Econometrica Supplementary Material

SUPPLEMENT TO "TESTING FOR REGIME SWITCHING: A COMMENT"

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We present proof of the inconsistency of the QMLE defined by Cho and White (2007). Inconsistency arises from the fact that the quasi-log-likelihood does not attain a maximum at the population parameter values.

S1. PROOF OF INCONSISTENCY OF A QMLE

IN THIS SUPPLEMENTAL section we prove that, for an autoregressive process, the gradient of the quasi-log-likelihood does not equal zero when evaluated at the population parameter values. Recall the mixture model for an autoregressive process

$$(1-\pi)\cdot N(\theta_1+\alpha X_{t-1},1)+\pi N(\theta_2+\alpha X_{t-1},1), \tag{S1}$$

on which the quasi-likelihood is based. For $\theta_1=\mu$ and $\theta_2=\mu+\gamma$, we have

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}[l_t(\pi,\alpha,\mu,\gamma)]=\mathbb{E}[l_t(\pi,\alpha,\mu,\gamma)]:=M(\pi,\alpha,\mu,\gamma),$$

where

$$M(\pi,\alpha,\mu,\gamma)=\mathbb{E}\log\left[\pi\exp\left[\gamma(X_t-\alpha X_{t-1}-\mu)-\frac{\gamma^2}{2}\right]+(1-\pi)\right] -\frac{1}{2}[\log 2pi+\mathbb{E}(X_t-\alpha X_{t-1}-\mu)^2].$$

Remaining variable definitions are given in the main paper.

The key is to calculate the first derivative of $M(\pi,\alpha,\mu,\gamma)$ with respect to the autoregressive coefficient α evaluated at the population values of the parameters $(\pi^*,\alpha^*,\mu^*,\gamma^*)$. To begin, let $Z_t=X_t-\alpha^*X_{t-1}-\mu^*$. The derivative for α is

$$\frac{\partial}{\partial\alpha}M(\pi,\alpha,\mu,\gamma)\bigg|_{(\pi^*,\alpha^*,\mu^*,\gamma^*)} =\mathbb{E}\left(\frac{-\pi^*\gamma^*X_{t-1}\exp\left[\gamma^*Z_t-\frac{(\gamma^*)^2}{2}\right]}{\pi^*\exp\left[\gamma^*Z_t-\frac{(\gamma^*)^2}{2}\right]+(1-\pi^*)}\right)+\mathbb{E}(X_{t-1}Z_t). \tag{S2}$$

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To calculate these expectations, we must account for the correlation between Z_t and X_{t-1}. The distribution of Z_t conditional on X_{t-1} depends on this previous observation through S_{t-1}. The conditional density is

$$\begin{aligned} f(z|X_{t-1}) &= \phi(z-\gamma^*)\mathbb{P}(S_t=2|X_{t-1}) + \phi(z)\mathbb{P}(S_t=1|X_{t-1}) \\ &= \left[\zeta_t \exp\left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)\right]\phi(z), \end{aligned}$$

where $\phi(z)$ is the density of a standard Gaussian random variable and $\zeta_t := \mathbb{P}(S_t = 2|\sigma(X^{t-1}))$.

The first expectation in (S2) is

$$\mathbb{E}\left(\frac{-\pi^*\gamma^* X_{t-1}\exp\left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2}\right]}{\pi^*\exp\left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*)}\right) \tag{S3}$$

$$= -\pi^*\gamma^*\mathbb{E}\left[X_{t-1}\int \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] \times \left(\frac{\zeta_t\exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)}{\pi^*\exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*)}\right)\phi(z)\,dz\right].$$

Because $\exp[\gamma^* z - \frac{(\gamma^*)^2}{2}]\phi(z) = \phi(z-\gamma^*)$,

$$\begin{aligned} &\int \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right]\left(\frac{\zeta_t\exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)}{\pi^*\exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*)}\right)\phi(z)\,dz \\ &= \int \left(\frac{\zeta_t\exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)}{\pi^*\exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*)}\right)\phi(z-\gamma^*)\,dz \\ &= \int \left(\frac{\zeta_t\exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)}{\pi^*\exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*)}\right)\phi(v)\,dv, \end{aligned}$$

and it follows that this integral is

$$\begin{aligned}
&\int \left(\frac{\zeta_t \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)}{\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*)} \right) \phi(v)\, dv \\
&\quad = \frac{\zeta_t}{\pi^*} + \int \Bigg[\left(\zeta_t \pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\zeta_t)\pi^* \right. \\
&\qquad - \left. \left(\zeta_t \pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + \zeta_t (1-\pi^*) \right) \right) \\
&\qquad \Big/ \left(\pi^* \left[\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*) \right] \right) \Bigg] \phi(v)\, dv \\
&\quad = \frac{\zeta_t}{\pi^*} + \frac{\pi^* - \zeta_t}{\pi^*} \int \left[\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*) \right]^{-1} \phi(v)\, dv.
\end{aligned}$$

Therefore, substituting this expression back into (S3),

$$\begin{aligned}
&\frac{\partial}{\partial \alpha} M(\pi, \alpha, \mu, \gamma) \bigg|_{(\pi^*, \alpha^*, \mu^*, \gamma^*)} \\
&\quad = -\gamma^* \mathbb{E}(X_{t-1}\zeta_t) + \gamma^* \mathbb{E}[X_{t-1}(\zeta_t - \pi^*)] C_{\pi^*, \gamma^*} + \mathbb{E}(X_{t-1} Z_t),
\end{aligned}$$

where C_{π^*,γ^*} is the expectation

$$C_{\pi^*,\gamma^*} := \mathbb{E}\left[\pi^* \exp\left[\gamma^* \xi + \frac{(\gamma^*)^2}{2}\right] + (1-\pi^*) \right]^{-1}$$

for ξ a standard Gaussian random variable. Clearly, C_{π^*,γ^*} does not depend on X_{t-1} or S_t and is a bounded positive quantity: $0 < C_{\pi^*,\gamma^*} \leq (1-\pi^*)^{-1}$.

Furthermore, we can use that $\mathbb{E}(Z_t|X_{t-1}) = \gamma^* \mathbb{P}(S_t = 2|X_{t-1}) = \gamma^* \zeta_t$ to simplify the expression

$$\frac{\partial}{\partial \alpha} M(\pi, \alpha, \mu, \gamma) \bigg|_{(\pi^*, \alpha^*, \mu^*, \gamma^*)} = \gamma^* C_{\pi^*,\gamma^*} \mathbb{E}[X_{t-1}(\zeta_t - \pi^*)].$$

The last factor in this expression is

$$\mathbb{E}[X_{t-1}(\zeta_t - \pi^*)] = \mathbb{E}(X_{t-1}\zeta_t) - \mathbb{E}X_{t-1}\mathbb{E}S_t = \mathrm{Cov}(X_{t-1}, S_t).$$

The last equality follows from $\mathbb{E}(X_{t-1}S_t) = \mathbb{E}(X_{t-1}\mathbb{E}(S_t|X_{t-1}))$ and $\mathbb{E}(S_t|X_{t-1}) = \zeta_t$.

To obtain an expression for the covariance of X_{t-1} and S_t, we use the following lemma.

LEMMA A: *Under model* (S1)

$$\mathrm{Cov}(X_{t-1}, S_t) = \gamma(1-\pi)(\pi - p_{12})\left(\frac{\pi}{\pi - \alpha(\pi - p_{12})}\right).$$

Therefore, the expression for the partial derivative of the M function with respect to α becomes

$$\frac{\partial}{\partial\alpha} M(\pi, \alpha, \mu, \gamma) = \gamma^2 C_{\pi,\gamma}(\pi - p_{12})\left(\frac{\pi(1-\pi)}{\pi - \alpha(\pi - p_{12})}\right).$$

PROOF OF LEMMA A: We first use the recursive expression

$$X_t = \mu + \alpha X_{t-1} + \gamma S_t + \xi_t = \sum_{k=0}^{\infty} \alpha^k (\mu + \gamma S_{t-k} + \xi_{t-k}),$$

where $\xi_t \sim$ i.i.d. $N(0, 1)$. This implies that the covariance is

$$\mathrm{Cov}(X_{t-1}, S_t) = \gamma \sum_{k=0}^{\infty} \alpha^k \,\mathrm{Cov}(S_{t-1-k}, S_t).$$

The covariance of the binary state variables is

$$\mathrm{Cov}(S_{t-k}, S_t) = \mathbb{P}(S_t = 2 | S_{t-k} = 2)\pi - \pi^2,$$

where $\pi = \mathbb{P}(S_t = 2) = \mathbb{P}(S_{t-k} = 2)$ is the stationary probability in the Markov chain. It can be shown that the conditional probability is

$$\mathbb{P}(S_t = 2 | S_{t-k} = 2) = \pi + (1-\pi)\left(\frac{\pi - p_{12}}{\pi}\right)^k.$$

Thus

$$\mathrm{Cov}(S_{t-k}, S_t) = \pi(1-\pi)\left(\frac{\pi - p_{12}}{\pi}\right)^k,$$

and

$$\begin{aligned}\mathrm{Cov}(X_{t-1}, S_t) &= \gamma\pi(1-\pi)\left(\frac{\pi - p_{12}}{\pi}\right)\sum_{k=0}^{\infty} \alpha^k \left(\frac{\pi - p_{12}}{\pi}\right)^k \\ &= \gamma(1-\pi)(\pi - p_{12})\left(\frac{\pi}{\pi - \alpha(\pi - p_{12})}\right).\end{aligned}$$

Q.E.D.

REFERENCE

CHO, J., AND H. WHITE (2007): "Testing for Regime-Switching," *Econometrica*, 75, 1671–1720. [1]

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