad + \sum_{f\in F_c^*} \Lambda_f^{Y_c} \big(\mathrm{d}\log\Lambda_f^{Y_c} + \mathrm{d}\log \mathrm{GDP}_c\big) - \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} \, \mathrm{d}\log\mu_i \\
&\quad - \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \big(\mathrm{d}\log\Lambda_f^{Y_c} + \mathrm{d}\log \mathrm{GDP}_c\big) \\
&\quad - \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big)\big(\mathrm{d}\log\Lambda_i^{Y_c} + \mathrm{d}\log \mathrm{GDP}\big) \\
&= \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} \, \mathrm{d}\log A_i + \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \, \mathrm{d}\log L_f + \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big) \, \mathrm{d}\log q_i \\
&\quad - \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} \, \mathrm{d}\log\mu_i - \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \, \mathrm{d}\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big)\big(\mathrm{d}\log\Lambda_i^{Y_c}\big) \\
&\quad + \left[1 - \left(\sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c}\right) - \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big)\right] \mathrm{d}\log \mathrm{GDP}_c \\
&= \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} \, \mathrm{d}\log A_i + \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \, \mathrm{d}\log L_f + \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big) \, \mathrm{d}\log q_i \\
&\quad - \sum_{i\in N_c} \tilde{\lambda}_i^{Y_c} \, \mathrm{d}\log\mu_i - \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \, \mathrm{d}\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c} \big(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\big)\big(\mathrm{d}\log\Lambda_i^{Y_c}\big).
\end{aligned}
$$

The last line follows from the fact that

$$
\sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} + \sum_{i\in N-N_c} \tilde{\Lambda}_i^{Y_c} = \left[1 + \sum_{i\in N-N_c} \Lambda_i^{Y_c}\right].
$$

Q.E.D.

PROOF OF THEOREM 2: Note that welfare is given by

$$
W_c = \frac{\sum_{f\in F^*} \Phi_{cf} w_f L_f + T_c}{P^{W_c}}.
$$

Hence, letting world GDP be the numeraire,

$$
\mathrm{d}\log W_c = \sum_f \Lambda_f^c (\mathrm{d}\log\Lambda_f) + \frac{dT}{\mathrm{GNE}_c} - (\tilde{\Omega}_{(W_c)})' \, \mathrm{d}\log p.
$$

Use the fact that

$$\mathrm{d}\log p_i = \sum_{j\in N} \tilde{\Psi}_{ij}\,\mathrm{d}\log A_j + \sum_{f\in F} \tilde{\Psi}_{if}(\mathrm{d}\log \Lambda_f - \mathrm{d}\log L_f)$$

to complete the proof. *Q.E.D.*

PROOF OF THEOREM 3: For each good,

$$\lambda_i = \sum_c \Omega_{W_c,i}\chi_c + \sum_i \Omega_{ji}\lambda_j,$$

where χ_c is the share of total income accruing to country c and $\Omega_{W_c,i}$ is the share of income household c spends on good i. This means

$$\lambda_i\,\mathrm{d}\log\lambda_i = \sum_c \chi_c\Omega_{W_c,i}\,\mathrm{d}\log\Omega_{W_c,i} + \sum_j \Omega_{ji}\lambda_j\,\mathrm{d}\log\Omega_{ji} + \sum_j \Omega_{ji}\,\mathrm{d}\lambda_j + \sum_c \Omega_{W_c,i}\chi_c\,\mathrm{d}\log\chi_c.$$

Now, note that

$$\begin{aligned}
\mathrm{d}\log\Omega_{W_c,i} &= (1-\theta_c)(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}),\\
\mathrm{d}\log\Omega_{ji} &= (1-\theta_j)(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log\mu_j) - \mathrm{d}\log\mu_j,\\
\mathrm{d}\log\chi_c &= \sum_{f\in F_c^*} \frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f + \sum_{i\in c}\frac{\lambda_i}{\mu_i}\,\mathrm{d}\log\mu_i,\\
\mathrm{d}\log p_i &= \tilde{\Psi}(\mathrm{d}\log\mu - \mathrm{d}\log A) + \tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log\Lambda,\\
\mathrm{d}\log P_{y_c} &= b'\tilde{\Psi}(\mathrm{d}\log\mu - \mathrm{d}\log A) + b'\tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log\Lambda.
\end{aligned}$$

For shock $\mathrm{d}\log\mu_k$, we have

$$\begin{aligned}
\mathrm{d}\log\Omega_{W_c,i} &= (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right),\\
\mathrm{d}\log\Omega_{ji} &= (1-\theta_j)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) - \theta_j\,\mathrm{d}\log\mu_j.
\end{aligned}$$

Putting this altogether gives

$$\begin{aligned}
\mathrm{d}\lambda_l = \sum_i\sum_c &(1-\theta_c)\chi_c\Omega_{W_c,i}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right.\\
&\left. - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\Psi_{il}\\
&+ \sum_i\sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il}\\
&- \theta_k\lambda_k\sum_i \Omega_{ki}\Psi_{il} + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\,\lambda_l = {} & \sum_c (1-\theta_c)\chi_c \Bigg[\sum_i \Omega_{W_c,i}\Big(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\Big)\Psi_{il} \\
& - \Big(\sum_i \Omega_{W_c,i}\Big(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\Big)\Big)\Big(\sum_i \Omega_{W_c,i}\Psi_{il}\Big)\Bigg] \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \tilde{\Omega}_{ji}\Big(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\Big)\Psi_{il} \\
& - \Big(\sum_i \tilde{\Omega}_{ji}\Psi_{il}\Big)\Big(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\Big) \\
& - \theta_k\lambda_k\big(\Psi_{kl} - \mathbf{1}(l=k)\big) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this further to get

$$\begin{aligned}
\mathrm{d}\,\lambda_l = {} & \sum_c (1-\theta_c)\chi_c\,\mathrm{Cov}_{b^{(c)}}\Big(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\Big) \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \tilde{\Omega}_{ji}\Big(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\Big)\Psi_{il} \\
& - \Big(\sum_i \tilde{\Omega}_{ji}\Psi_{il}\Big)\Big(\sum_i \tilde{\Omega}_{ji}\tilde{\Psi}_{ik} + \sum_i \tilde{\Omega}_{ji}\sum_f \Psi_{if}\,\mathrm{d}\log\Lambda_f\Big) \\
& - \theta_k\lambda_k\big(\Psi_{kl} - \mathbf{1}(l=k)\big) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Using the input–output covariance notation, write

$$\begin{aligned}
\mathrm{d}\,\lambda_l = {} & \sum_c (1-\theta_c)\chi_c\,\mathrm{Cov}_{\Omega_{(W_c)}}\Big(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\Big) \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\,\mathrm{Cov}_{\tilde{\Omega}^{(j)}}\Big(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\Big) \\
& - (1-\theta_k)\lambda_k\big(\Psi_{kl} - \mathbf{1}(l=k)\big) - \theta_k\lambda_k\big(\Psi_{kl} - \mathbf{1}(l=k)\big) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

This then simplifies to give from the fact that $\sum_i \Omega_{W_c,i}\Psi_{il} = \lambda_l^{W_c}$:

$$\begin{aligned}
\lambda_l\,\mathrm{d}\log\lambda_l = {} & \sum_{j\in N,C} (1-\theta_j)\lambda_j\mu_j^{-1}\,\mathrm{Cov}\Bigg(\tilde{\Psi}_{(k)} + \sum_f^F \mathrm{d}\log\Lambda_f, \Psi_{(l)}\Bigg) \\
& - \lambda_k\big(\Psi_{kl} - \mathbf{1}(k=l)\big) + \sum_c \chi_c\lambda_l^{W_c}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

To complete the proof, note that

$$P_{y_c} Y_c = \sum_f w_f L_f + \sum_{i \in N_c} \left(1 - \frac{1}{\mu_i}\right) p_i y_i.$$

Hence,

$$\mathrm{d}(P_{y_c} Y_c) = \sum_{f \in F_c} w_f L_f \,\mathrm{d}\log w_f + \sum_{i \in N_c} \left(1 - \frac{1}{\mu_i}\right) p_i y_i \,\mathrm{d}\log(p_i y_i) + \sum_{i \in N_c} \frac{\mathrm{d}\left(1 - \frac{1}{\mu_i}\right)}{\mathrm{d}\log \mu_i} p_i y_i \,\mathrm{d}\log \mu_i.$$

In other words, since $P_y Y = 1$, we have

$$\mathrm{d}\,\chi_c = \sum_{f \in F_c} \Lambda_f \,\mathrm{d}\log w_f + \sum_{i \in N_c} \left(1 - \frac{1}{\mu_i}\right) \lambda_i \,\mathrm{d}\log \lambda_i + \sum_{i \in N_c} \frac{\mathrm{d}\left(1 - \frac{1}{\mu_i}\right)}{\mathrm{d}\log \mu_i} \lambda_i \,\mathrm{d}\log \mu_i.$$

Hence,

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \,\mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\chi_c} \,\mathrm{d}\log \mu_i.$$

Q.E.D.

PROOF OF THEOREM 5: Proof of Part(1):

The expression for $\mathrm{d}^2 \log Y$ follows from applying part (2) to the whole world. The equality of real GNE and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then we can write

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} = \sum_{f \in F_c} \Lambda_f^{Y_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_j \frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_i} \frac{\left(1 - \frac{1}{\mu_j}\right)}{P_{Y_c} Y_c} + \frac{\lambda_i^{Y_c}}{\mu_i} - \frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log \mu_i},$$

where

$$\frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log \mu_i} = \sum_{f \in F_c} \tilde{\Lambda}_f^{Y_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_{m \in M_c} \tilde{\lambda}_m^{Y_c} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i} - \tilde{\lambda}_i^{Y_c} - \sum_{m \in M_c} \Lambda_m^{Y_c} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i},$$

and

$$\tilde{\lambda}_i^{Y_c} = \sum_j \Omega_{Y_c, j} \tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned} \frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} = & \sum_{f \in F_c} \left(\Lambda_f^{Y_c} - \tilde{\Lambda}_f^{Y_c}\right) \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_{m \in M_c} \left(\lambda_m^{Y_c} - \tilde{\lambda}_m^{Y_c}\right) \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i} \\ & + \sum_{j \in N_c} \lambda_j^{Y_c} \frac{\mathrm{d}\log \lambda_j}{\mathrm{d}\log \mu_i} \left(1 - \frac{1}{\mu_j}\right) + \frac{\lambda_i^{Y_c}}{\mu_i} - \tilde{\lambda}_i^{Y_c}. \end{aligned}$$

At the efficient point,

$$\frac{\mathrm{d}^2 \log Y_c}{\mathrm{d}\log \mu_i \,\mathrm{d}\log \mu_k} = \sum_{f\in F_c}\left(\frac{\mathrm{d}\,\Lambda_f^{Y_c}}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\,\tilde{\Lambda}_f^{Y_c}}{\mathrm{d}\log \mu_i}\right)\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}$$
$$+ \sum_{m\in M_c}\left(\frac{\mathrm{d}\,\lambda_m^{Y_c}}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\,\tilde{\lambda}_m^{Y_c}}{\mathrm{d}\log \mu_i}\right)\frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_k^{Y_c}}{\mathrm{d}\log \mu_i}$$
$$+ \lambda_k^{Y_c}\left(\frac{\mathrm{d}\log \lambda_k^{Y_c}}{\mathrm{d}\log \mu_i} - \delta_{ki}\right) + \frac{1}{P_{Y_c} Y_c}\frac{\mathrm{d}\,\lambda_i^{Y_c}}{\mathrm{d}\log \mu_k},$$

where δ_{ki} is the a Kronecker delta.

Using Lemma 8,

$$\frac{\mathrm{d}^2 \log Y_c}{\mathrm{d}\log \mu_i \,\mathrm{d}\log \mu_k} = -\sum_{f\in F_c}\lambda_i^{Y_c}\Psi_{if}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \sum_{m\in M_c}\lambda_i^{Y_c}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_k} - \lambda_i^{Y_c}(\Psi_{ik} - \delta_{ik})$$
$$- \lambda_k^{Y_c}\delta_{ik} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k}\frac{1}{P_{Y_c} Y_c},$$
$$= -\sum_{f\in F_c}\lambda_i^{Y_c}\Psi_{if}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \sum_{m\in M_c}\lambda_i^{Y_c}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_k} - \lambda_i^{Y_c}\Psi_{ik}$$
$$+ \lambda_i^{Y_c}\left(\frac{\mathrm{d}\log p_i}{\mathrm{d}\log \mu_k} + \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k}\right),$$
$$= \lambda_i^{Y_c}\frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k}.$$

Q.E.D.

LEMMA 7: *Let χ_h be the income share of country h at the initial equilibrium. Then*

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_k} - \sum_h \overline{\chi}_h \frac{\mathrm{d}\log \tilde{\lambda}_j^{W_h}}{\mathrm{d}\log \mu_k} = \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i}\lambda_j^{W_h} - \lambda_i(\Psi_{ij} - \delta_{ij}).$$

PROOF: Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its kth diagonal element. Then

$$\overline{\chi}'\frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log \mu_k} = \chi'\frac{\mathrm{d}\,\tilde{\Omega}_{(W)}}{\mathrm{d}\log \mu_k} + \chi'\frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log \mu_k}\mu\Omega + \chi'\tilde{\lambda}I_{\mu_k}\Omega + \chi'\tilde{\lambda}\mu\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k},$$

where $\tilde{\Omega}_{(W)}$ is a matrix whose cith element is household c's expenditure share $\tilde{\Omega}_{W_c,i}$ on good i.

On the other hand,

$$\lambda = \chi'\tilde{\Omega}_{(W)} + \lambda\Omega.$$

Form this, we have

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k} = \frac{\mathrm{d}\,\chi'}{\mathrm{d}\log \mu_k}\tilde{\Omega}_{(W)} + \chi'\frac{\mathrm{d}\,\tilde{\Omega}_{(W)}}{\mathrm{d}\log \mu_k} + \lambda\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log \mu_k} + \frac{\mathrm{d}\,\lambda}{\mathrm{d}\log \mu_k}\Omega.$$

Combining these two expressions,

$$\left(\frac{d\lambda}{d\log\mu_k} - \overline{\chi}'\frac{d\log\tilde{\lambda}}{d\log\mu_k}\right) = \left(\frac{d\lambda}{d\log\mu_k} - \overline{\chi}'\frac{d\log\tilde{\lambda}}{d\log\mu_k}\right)\Omega + \frac{d\chi}{d\log\mu_k}\tilde{\Omega}_{(W)} - \chi'\tilde{\lambda}^{(h)}I_{\mu_k}\Omega.$$

Rearrange this to get

$$\left(\frac{d\lambda}{d\log\mu_k} - \overline{\chi}'\frac{d\log\tilde{\lambda}}{d\log\mu_k}\right) = \frac{d\chi}{d\log\mu_k}\tilde{\Omega}_{(W)}\Psi - \chi'\tilde{\lambda}^{(h)}I_{\mu_k}(\Psi - I),$$

or

$$\left(\frac{d\lambda}{d\log\mu_k} - \overline{\chi}'\frac{d\log\tilde{\lambda}}{d\log\mu_k}\right) = \frac{d\chi}{d\log\mu_k}\tilde{\Omega}_{(W)}\Psi - \lambda I_{\mu_k}(\Psi - I). \qquad \textit{Q.E.D.}$$

LEMMA 8: *At the efficient steady state*

$$\frac{d\lambda_j^{Y_c}}{d\log\mu_k} - \frac{d\tilde{\lambda}_j^{Y_c}}{d\log\mu_k} = -\lambda_k^{Y_c}(\Psi_{kj} - \delta_{kj}).$$

PROOF: Start from the relations

$$\lambda_j^{Y_c} = \chi_j^{Y_c} + \sum_i \lambda_i^{Y_c}\Omega_{ij},$$

and

$$\tilde{\lambda}_j^{Y_c} = \chi_j^{Y_c} + \sum_i \tilde{\lambda}_i^{Y_c}\mu_i\Omega_{ij}.$$

Differentiate both to get

$$\frac{d\lambda_j^{Y_c}}{d\log\mu_k} - \frac{d\tilde{\lambda}_j^{Y_c}}{d\log\mu_k} = \sum_i\left(\frac{d\lambda_j^{Y_c}}{d\log\mu_k} - \frac{d\tilde{\lambda}_j^{Y_c}}{d\log\mu_k}\right)\Omega_{ij} - \lambda_k^{Y_c}\Omega_{ki}.$$

Rearrange this to get the desired result. *Q.E.D.*

PROOF OF COROLLARY 4: Let $\overline{\chi}_h^W$ be the elasticity of social welfare with respect to the consumption of country h (i.e., log Pareto weight). Then

$$\frac{d\log W^{\mathrm{BS}}}{d\log\mu_k} = \sum_{h\in H}\overline{\chi}_h^W\frac{d\log W_h}{d\log\mu_k} = \sum_h\overline{\chi}_h^W\left(\frac{d\log\chi_h^W}{d\log\mu_k} - \frac{d\log P_{cpi,h}}{d\log\mu_k}\right),$$

$$\frac{d\log\chi_h^W}{d\log\mu_k} = \sum_{f\in F_c}\frac{\Lambda_f}{\chi_h}\frac{d\log\Lambda_f}{d\log\mu_k} + \sum_{i\in N_h}\frac{d\lambda_i}{d\log\mu_k}\frac{\left(1-\frac{1}{\mu_i}\right)}{\chi_h},$$

$$\frac{d\log P_{cpi,h}}{d\log\mu_k} = \sum_{f\in F}\tilde{\Lambda}_f^{W_h}\frac{d\log\Lambda_f}{d\log\mu_k} + \tilde{\lambda}_k^{W_h}.$$

Hence, assuming the normalization $P_Y Y = 1$ gives

$$\begin{aligned}
&\frac{d^2 \log W^{BS}}{d\log\mu_k \, d\log\mu_i} \\
&\quad = \sum_h \overline{\chi}_h^W \Bigg(\sum_f \frac{d\Lambda_f}{d\log\mu_i} \frac{d\log\Lambda_f}{d\log\mu_k} \frac{1}{\chi_h^W} + \sum_f \frac{\Lambda_f}{\chi_h^W} \frac{d^2\log\Lambda_f}{d\log\mu_i \, d\log\mu_k} \\
&\qquad - \sum_f \frac{\Lambda_f}{\chi_h^W} \frac{d\log\Lambda_f}{d\log\mu_k} \frac{d\log\chi_h^W}{d\log\mu_i} + \frac{d\lambda_k}{d\log\mu_i} \frac{1}{\chi_h^W \mu_k} - \frac{\lambda_k}{\chi_h^W \mu_k} \frac{d\log\chi_h^W}{d\log\mu_i} - \frac{\lambda_k}{\chi_h^W \mu_k} \delta_{ki} \\
&\qquad + \sum_i \frac{d^2\lambda_j}{d\log\mu_i \, d\log\mu_k} \frac{1 - \frac{1}{\mu_j}}{\chi_h} + \frac{d\lambda_i}{d\log\mu_k} \frac{1}{\mu_i \chi_h^W} + \sum_j \frac{d\lambda_j}{d\log\mu_k} \frac{1 - \frac{1}{\mu_j}}{\chi_h^W} \frac{d\log\chi_h^W}{d\log\mu_i} \\
&\qquad - \sum_f \frac{d\tilde{\Lambda}_f^{W_h}}{d\log\mu_i} \frac{d\log\Lambda_f}{d\log\mu_k} - \sum_f \tilde{\Lambda}_f^{W_h} \frac{d^2\log\Lambda_f}{d\log\mu_i \, d\log\mu_k} - \frac{d\tilde{\lambda}_k^{W_h}}{d\log\mu_i} \Bigg).
\end{aligned}$$

At the efficient point, this simplifies to

$$\begin{aligned}
\frac{d^2 \log W^{BS}}{d\log\mu_k \, d\log\mu_i} &= \sum_f \frac{d\log\Lambda_f}{d\log\mu_k} \left(\frac{d\Lambda_f}{d\log\mu_i} - \sum_h \overline{\chi_h^W} \frac{d\tilde{\Lambda}_f^{W_h}}{d\log\mu_i} \right) \\
&\quad + \frac{d\lambda_k}{d\log\mu_i} - \sum_h \overline{\chi_h^W} \frac{d\tilde{\lambda}_k^{W_h}}{d\log\mu_i} - \sum_{f,h} \Lambda_f \frac{d\log\Lambda_f}{d\log\mu_k} \frac{d\log\chi_h^W}{d\log\mu_i} \\
&\quad - \lambda_k \frac{d\log\chi_h^W}{d\log\mu_i} - \lambda_k \delta_{ki} + \frac{d\lambda_i}{d\log\mu_k}.
\end{aligned}$$

By Lemma 7, at the efficient point,

$$\frac{d\lambda_j}{d\log\mu_i} - \sum_h \overline{\chi}_h^W \frac{d\tilde{\lambda}_j^{W_h}}{d\log\mu_i} = \sum_h \frac{d\chi_h^W}{d\log\mu_i} \tilde{\lambda}_j^{W_h} - \lambda_i(\Psi_{ij} - \delta_{ij}).$$

Whence, we can further simplify the previous expression to

$$\begin{aligned}
\frac{d^2 \log W^{BS}}{d\log\mu_k \, d\log\mu_i} &= \sum_f \frac{d\log\Lambda_f}{d\log\mu_k} \left(\sum_h \frac{d\chi_h^W}{d\log\mu_i} \tilde{\Lambda}_f^{W_h} - \lambda_i \Psi_{if} \right) \\
&\quad + \sum_h \frac{d\chi_h}{d\log\mu_i} \tilde{\lambda}_k^{W_h} - \lambda_i(\Psi_{ik} - \delta_{ik}) - \sum_{f,h} \Lambda_f \frac{d\log\Lambda_f}{d\log\mu_k} \frac{d\log\chi_h}{d\log\mu_i} \\
&\quad - \frac{\lambda_k}{d\log\chi_h} d\log\mu_i - \lambda_k \delta_{ki} + \frac{d\lambda_i}{d\log\mu_k} \\
&= \sum_f \frac{d\log\Lambda_f}{d\log\mu_k} \left(\sum_h \frac{d\chi_h}{d\log\mu_i} \tilde{\Lambda}_f^{W_h} - \lambda_i \Psi_{if} \right)
\end{aligned}$$

$$+\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}$$

$$-\frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},$$

and using $\mathrm{d}\log\lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$\begin{aligned}
&= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right) \\
&\quad + \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&\quad - \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \lambda_i\frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k} + \lambda_i\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} \\
&= \sum_{f,h}\chi_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \lambda_i\sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&\quad + \sum_h \chi_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&\quad - \lambda_k\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} + \lambda_i\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} \\
&\quad + \lambda_i\left(\sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \Psi_{ik}\right) \\
&= \sum_{f,h}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\chi_h\tilde{\Lambda}_f^{W_h} - \Lambda_f\right) \\
&\quad + \lambda_i\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_k\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&= \lambda_i\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\left(\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^{W_h}\right) \\
&\quad - \sum_{f,h}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f - \lambda_k\sum_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&= \lambda_i\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} \\
&\quad - \left(\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f\right)\left(\sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\right) - \lambda_k\sum_h\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&= \lambda_i\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \mathrm{Cov}_\chi\left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right),
\end{aligned}$$

since

$$-\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f = -\sum_f \frac{\mathrm{d}\Lambda_f}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\left(1-\sum_j \lambda_j\left(1-\frac{1}{\mu_j}\right)\right)}{\mathrm{d}\log\mu_k} = -\lambda_k$$

at the efficient point, and

$$\sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} = 0. \qquad \textit{Q.E.D.}$$

PROOF OF THEOREM 6: From Theorem 5, we have

$$\mathcal{L} = -\frac{1}{2}\sum_l (\mathrm{d}\log\mu_l)\lambda_l \,\mathrm{d}\log y_l.$$

With the maintained normalization $PY = 1$, we also have

$$\begin{aligned}
\mathrm{d}\log y_l &= \mathrm{d}\log\lambda_l - \mathrm{d}\log p_l, \\
\mathrm{d}\log p_l &= \sum_f \Psi_{lf}\,\mathrm{d}\log\Lambda_f + \sum_k \Psi_{lk}\,\mathrm{d}\log\mu_k,
\end{aligned}$$

where, from Theorem 3,

$$\begin{aligned}
\mathrm{d}\log\lambda_l = {} & \sum_k \left(\delta_{lk} - \frac{\lambda_k}{\lambda_l}\Psi_{kl}\right)\mathrm{d}\log\mu_k \\
& - \sum_j \frac{\lambda_j}{\lambda_l}(\theta_j - 1)\,\mathrm{Cov}_{\Omega^{(j)}}\left(\sum_k \Psi_{(k)}\,\mathrm{d}\log\mu_k - \sum_g \Psi_{(g)}\,\mathrm{d}\log\Lambda_g, \Psi_{(l)}\right) \\
& + \frac{1}{\lambda_l}\sum_{g\in F^*}\sum_c \left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g \mathrm{d}\log\Lambda_g,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}\log\Lambda_f = {} & -\sum_k \lambda_k \frac{\Psi_{kf}}{\Lambda_f}\,\mathrm{d}\log\mu_k \\
& - \sum_j \lambda_j(\theta_j - 1)\,\mathrm{Cov}_{\Omega^{(j)}}\left(\sum_k \Psi_{(k)}\,\mathrm{d}\log\mu_k - \sum_g \Psi_{(g)}\,\mathrm{d}\log\Lambda_g, \frac{\Psi_{(f)}}{\Lambda_f}\right) \\
& + \frac{1}{\Lambda_f}\sum_{g\in F^*}\sum_c \left(\Lambda_i^{W_c} - \Lambda_f\right)\Phi_{cg}\Lambda_g \mathrm{d}\log\Lambda_g.
\end{aligned}$$

We will now use these expressions to replace in formula for the second-order loss function. We get

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}\sum_l \sum_k \left(\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k}\right)\lambda_k \lambda_l \, \mathrm{d}\log\mu_k \, \mathrm{d}\log\mu_l \\
& + \frac{1}{2}\sum_l \lambda_l \, \mathrm{d}\log\mu_l \sum_f \Psi_{lf} \, \mathrm{d}\log\Lambda_f \\
& + \frac{1}{2}\sum_l \sum_j (\mathrm{d}\log\mu_l)\lambda_j(\theta_j - 1)\,\mathrm{Cov}_{\Omega^{(j)}}\left(\sum_k \Psi_{(k)}\,\mathrm{d}\log\mu_k - \sum_g \Psi_{(g)}\,\mathrm{d}\log\Lambda_g, \Psi_{(l)}\right) \\
& - \frac{1}{2}\sum_l \mathrm{d}\log\mu_l \left(\sum_g \sum_c (\lambda_l^{W_c} - \lambda_l)\Phi_{cg}\Lambda_g \, \mathrm{d}\log\Lambda_g\right), \\
\mathcal{L} = & -\frac{1}{2}\sum_l \sum_k \left(\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k}\right)\lambda_k \lambda_l \, \mathrm{d}\log\mu_k \, \mathrm{d}\log\mu_l \\
& + \frac{1}{2}\sum_l \lambda_l \, \mathrm{d}\log\mu_l \sum_f \Psi_{lf} \, \mathrm{d}\log\Lambda_f \\
& + \frac{1}{2}\sum_l \sum_j (\mathrm{d}\log\mu_l)\lambda_j(\theta_j - 1)\,\mathrm{Cov}_{\Omega^{(j)}}\left(\sum_k \Psi_{(k)}\,\mathrm{d}\log\mu_k - \sum_g \Psi_{(g)}\,\mathrm{d}\log\Lambda_g, \Psi_{(l)}\right) \\
& - \frac{1}{2}\sum_l \left(\sum_c (\lambda_l^{W_c} - \lambda_l)\chi_c \, \mathrm{d}\log\chi_c\right)\mathrm{d}\log\mu_l.
\end{aligned}$$

We can rewrite this expression as

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_X + \mathcal{L}_H,$$

where

$$\begin{aligned}
\mathcal{L}_I = & \frac{1}{2}\sum_k \sum_l \left[\frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\Psi_{lk} - \delta_{lk}}{\lambda_k} + \frac{\delta_{kl}}{\lambda_l} - 1\right]\lambda_k \lambda_l \, \mathrm{d}\log\mu_k \, \mathrm{d}\log\mu_l \\
& + \frac{1}{2}\sum_k \sum_l \sum_j \mathrm{d}\log\mu_k \, \mathrm{d}\log\mu_l \lambda_j(\theta_j - 1)\,\mathrm{Cov}_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}), \\
\mathcal{L}_X = & \frac{1}{2}\sum_l \sum_f \left(\frac{\Psi_{lf}}{\Lambda_f} - 1\right)\lambda_l \Lambda_f \, \mathrm{d}\log\mu_l \, \mathrm{d}\log\Lambda_f \\
& - \frac{1}{2}\sum_l \sum_g \mathrm{d}\log\mu_l \, \mathrm{d}\log\Lambda_g \sum_j \lambda_j(\theta_j - 1)\,\mathrm{Cov}_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}), \\
\mathcal{L}_H = & -\frac{1}{2}\sum_l \left(\sum_c (\lambda_l^{W_c} - \lambda_l)\chi_c \, \mathrm{d}\log\chi_c\right)\mathrm{d}\log\mu_l,
\end{aligned}$$

where $d\log\Lambda$ is given by the usual expression.[1] Finally, using Lemma 10, we can write

$$\mathcal{L}_{\mathcal{I}} = \frac{1}{2}\sum_l\sum_k(d\log\mu_l)(d\log\mu_k)\sum_j\lambda_j\theta_j\,\mathrm{Cov}_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}),$$

and

$$\mathcal{L}_X = -\frac{1}{2}\sum_l\sum_g d\log\mu_l\,d\log\Lambda_g\sum_j\lambda_j\theta_j\,\mathrm{Cov}_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$ *Q.E.D.*

LEMMA 9: *The following identity holds*:

$$\sum_j\lambda_j\left(\tilde{\Psi}_{jk}\Psi_{jl}-\sum_m\Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml}\right)=\tilde{\lambda}_k\lambda_l.$$

PROOF: Write Ω so that it contains all the producers, all the households, and all the factors *as well as* a new row (indexed by 0) where $\Omega_{0i}=\chi_i$ if $i\in C$ and 0 otherwise. Then, letting e_0 be the standard basis vector corresponding to the 0th row, we can write

$$\lambda' = e_0' + \lambda'\Omega,$$

or equivalently

$$\lambda'(I-\Omega) = e_0'.$$

Let X^{kl} be the vector where $X^{kl}_m = \tilde{\Psi}_{mk}\Psi_{ml}$. Then

$$\begin{aligned}\sum_j\lambda_j\left(\tilde{\Psi}_{jk}\Psi_{jl}-\sum_m\Omega_{jm}\tilde{\Psi}_{mk}\Psi_{ml}\right) &= \lambda'(I-\Omega)X^{kl}\\ &= e_0'(I-\Omega)^{-1}(I-\Omega)X^{kl}\\ &= e_0'X^{kl} = \tilde{\Psi}_{0k}\Psi_{0l} = \tilde{\lambda}_k\lambda_l.\end{aligned}$$ *Q.E.D.*

LEMMA 10: *The following identity holds*:

$$\sum_j\lambda_j\mu_j^{-1}\,\mathrm{Cov}_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)},\Psi_{(l)}) = \lambda_l\lambda_k\left[\frac{\tilde{\Psi}_{lk}-\delta_{lk}}{\lambda_k}+\frac{\Psi_{kl}-\delta_{kl}}{\lambda_l}+\frac{\delta_{lk}}{\lambda_k}-\frac{\tilde{\lambda}_k}{\lambda_k}\right].$$

[1] We have used the intermediate step:

$$\begin{aligned}\mathcal{L}_X = {}& \frac{1}{2}\sum_l\sum_k\lambda_k\lambda_l\,d\log\mu_k\,d\log\mu_l + \frac{1}{2}\sum_l\sum_f d\log\mu_l\,d\log\Lambda_f\lambda_l\Psi_{lf}\\ &-\frac{1}{2}\sum_l\sum_g d\log\mu_l\,d\log\Lambda_g\sum_j\lambda_j(\theta_j-1)\,\mathrm{Cov}_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).\end{aligned}$$

PROOF: We have

$$\sum_j \lambda_j \mu_j^{-1} \operatorname{Cov}_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \sum_j \lambda_j \mu_j^{-1} \left[\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right) \right],$$

or

$$\sum_j \lambda_j \mu_j^{-1} \operatorname{Cov}_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),$$

or

$$\begin{aligned}\sum_j \lambda_j \mu_j^{-1} \operatorname{Cov}_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) &= \sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} \\ &\quad + \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),\end{aligned}$$

or using, Lemma 9

$$\sum_j \lambda_j \mu_j^{-1} \operatorname{Cov}_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = -\tilde{\lambda}_k \lambda_l + \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j (\tilde{\Psi}_{jk} - \delta_{jk})(\Psi_{jl} - \delta_{jl}),$$

and finally

$$\sum_j \lambda_j \mu_j^{-1} \operatorname{Cov}_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k \left[\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k} \right].$$

Q.E.D.

PROPOSITION 1—Structural Output Loss: *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions*,

$$\begin{aligned}\Delta \log Y \approx &-\frac{1}{2} \sum_{l \in N} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j \operatorname{Cov}_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}) \\ &- \frac{1}{2} \sum_{l \in N} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j \operatorname{Cov}_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}) \\ &+ \frac{1}{2} \sum_{l \in N} \sum_{c \in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l \big(\lambda_l^{W_c} - \lambda_l \big).\end{aligned}$$

PROOF: The proof follows along the same lines as Theorem 6. *Q.E.D.*

APPENDIX C: ADDITIONAL EXAMPLES

C.1. *Writing an Economy in Standard Form*

We use a two-country example to show how to map a specific nested-CES model into *standard-form* required by Theorem 3. Suppose there are n industries at home and foreign. The utility function of home and foreign consumers is

$$W=\prod_{i=1}^{n}(x_{0i})^{\Omega_{0i}}, \qquad W_*=\prod_{i=1}^{n}\left(x_{0i}^*\right)^{\Omega_{0i}},$$

where x_{0i} and x_{0i}^* are home and foreign consumption of goods from industry i. The production function of industry i (at home or foreign) is a Cobb–Douglas aggregate of intermediates and the local factor

$$y_i=L_{ij}^{\Omega_{iL}}\prod_{i=1}^{n}x_{ij}^{\Omega_{ij}}.$$

Suppose that the intermediate good x_{ij} is a CES combination of domestic and foreign varieties of j, with initial home share Ω_j and foreign share $\Omega_j^*=1-\Omega_j$, and elasticity of substitution ε_j+1. Since the market share of home and foreign in industry j does not vary by consumer i, this means there is no home-bias.

In standard form, this economy has $N=3n$ producers: the first n are industries at home, the second n are industries in foreign, and the last n are CES aggregates of domestic and foreign varieties that every other industry buys. The HAIO matrix for this economy, in standard form, is $(2+3n+2)\times(2+3n+2)$:

$$\Omega=\left[\begin{array}{c|c|c|c|cc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{0i}\right]_{i=1}^{n} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{0i}\right]_{i=1}^{n} & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{ij}\right]_{i,j=1}^{n} & \left[\Omega_{iL}\right]_{i=1}^{n} & \mathbf{0}\\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{ij}\right]_{i,j=1}^{n} & \mathbf{0} & \left[\Omega_{iL}\right]_{i=1}^{n}\\
\hline
\mathbf{0} & \begin{matrix}\Omega_1 & \cdots & 0\\ & \ddots & \\ 0 & & \Omega_n\end{matrix} & \begin{matrix}\Omega_1^* & \cdots & 0\\ & \ddots & \\ 0 & & \Omega_n^*\end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right].$$

The first two rows and columns correspond to the households, the next $2n$ rows and columns correspond to home industries and foreign industries, respectively. The next n rows and columns correspond to bundles of home and foreign varieties. The last two rows and columns correspond to the home and foreign factor. The vector elasticities of substitution θ for this economy is a vector with $2+3n$ elements $\theta=(1,\dots,1,\varepsilon_1+1,\dots,\varepsilon_n+1)$, where ε_i is the trade elasticity in industry i.

Now that we have written this economy in standard form, we can use Theorem 3 to study the change in home's share of income following a productivity shock $d\log A_j$ to some *domestic* producer j:

$$\frac{d\log\Lambda_L}{d\log A_j}=\frac{\lambda_j}{\Lambda_L}\frac{\varepsilon_j\Omega_j^*\Omega_{jL}}{1+\sum_i\varepsilon_i\frac{\lambda_i\Omega_{iL}}{\Lambda_L}\frac{\Omega_{iL}}{1-\Lambda_L}\Omega_i^*}\geq 0,$$

which is positive as long as domestic and foreign varieties are substitutes $\varepsilon_j > 0$ for every j. The numerator captures the fact that a shock to j will increase demand for the home factor if j uses the home factor $\Omega_{jL} > 0$. The denominator captures the fact that an increase in the price of the home factor attenuates the increase in demand for the home factor by bidding up the price of home goods.

The positive productivity shock to j will therefore shrink the market share of every other domestic producer, a phenomenon known as Dutch disease. To see this, apply Theorem 3 to some domestic producer $i \neq j$ to get

$$\frac{\mathrm{d}\log\lambda_i}{\mathrm{d}\log A_j} = -\varepsilon_i\Omega_i^* \frac{\Omega_{iL}}{1-\Lambda_L}\frac{\mathrm{d}\log\Lambda_L}{\mathrm{d}\log A_j} < 0.$$

In words, the shock to j boosts the price of the home factor, which makes i less competitive in the world market if i relies on the home factor $\Omega_{iL} > 0$. Hence, if $\varepsilon_j > 0$ for every j, a domestic productivity shock to one sector will cause Dutch disease and shrink the market share of other domestic producers by bidding up home wages.

C.2. *More Details on Example IV From Section 6*

First, the forward propagation equations (7) from Theorem 3 imply that the change in the price of each good is

$$\mathrm{d}\log p = \sum_{k\in N}\Psi_{(k)}\,\mathrm{d}\log\mu_k + \frac{\Psi_{(L)}}{\Lambda_L}\,\mathrm{d}\Lambda_L - \frac{(1-\Psi_{(L)})}{1-\Lambda_L}\left[\mathrm{d}\Lambda_L + \sum_i \lambda_i\,\mathrm{d}\log\mu_i\right].$$

The first term captures the direct effect of the tariff on the price of each good, the second term captures the effect of the change in the wage of domestic workers, and the last term captures the effect of changes in the foreign wage. Here, we use the fact that the change in the foreign wage relative to world GDP is the negative of the change in the home wage and the tax revenues collected (the expression in square brackets).

Substituting the expression for prices into the backward propagation equations from Theorem 3 yields the following expression for the home factor's change in aggregate income:

$$\mathrm{d}\Lambda_L = \frac{-\mathrm{d}\log\mu_L + \sum_{k\in N}\lambda_k(1-\theta_k)\,\mathrm{Cov}_{\Omega^{(k)}}\left(\Psi_{(L)}, \Psi_{(M)}\,\mathrm{d}\log\mu + \Psi_{(L)}\frac{\mathrm{d}\Lambda_R}{1-\Lambda_L}\right) + \left(\Lambda_L^{W_L} - \Lambda_L^{W_{L*}}\right)\mathrm{d}\Lambda_R}{1 - \frac{1}{\Lambda_L(1-\Lambda_L)}\sum_{k\in N}\lambda_k(1-\theta_k)\,\mathrm{Var}_{\Omega^{(k)}}(\Psi_{(L)}) - \left(\Lambda_L^{W} - \Lambda_L^{W_*}\right)}, \tag{20}$$

where $\mathrm{d}\log\mu_L = \sum_k \lambda_k\Psi_{kL}\,\mathrm{d}\log\mu_k$ and $\Psi_{(M)}\,\mathrm{d}\log\mu = \sum_{k\in N}\Psi_{(k)}\,\mathrm{d}\log\mu_k$. The tariff revenues are $\mathrm{d}\Lambda_R = \sum_k \lambda_k\,\mathrm{d}\log\mu_k$. Each term in (20) is intuitive: the numerator is the effect of the tax in partial equilibrium, holding fixed factor prices in terms of world GDP. The denominator is the general equilibrium effect capturing the endogenous substitution and income redistribution effects triggered by changes in factor prices. This comes from solving the fixed point for factor shares $\mathrm{d}\log\Lambda$ in Theorem 3.

To understand the intuition, consider the numerator, which consists of three effects. The first summand in the numerator is the direct incidence of the tax on the home labor, taking into account supply chains. The second term, involving the covariance, is how the tax causes substitution by changing relative prices of goods, and the covariance captures whether or not goods whose relative prices rise tend to be reliant on home labor. The final term in the numerator captures the fact that the tariff revenues, by redistributing income between home and foreign, change demand for the domestic factor. The denominator then accounts for the fact that the partial equilibrium change in factor prices result in additional rounds of expenditure-switching due to substitution and income redistribution.

From home's perspective, the ideal tariff, which raises home wages relative to foreign wages, is one which is imposed on goods that do not directly or indirectly use the domestic factor. For such goods, $\mathrm{d}\log\mu_L = 0$. Furthermore, if substitution elasticities are greater than one, $\theta_k \geq 1$, then the ideal tariff should be levied on goods, which negatively correlate with domestic factor usage, in which case $\mathrm{Cov}_{\Omega^{(k)}}(\Psi_{(L)}, \Psi_{(M)}\,\mathrm{d}\log\mu) < 0$. In other words, if a good is heavily exposed to the tax, then it should also be heavily exposed to foreign (rather than domestic) labor.

APPENDIX D: COMPUTATIONAL APPENDIX

This Appendix describes our computational procedure, as well as the Matlab code in our replication files. Before running the code, customize your folder directory in the code accordingly. Notice that the description below is based on the generic version of the code under flexible wages stored in "Generic" folder.

Writing nested-CES economies in standard form is useful for intuition, but it is computationally inefficient since it greatly expands the size of the input–output matrix. Therefore, for computational efficiency, we instead use the generalization in Appendix E to directly linearize the nested-CES production functions without first putting them into standard form.

Overview

First, we provide an overview of the different files before providing an in depth description of each.

1. **main_load_data_rev.m**: Function that calculates expenditure shares from data.
2. **main_dlogW_org.m**: Main code that loads inputs and calls functions to iterate.
3. **AES_func.m**: Function that calculates Allen–Uzawa elasticities of substitution.
4. **Nested_CES_linear_final_rev.m**: Function that solves the system of linear equations described in Theorem 3.
5. **Nested_CES_linear_result_final.m**: Function that calculates derivatives that are used to derive welfare changes or iterate for large shocks.

While 1. and 3. are specific to our quantitative application, 2., 4., and 5. are general purpose functions that can be used to derive comparative statics and solve any model in the class we study. We now describe each part of the code in some detail.

1. Function Code That Loads Aata

The data used here is based on 2013 release of World Input–Output Database in year 2008. According to Appendix A, there are $C = 41$ countries including ROW (rest-of-world), and $N = 30$ sectors in each country. The code is flexible in terms of which countries to be included in the analysis by `keep_c` input variable. Any countries not included in `keep_c` are put into an aggregate rest-of-the-world composite country. The order of countries are in `countries` variable in the main code. Notice that we always exclude 35th country for ROW for `keep_c` input and put ROW in the last for welfare output. For example, this is why USA is 41st country for input, and 40th country for output.

Code: main_load_data_rev.m

Data Input:

1. Trade elasticity when a country imports or buys inputs in each sector from different destinations (`trade_elast`: N by 1 vector)
2. Input–output matrix across country and sectors (`Omega_tilde`: CN by CN matrix, (i, j) element: expenditure share of sector i on sector j)
3. Household expenditure share on heterogenous goods (`beta`: CN by C matrix, (i, c) element: expenditure share of household c on sector i)
4. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i), Primary Factor share (`alpha_VA`: CN by F matrix, (i, f) element: expenditure share of sector i on factor f out of factor usage)
5. GNE of each country relative to world GNE (`GNE_weights`: C by 1 vector)
6. (Optional) If there are initial tariffs:
   (a) Tariff matrix when household (column) buys goods (row) – `Tariff_cons_matrix_new`: CN by C matrix ((i, c) element: tariff rate of household c, destination, on sector i, origin)
   (b) Tariff matrix when a sector (row) buys goods (column) – `Tariff_matrix_new`: CN by CN matrix ((i, j) element: tariff rate of sector i, destination, on sector j, origin)

User Input:

1. `keep_c` controls which countries to be included. For example,
   Command `keep_c = (1:41); keep_c(35) = [];` include all 41 countries in the data.
2. If the economy does not have initial tariff, `initial_tariff_index` = 1. Otherwise, if the economy has initial tariff, = 2.
3. If factors are country-specific (4 factors per country), `factor_index` = 1. Otherwise, if factors are country-sector-specific (N factors per country), = 2.

Outputs:

1. `data`, `shock` struct

From the inputs, the code automatically calculates input shares (`beta_s`, `beta_disagg`, `Omega_s`, `Omega_disagg`, `Omega_total_C`, `Omega_total_N`) and the input–output matrix (`Omega_total_tilde`, `Omega_total`). These variables are used to calculate Allen–Uzawa elasticities of substitution and solve system of linear equations.

2. Main Code That Loads Inputs and Calls Functions

Code: main_dlogW_org.m

Data Input:

1. `data`, `shock` struct from **main_load_data_rev.m**

User Input:

1. Elasticity of substitution parameters for nested CES structure: Elasticity of substitution (1) across sectors in consumption (`sigma`), (2) across composite of value-added

and intermediates (theta), (3) across primary factors (gamma), and (4) across intermediate inputs (epsilon). In the text of the paper, these elasticities are relabeled as $(\sigma, \epsilon, \theta, \gamma) = (\theta_0, \theta_1, \theta_2, \theta_3)$.
2. If the economy gets universal iceberg trade cost shock, shock_index $= 1$. Otherwise, if the economy gets universal tariff shock, $= 2$.
3. When intensity of shock is $x\%$, intensity $= x$.
4. When shock is discretized by $x/y\%$ and model cumulates the effect of shocks y times, ngrid $= y$.
5. Ownership structure
   (a) Ownership structure of factor (Phi_F: C by CF matrix, (c, f) element: Factor income share of factor f owned by household c)
   (b) Ownership structure of tariff revenue (Phi_T: C + CN by CN + CF by C 3-D matrix, (i, j, c) element: Tariff revenue share owned by household c when household/sector i buys from sector/factor j)
6. (Optional) Technical details about how to customize iceberg trade cost shock matrix $d\log\tau$ and tariff shock matrix $d\log t$ are described in **Nested_CES_linear_final_rev.m**

Output:

1. dlogW (C by ngrid matrix) collects change in real income of each country for each iteration of discretized shocks
2. dlogW_sum (C by 1 vector) shows change in real income of each country from linearized system by summing up dlogW
3. dlogW_world (1 by ngrid vector) is change in real income of world for each iteration of discretized shocks
4. dlogR (C by ngrid matrix) collects reallocation terms of each country for each iteration of discretized shocks
5. dlogR_sum (C by 1 vector) shows reallocation terms of each country from linearized system by summing up dlogR

3. Allen–Uzawa Elasticity of Substitution (AES)

This code computes Allen–Uzawa elasticities of substitution for each sector. These are then used following Appendix E.

Code: AES_func.m

Inputs:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)
2. Elasticity of substitution parameters for nested CES structure: That is, $(\sigma, \epsilon, \theta, \gamma) = (\theta_0, \theta_1, \theta_2, \theta_3)$.
3. Trade elasticity when a country imports or buys domestic product (trade_elast: N by 1 vector).
4. Value-added share (alpha: CN by 1 vector, $(i, 1)$ element: value-added share of sector i).
5. Expenditure shares:
   (a) b_{ic}(beta_s: C by N matrix, (c, i) element: How much household c consumes sector i good).

(b) ω_j^{ic}(`Omega_s`: CN by N matrix, (ic, j) element: How much sector i in country c uses sector j good).
(c) $\tilde{\Omega}_{jm}^{0c}$(`Omega_total_C`: C by CN matrix, (c, jm) element: How much household c buys from sector j in country m).
(d) $\tilde{\Omega}_{jm}^{ic}$(`Omega_total_N`: CN by CN + CF matrix, (ic, jm) element: How much sector i in country c buys from good/factor j in country m).

Outputs:

1. $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix, (ic', jm, c) element: AES of household in country c that substitutes good i in country c' and good j in country m)
2. $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN + CF by CN 3-D matrix, (ic', jm, kc) element: AES of producer of sector k in country c that substitutes good i in country c' and good/factor j in country m)
3. $\theta_{kc}(fc, jm)$ (`AES_F_Mat`: CF by CN + CF by CN 3-D matrix, (fc, jm, kc) element: AES of producer of sector k in country c that substitutes factor f in country c and good j in country m)

To describe how this code functions, we introduce the following notation.

Notation:

Let $p_{ic'}^{kc}$ be the bilateral price when industry or household k in country c buys from industry i in country c'. That is,

$$p_{ic'}^{kc} = \tau_{ic'}^{kc} t_{ic'}^{kc} p_{ic'},$$

where $\tau_{ic'}^{kc}$ is an iceberg cost on kc purchasing goods from ic' and $t_{ic'}^{kc}$ is a tariff on kc purchasing goods from ic', and where $p_{ic'}$ is the marginal cost of producer i in country c'. Define

$$\Omega_{jm}^{ic} = \frac{p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}}, \qquad \tilde{\Omega}_{jm}^{ic} = \frac{t_{jm}^{ic} p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}},$$

where $p_{jm} x_{jm}^{ic}$ is expenditures of ic on jm not including the import tariff. Notice that every row of $\tilde{\Omega}_{jm}^{ic}$ should always sum up to 1. Also, assume that C is a set of countries, and F_c is the factors owned by Household in country c. Then, we having the following:

Households: The price of final consumption in country c

$$P_{0c} = \left(\sum_i b_{ic} \left(P_i^{0c}\right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

where $b_{ic} = \sum_{m \in C} \tilde{\Omega}_{im}^{0c}$. The price of consumption good from industry i in country c

$$P_i^{0c} = \left(\sum_{m \in C} \delta_m^{0c} \left(t_{im}^{0c} \tau_{im}^{0c} p_{im} \right)^{1-\varepsilon_i} \right)^{\frac{1}{1-\varepsilon_i}},$$

where $\varepsilon_i + 1$ is the trade elasticity for industry i and $\delta_m^{0c} = \tilde{\Omega}_{im}^{0c} / (\sum_{v \in C} \tilde{\Omega}_{iv}^{0c})$.

Producers: The marginal cost of good i produced by country c,

$$p_{ic} = \left(\alpha_{ic} P_{w_{ic}}^{1-\theta} + (1 - \alpha_{ic}) P_{M_{ic}}^{1-\theta} \right)^{\frac{1}{1-\theta}},$$

where $\alpha_{ic} = \sum_{f\in F_c} \tilde{\Omega}^{ic}_{fc}$. The price of value-added bundle used by producer i in country c,

$$p_{w_{ic}} = \left(\sum_{f\in F_c} \alpha^{ic}_f w^{1-\gamma}_{fc}\right)^{\frac{1}{1-\gamma}},$$

where $\alpha^{ic}_f = \tilde{\Omega}^{ic}_{fc}/(\sum_{d\in F_c} \tilde{\Omega}^{ic}_{dc})$. The price of intermediate bundle used by producer i in country c,

$$p_{M_{ic}} = \left(\sum_j \omega^{ic}_j \left(q^{ic}_j\right)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}},$$

where $\omega^{ic}_j = (\sum_{m\in C} \tilde{\Omega}^{ic}_{jm})/(1-\alpha_{ic})$. The price of intermediate bundle good j used by producer i in country c,

$$q^{ic}_j = \left(\sum_{m\in C} \delta^{ic}_{jm}\left(\tau^{ic}_{jm} t^{ic}_{jm} p_{jm}\right)^{1-\varepsilon_i}\right)^{\frac{1}{1-\varepsilon_i}},$$

where $\varepsilon_i + 1$ is the trade elasticity for good i and $\delta^{ic}_{jm} = \tilde{\Omega}^{ic}_{jm}/(\sum_{v\in C} \tilde{\Omega}^{ic}_{iv})$.

Deriving Allen–Uzawa elasticities for nested-CES models is straightforward. To do so, we proceed as follows.

Derivation:

(1) $\theta_{0c}(ic', jm)$. Household demand in country c for good i from c' is

$$x^{0c}_{ic'} = \tilde{\Omega}^{0c}_{ic'}\left(\frac{p^{0c}_{ic'}}{P^{0c}_i}\right)^{-\varepsilon_i}\left(\frac{P^{0c}_i}{P^{0c}}\right)^{-\sigma} C_c.$$

Hence,

$$\theta_{0c}(ic', jm) = \frac{1}{\tilde{\Omega}^{0c}_{jm}}\frac{\partial \log x^{0c}_{ic'}}{\partial \log p^{0c}_{jm}} = -\varepsilon_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j=i)\delta^{0c}_{jm}\right)}{\tilde{\Omega}^{0c}_{jm}} - \frac{\sigma\left(\mathbf{1}(j=i)\delta^{0c}_{jm} - \tilde{\Omega}^{0c}_{jm}\right)}{\tilde{\Omega}^{0c}_{jm}}.$$

This can be simplified as

$$\theta_{0c}(ic', jm) = \frac{\varepsilon_i}{\sum_{v\in C}\tilde{\Omega}^{0c}_{iv}} + \sigma\left(1 - \frac{1}{\sum_{v\in C}\tilde{\Omega}^{0c}_{iv}}\right) = \frac{\varepsilon_i}{b_{ic}} + \sigma\left(1 - \frac{1}{b_{ic}}\right) \quad \text{when } i = j \ \&\ ic' \neq jm,$$

$$\theta_{0c}(ic', jm) = -\frac{\varepsilon_i}{\tilde{\Omega}^{0c}_{jm}} + \frac{\theta_i}{b_{ic}} + \sigma\left(1 - \frac{1}{b_{ic}}\right) \quad \text{when } ic' = jm.$$

Otherwise, $\theta_{0c}(ic', jm) = \sigma$.

(2) $\theta_{kc}(ic', jm)$. When k is not a household, demand by k in country c for good i from c' is

$$x^{kc}_{ic'} = \tilde{\Omega}^{kc}_{ic'}\left(\frac{p^{kc}_{ic'}}{P^{kc}_i}\right)^{-\varepsilon_i}\left(\frac{P^{kc}_i}{P^{kc}_M}\right)^{-\epsilon}\left(\frac{P^{kc}_M}{p_{kc}}\right)^{-\theta} Y_{kc}.$$

Hence,

$$\theta_{kc}(ic', jm) = \frac{1}{\tilde{\Omega}^{kc}_{jm}} \frac{\partial \log x^{kc}_{ic'}}{\partial \log p^{kc}_{jm}}$$
$$= -\varepsilon_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i)\delta^{kc}_{jm}\right)}{\tilde{\Omega}^{kc}_{jm}} - \frac{\epsilon\left(\mathbf{1}(j = i)\delta^{kc}_{jm} - \mathbf{1}(j \notin F)\delta^{kc}_{jm}\omega^{kc}_j\right)}{\tilde{\Omega}^{kc}_{jm}}$$
$$- \frac{\theta\left(\mathbf{1}(j \notin F)\delta^{kc}_{jm}\omega^{kc}_j - \tilde{\Omega}^{kc}_{jm}\right)}{\tilde{\Omega}^{kc}_{jm}}.$$

This can be simplified as

$$\theta_{kc}(ic', jm) = \frac{\varepsilon_i}{(1-\alpha_{kc})\omega^{kc}_j} + \epsilon\left(\frac{1}{1-\alpha_{kc}} - \frac{1}{(1-\alpha_{kc})\omega^{kc}_j}\right)$$
$$+ \theta\left(1 - \frac{1}{1-\alpha_{kc}}\right) \quad \text{when } i = j \in N \ \&\ ic' \neq jm,$$
$$\theta_{kc}(ic', jm) = -\frac{\varepsilon_i}{\tilde{\Omega}^{kc}_{jm}} + \frac{\varepsilon_i}{(1-\alpha_{kc})\omega^{kc}_j} + \epsilon\left(\frac{1}{1-\alpha_{kc}} - \frac{1}{(1-\alpha_{kc})\omega^{kc}_j}\right)$$
$$+ \theta\left(1 - \frac{1}{1-\alpha_{kc}}\right) \quad \text{when } ic' = jm,$$
$$\theta_{kc}(ic', jm) = \frac{\epsilon}{1-\alpha_{kc}} + \theta\left(1 - \frac{1}{1-\alpha_{ic}}\right) \quad \text{when } i \neq j \in N,$$

and when $j \in F$, $\theta_{kc}(ic', jm) = \theta$.

(3) $\theta_{kc}(fc, jm)$. Lastly, when k is not a household, demand by k in country c for factor f is

$$x^{kc}_{fc} = \tilde{\Omega}^{kc}_{fc}\left(\frac{p_{fc}}{p_{w_{kc}}}\right)^{-\gamma}\left(\frac{p_{w_{kc}}}{p^{kc}}\right)^{-\theta} Y_{kc}.$$

Hence,

$$\theta_{kc}(fc, jm) = \frac{1}{\tilde{\Omega}^{kc}_{jm}} \frac{\partial \log x^{kc}_{fc}}{\partial \log p^{kc}_{jm}}$$
$$= -\gamma \frac{\left(\mathbf{1}(jm = fc) - \mathbf{1}(jm \in F_c)\alpha^{ic}_j\right)}{\tilde{\Omega}^{kc}_{jm}} - \theta \frac{\left(\mathbf{1}(jm \in F_c)\alpha^{ic}_j - \tilde{\Omega}^{kc}_{jm}\right)}{\tilde{\Omega}^{kc}_{jm}}.$$

Notice that $\theta_{kc}(fc, jm) = \theta$ if $j \in N$. Also,

$$\theta_{kc}(fc, jc) = \frac{\gamma}{\sum_{g \in F_c} \tilde{\Omega}^{kc}_{gc}} + \theta\left(1 - \frac{1}{\sum_{g \in F_c} \tilde{\Omega}^{kc}_{gc}}\right) = \frac{\gamma}{\alpha_{kc}} + \theta\left(1 - \frac{1}{\alpha_{kc}}\right) \quad \text{when } j \in F \ \&\ m = c,$$

$$\theta_{kc}(fc, jc) = -\frac{\gamma}{\tilde{\Omega}^{kc}_{fc}} + \frac{\gamma}{\alpha_{kc}} + \theta\left(1 - \frac{1}{\alpha_{kc}}\right) \quad \text{when } fc = jm.$$

4. Solving the System of Linear Equations

This code takes the following inputs, forms the linear system of market clearing conditions in factor markets in Theorem 3, and computes the change in factor shares in equilibrium.

Code: Nested_CES_linear_final_rev.m

Input:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)
2. Allen–Uzawa elasticities of substitution:
   (a) $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix)
   (b) $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN + CF by CN 3-D matrix)
   (c) $\theta_{kc}(fc, jm)$ (`AES_F_Mat` CF by CN + CF by CN 3-D matrix)
3. Input–output matrix and Leontief inverse
   (a) $\tilde{\Omega}^{ic}_{jm}$ (`Omega_total_tilde`: C + CN + CF by C + CN + CF matrix): Standard form of Cost-based IO matrix
   (b) Ω^{ic}_{jm} (`Omega_total`: C + CN + CF by C + CN + CF matrix): Standard form of Revenue-based IO matrix
   (c) $\tilde{\Psi}^{ic}_{jm}$ (`Psi_total_tilde`): Leontief inverse of $\tilde{\Omega}^{ic}_{jm}$
   (d) Ψ^{ic}_{jm} (`Psi_total`): Leontief inverse of Ω^{ic}_{jm}
4. Initial sales share λ_{CN} (`lambda_CN`: C + CN by 1 vector) and factor income Λ_F (`lambda_F`: CF by 1 vector)
5. Ownership structure of factor (`Phi_F`: C by CF matrix) and tariff revenue (`Phi_T`: C + CN by CN by C 3-D matrix) defined in **main_dlogW_org.m**
6. If factors are country-specific (4 factors per country), `factor_index` = 1. Otherwise, if factors are country-sector-specific (N factors per country), = 2.
7. (Optional) If economy has initial tariff, initial tariff matrix (`init_t`: C + CN by CN matrix) defined in **main_load_data_rev.m**

Current version of code simulates universal iceberg trade cost or tariff shock. If the user wants to specify the shocks, customize:

1. universal iceberg trade cost shock matrix (`dlogtau`: C + CN by CN + CF matrix, (i, j) element: log change in iceberg trade cost when household/sector i buys from sector/factor j) or
2. tariff shock matrix (`dlogt`: C + CN by CN + CF matrix, (i, j) element: log change in tariff when household/sector i buys from sector/factor j).

Output:

Let $\mathrm{d}\,\Lambda_F$ be the vector of changes in the sales of primary factors and

$$\mathrm{d}\,\Lambda_{F,c',*} = \sum_{ic}\sum_{jm} \Phi_{c',ic,jm}\Omega^{ic}_{jm}\left(t^{ic}_{jm} - 1\right)\mathrm{d}\,\lambda_{ic}$$

be the change in wedge revenues of household c' due to changes in sales shares, where $\Phi_{c',ic,jm}$ is the share of tax revenues on ic's purchases of jm that go to household c'. The linear system in Theorem 3 can be written as

$$\begin{bmatrix} \mathrm{d}\,\Lambda_F \\ \mathrm{d}\,\Lambda_{F_*} \end{bmatrix} = A \begin{bmatrix} \mathrm{d}\,\Lambda_F \\ \mathrm{d}\,\Lambda_{F_*} \end{bmatrix} + B.$$

This code outputs:
1. `A` (C + CF by C + CF matrix) and `B` (C + CF by 1 vector).

Using these outputs, the code inverts the system and solves for $\mathrm{d}\,\Lambda_F$(`dlambda_F`) and $\mathrm{d}\,\Lambda_{F_*}$(`dlambda_F_star`), which are used to obtain derivatives calculated by **Nested_CES_linear_result_final.m**. It updates $\tilde{\Omega}$ and other variables, which are used in the next iteration.

5. Calculate Derivatives

This code takes the equilibrium factor market response calculated in the previous step and uses these to update all endogenous variables so that the whole process can be repeated.

Code: Nested_CES_linear_result_final.m

Input:

All inputs used in **Nested_CES_linear_final_rev.m** are also used in this code. Additionally, it requires:
1. `GNE_weights` (C by 1 vector): A ratio of GNE of each country to world GNE
2. $\mathrm{d}\,\Lambda_F$(`dlambda_F`) and $\mathrm{d}\,\Lambda_{F*}$(`dlambda_F_star`): Solutions from **Nested_CES_linear_final_rev.m**

Output:

1. $\mathrm{d}\,\lambda$ (`dlambda_result`: C + CN + CF by 1 vector): Change in sales shares;
2. $\mathrm{d}\,\chi$ (`dchi_std`: C + CN + CF by 1 vector): Change in household income shares;
3. $\mathrm{d}\log P$ (`dlogP_Vec`: C + CN + CF by 1 vector): Change in either the price index (household), marginal cost (sector), or factor price;
4. $\mathrm{d}\,\tilde{\Omega}^{ic}_{jm}$ (`dOmega_total_tilde`: C + CN + CF by C + CN + CF matrix): Change in Cost-based IO matrix;
5. $\mathrm{d}\,\Omega^{ic}_{jm}$ (`dOmega_total`: C + CN + CF by C + CN + CF matrix): Change in Revenue-based IO matrix.

For each iteration, change in real income of country c is

$$\mathrm{d}\log W_c = \mathrm{d}\log \chi_c - \mathrm{d}\log P_c,$$

where $\mathrm{d}\log P_c$ is change in price index of household c. Meanwhile, outputs are used to update λ, χ, Ω, $\tilde{\Omega}$, which are used as a simulated data with discretized shock in next iteration.

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