Econometrica Supplementary Material

SUPPLEMENT TO "VERY SIMPLE MARKOV-PERFECT INDUSTRY DYNAMICS: THEORY"

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THIS SUPPLEMENT to Abbring, Campbell, Tilly, and Yang (2018) (hereafter referred to as the "main text") (i) proves a theorem we rely upon for the characterization of equilibria and (ii) develops results for an alternative specification of the model.

Section S1's Theorem S1 establishes that a strategy profile is subgame perfect if no player can benefit from deviating from it in one stage of the game and following it faithfully thereafter. Our proof very closely follows that of the analogous Theorem 4.2 in Fudenberg and Tirole (1991). That theorem only applies to games that are "continuous at infinity" (Fudenberg and Tirole, p. 110), which our game is not. In particular, we only bound payoffs from above (Assumption A1 in the main text) and not from below, because we want the model to encompass econometric specifications like Abbring, Campbell, Tilly, and Yang's (2017) that feature arbitrarily large cost shocks. Instead, our proof leverages the presence of a repeatedly-available outside option with a fixed and bounded payoff, exit.

Section S2 presents the primitive assumptions and analysis for an alternative model of Markov-perfect industry dynamics in which one potential entrant makes its entry decision at the same time as incumbents choose between continuation and exit. We show that the alternative model always has a unique symmetric Markov-perfect equilibrium that satisfies an intuitive refinement criterion. This equilibrium can be computed rapidly by a simple algorithm based on contraction mappings similar to that in the main text.

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S1. ONE-SHOT DEVIATIONS AND SUBGAME PERFECTION

To prove that a strategy forms a subgame-perfect equilibrium if and only if one-shot deviations from it increase no player's payoff, we analyze a *general game* that encompasses that described in the recursive extensive form in Figure 1 of the main text. Like the main text's game, the general game

- is specified in discrete time $t \in \mathbb{N}$,
- is played by firms with names $f \in \mathcal{F}$,
- places an upper bound on each player's flow payoff, and
- regularly offers firms the option to collect a continuation value of zero.

The general game allows for arbitrary differences across players' payoffs, because symmetry is *not* required for this section's result.

Since a subgame of the general game may initialize from multiple points within a given period t, it is useful to focus on *stages* instead of periods. We index these stages with $k \in \mathbb{N}$ and define a function $\theta : \mathbb{N} \to \mathbb{N}$ such that $\theta(k) = t$ when stage k is located in period t. Each period contains the same, finite number $\check{k}$ of stages. In the main text's game, there are $\check{k} = \check{j} + 2$ nontrivial stages each period: the $\check{j} \in \mathbb{N}$ stages at which potential entrants $(t, 1), \dots, (t, \check{j})$ make their entry decisions, the stage at which all incumbents and new entrants make continuation decisions, and the stage in which nature draws Y'.

Denote the random variable for the entire *history* at stage k with H^k and its realization with h^k. At history h^k, firm f chooses an *action* from the finite (and possibly empty) set $\mathbb{A}^f(h^k)$. At each history h^k, only finitely many firms $f \in \mathcal{F}$ have $\mathbb{A}^f(h^k) \neq \emptyset$. For example, in the main text's game, $\mathbb{A}^f(h^k) = \{0, 1\}$ if firm f gets to decide on entry or survival at history h^k, and $\mathbb{A}^f(h^k) = \emptyset$ otherwise.

A *strategy* σ^f for firm f assigns a probability distribution $\sigma^f(h^k)$ over actions in $\mathbb{A}^f(h^k)$ to each history h^k. A *strategy profile* σ is a collection of strategies, one for each firm. We denote the random action taken by firm f in stage k with A_k^f, with possible realizations $a_k^f \in \mathbb{A}^f(h^k)$. We collect all firms' actions in stage k in the *action profile* A_k, with realizations a_k.

Firms can receive flow payoffs in one or more stages within each period. Let $g^f(a_k, h^k)$ denote the *flow payoff* that firm f receives in stage k with history h^k if firms take actions a_k. The analogue to Assumption A1 for this general game is the following:

ASSUMPTION S1—Flow Payoff Bounded From Above: *There is a $\check{g} < \infty$ such that for any firm f, action profile a_k, and history h^k, we have $g^f(a_k, h^k) \leq \check{g}$.*

In the main text's game, $g^f(a_k, h^k) \equiv \rho \mathbb{E}[\pi(n', Y') | Y = y] \leq \rho \check{\pi} < \infty$ if stage k is the survival stage, the implied (by a_k and h^k) number of firms surviving that stage is n', the current demand equals y, and firm f continues ($a_k^f = 1$). If instead stage k contains f's entry decision, the number of active firms prior to f's entry is n, and the current demand is y, then $g^f(a_k, h^k) \equiv -\varphi(n+1, y) \leq 0$ if firm f enters ($a_k^f = 1$). In all other cases, $g^f(a_k, h^k) \equiv 0$.

To complete the general model's specification, let $u^f(\sigma, h^k)$ denote firm f's expected payoff at history h^k, discounted to stage k, when all firms use the strategies in σ. This *continuation value* is defined as

$$u^f(\sigma, h^k) = \lim_{Q \to \infty} \mathbb{E}_\sigma \left[\sum_{q=k}^{Q} \rho^{\theta(q) - \theta(k)} g^f(A_q, H^q) \middle| H^k = h^k \right].$$

Here, the expectation operator's subscript indicates its dependence on all firms following their strategies in σ. Since Assumption S1 ensures that the flow payoff is bounded from above, the limit in the right-hand side is always well defined, either as a real number, which cannot exceed $\check{u} \equiv \frac{\check{k}\check{g}}{1-\rho}$, or as $-\infty$.

A formal statement of the assumption that each firm f in each stage k can collect a continuation payoff of zero within a finite number of stages, irrespective of the strategies followed by the other players, requires the following definition.

DEFINITION S1—l-Shot Deviation: Given a firm f and a strategy σ^f, we say that an alternative strategy $\hat{\sigma}^f$ prescribes an *l-shot deviation* from σ^f starting in stage k if

$$\hat{\sigma}^f\left(h^{k'}\right) = \sigma^f\left(h^{k'}\right)$$

for all possible histories $h^{k'}$; $k' = k + l, k + l + 1, \ldots$.

If $l = 1$, $\hat{\sigma}^f$ prescribes a one-shot deviation. Note that Definition S1 only excludes deviations beyond stage $k + l$ and allows the deviation in earlier stages to be trivial.

ASSUMPTION S2—Exit Option: *For each stage k and firm f, there exists a finite $k' \geq k$ such that, for all possible histories $h^{k'}$ at stage k' and strategy profiles σ, $u^f(\hat{\sigma}, h^{k'}) = 0$ for a strategy profile $\hat{\sigma}$ obtained from σ by replacing σ^f with some strategy $\hat{\sigma}^f$ that prescribes a (possibly trivial) one-shot deviation from it in stage k'.*

The main text's game satisfies this assumption. In particular, in that game, a firm f active in stage k will have an option to exit and collect a zero continuation value within $\check{k}$ stages. A firm f that will have an entry opportunity in or after stage k will be able to forgo that entry opportunity and collect zero within a finite number of periods (and therefore stages). A firm f that has exited before stage k will trivially collect $u^f(\sigma, h^k) = 0$ for all possible histories at stage k (in this trivial case, k' can simply be taken equal to k, and no one-shot deviation is needed).

With the general game's specification complete, we begin its analysis with two further definitions.

DEFINITION S2—l-Shot-Deviation Proof: A strategy profile σ is *l-shot-deviation proof* if for any stage k, history h^k, and firm f, there is no strategy $\hat{\sigma}^f$ that prescribes an l-shot deviation from σ^f starting in stage k such that

$$u^f\left(\hat{\sigma}, h^k\right) > u^f\left(\sigma, h^k\right),$$

where $\hat{\sigma}$ is the strategy profile obtained by replacing σ^f with $\hat{\sigma}^f$ in σ.

DEFINITION S3—Subgame Perfection: A strategy profile σ is *subgame perfect* if for any stage k, history h^k, and firm f, there is no strategy $\hat{\sigma}^f$ such that

$$u^f\left(\hat{\sigma}, h^k\right) > u^f\left(\sigma, h^k\right),$$

where $\hat{\sigma}$ is the strategy profile obtained by replacing σ^f with $\hat{\sigma}^f$ in σ.

We begin our demonstration that a strategy profile is one-shot-deviation proof if and only if it is subgame perfect with the following lemma. Its proof mimics Fudenberg and Tirole's (1991) inductive proof of their Theorem 4.1.

LEMMA S1: *Any strategy profile σ that is one-shot-deviation proof is also l-shot-deviation proof for any $l \in \mathbb{N}$.*

PROOF: The lemma is true by assumption for $l = 1$. Suppose that it is true for some $l \in \mathbb{N}$. We wish to demonstrate that it is also true for $l + 1$. Consider a strategy for some firm f, $\hat{\sigma}^f$, that prescribes a single $(l + 1)$-shot deviation starting in some stage k. Use $\hat{\sigma}$ to denote the strategy profile obtained from σ by replacing σ^f with $\hat{\sigma}^f$. Next, construct a second strategy $\tilde{\sigma}^f$ that agrees with $\hat{\sigma}^f$ in all stages $k' \leq k + l - 1$ and agrees with σ^f otherwise. The strategy profile obtained from σ by replacing σ^f with $\tilde{\sigma}^f$ is $\tilde{\sigma}$.

Note that $\tilde{\sigma}^f$ prescribes an l-shot deviation from σ^f starting in stage k. Fix an arbitrary history h^k. The presumption that the proposition is true for l implies that

$$\begin{aligned} u^f(\hat{\sigma}, h^k) - u^f(\sigma, h^k) &= u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k) + u^f(\tilde{\sigma}, h^k) - u^f(\sigma, h^k) \\ &\leq u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k). \end{aligned} \tag{S1}$$

The definition of $u^f(\cdot, h^k)$ gives us

$$\begin{aligned} u^f(\hat{\sigma}, h^k) = \sum_{q=k}^{k+l-1} \rho^{\theta(q)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[g^f(A_q, H^q)|H^k = h^k\big] \\ + \rho^{\theta(k+l)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\hat{\sigma}, H^{k+l})|H^k = h^k\big]. \end{aligned}$$

The same definition and the construction of $\tilde{\sigma}$ yield

$$\begin{aligned} u^f(\tilde{\sigma}, h^k) &= \sum_{q=k}^{k+l-1} \rho^{\theta(q)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[g^f(A_q, H^q)|H^k = h^k\big] \\ &\quad + \rho^{\theta(k+l)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\sigma, H^{k+l})|H^k = h^k\big] \\ &= u^f(\hat{\sigma}, h^k) + \rho^{\theta(k+l)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\sigma, H^{k+l})|H^k = h^k\big] \\ &\quad - \rho^{\theta(k+l)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\hat{\sigma}, H^{k+l})|H^k = h^k\big]. \end{aligned}$$

Since $\hat{\sigma}^f$ prescribes a one-shot deviation from σ^f in stage $k + l$, we know that $u^f(\hat{\sigma}, h^{k+l}) \leq u^f(\sigma, h^{k+l})$ for all possible histories h^{k+l} in stage $k + l$. Therefore,

$$u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k) = \rho^{\theta(k+l)-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\hat{\sigma}, H^{k+l}) - u^f(\sigma, H^{k+l})|H^k = h^k\big] \leq 0.$$

The above inequality and (S1) jointly imply that $u^f(\hat{\sigma}, h^k) - u^f(\sigma, h^k) \leq 0$. That is, the proposed $(l + 1)$-shot deviation fails to increase firm f's payoff. Continuing inductively establishes the desired conclusion. *Q.E.D.*

With this lemma in hand, we state and prove this section's central theorem, which establishes the necessity and sufficiency of one-shot-deviation proofness for subgame perfection.

THEOREM S1: *A strategy profile σ is subgame perfect if and only if it is one-shot-deviation proof.*

PROOF: Necessity directly follows from the definition of subgame perfection. To demonstrate sufficiency, assume that firm f uses a strategy $\hat{\sigma}^f$ that prescribes a deviation from some stage k and use $\hat{\sigma}$ to denote the strategy profile obtained from σ by replacing σ^f with $\hat{\sigma}^f$. If $\hat{\sigma}^f$ is an l-shot deviation from σ^f, Lemma S1 ensures that it cannot improve on σ^f.

Suppose that $\hat{\sigma}^f$ instead prescribes a deviation in infinitely many stages. Fix an arbitrary history h^k and some finite $k' > k$. Assumption S2 guarantees that there is a finite stage $k'' \geq k'$, such that firm f can guarantee a payoff $u^f(\cdot, h^{k''}) = 0$ for all possible histories $h^{k''}$ and all strategy profiles by a (possibly trivial) one-shot deviation in stage k''. Now construct an alternative deviating strategy $\tilde{\sigma}^f$ that agrees with $\hat{\sigma}^f$ until (but not including) k'' and agrees with σ^f afterwards. The strategy profile obtained from σ by replacing σ^f with $\tilde{\sigma}^f$ is $\tilde{\sigma}$. The construction of $\tilde{\sigma}$ gives us

$$u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k) = \rho^{\theta(k'')-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\hat{\sigma}, H^{k''}) - u^f(\sigma, H^{k''}) | H^k = h^k\big].$$

Since σ is one-shot-deviation proof and Assumption S2 ensures that firm f can earn a continuation value of zero by a one-shot deviation from σ in stage k'', we have

$$\mathbb{E}_{\hat{\sigma}}\big[u^f(\sigma, H^{k''}) | H^k = h^k\big] \geq 0.$$

So

$$u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k) \leq \rho^{\theta(k'')-\theta(k)} \mathbb{E}_{\hat{\sigma}}\big[u^f(\hat{\sigma}, H^{k''}) | H^k = h^k\big] \leq \rho^{\theta(k'')-\theta(k)} \check{u}.$$

With this inequality in hand, we can write

$$\begin{aligned} u^f(\hat{\sigma}, h^k) - u^f(\sigma, h^k) &= u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k) + u^f(\tilde{\sigma}, h^k) - u^f(\sigma, h^k) \\ &\leq u^f(\hat{\sigma}, h^k) - u^f(\tilde{\sigma}, h^k) \\ &\leq \rho^{\theta(k'')-\theta(k)} \check{u} \leq \rho^{\theta(k')-\theta(k)} \check{u}. \end{aligned}$$

Here, the first inequality follows from Lemma S1. Since k' was arbitrary and $\lim_{k' \to \infty} \theta(k') = \infty$, we conclude that $u^f(\hat{\sigma}, h^k) - u^f(\sigma, h^k) \leq 0$. Therefore, the proposed deviation fails to improve firm f's payoff and the strategy profile is subgame perfect. *Q.E.D.*

S2. AN ALTERNATIVE MODEL

The remainder of this supplement presents an alternative model in which one potential entrant per period makes an entry decision at the same time incumbent firms choose between continuation and exit.

S2.1. *Primitives*

The alternative model considered here differs from the one presented in the main text in two primitive assumptions. First, each period has exactly one potential entrant. Second, the single potential entrant makes its entry decision at the same time incumbents make their continuation decisions. The recursive extensive form in Figure S1 presents the timing. All primitive assumptions from the main text on per period profit $\pi(n, y)$ and the cost of entry $\varphi(n, y)$ remain in place.

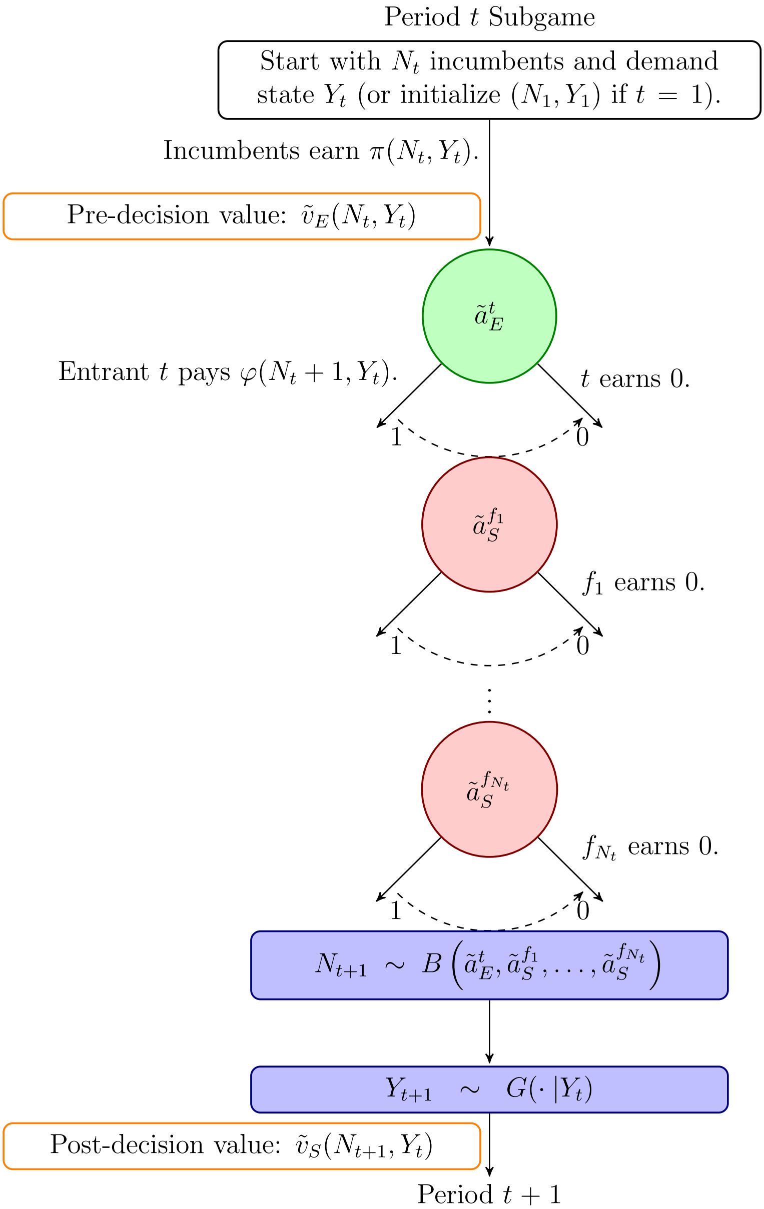


FIGURE S1.—The alternative model's recursive extensive form.

In this model, there is no distinction between the entry stage and the survival stage. Each period of the alternative model starts with incumbent firms earning the per period profit $\pi(N_t, Y_t)$. Then, one potential entrant with name t contemplates becoming active *at the same time* incumbents make their continuation decisions. (Incumbent firms in period 0 have arbitrary names in $\mathcal{F}_0$.) Entry requires paying the sunk cost $\varphi(N_t + 1, Y_t)$. As in the model presented in the main text, simultaneously moving firms, including the potential entrant, can use mixed strategies. All entry and survival outcomes are realized independently across firms according to the chosen Bernoulli distributions. Firms that exit or fail to enter earn zero and never again participate in the market. Active firms continue to the next period.

S2.2. *Equilibrium*

For incumbent firms contemplating survival, the payoff-relevant variables are the number of active firms at the beginning of the period and the current demand state. For a potential entrant, they are this same number of firms plus 1 (i.e., the number of firms that would serve the market next period if this firm would enter and all incumbents would continue) and the current demand state. Note that this implies that incumbents that decide on survival in a state (n, y) move jointly with a potential entrant in state $(n + 1, y)$. We have made this, perhaps counterintuitive, choice, because it is consistent with the specification of the state variables in the main text and will simplify notation later.

As in the main text, we focus our analysis on symmetric Markov-perfect equilibria: subgame-perfect equilibria in which all players use a common Markov strategy. In this context, a Markov strategy is a pair of functions, $\tilde{a}_E : \mathbb{N} \times \mathcal{Y} \to [0, 1]$ and $\tilde{a}_S : \mathbb{N} \times \mathcal{Y} \to [0, 1]$. Firms' values at two nodes of the game tree—just before firms' entry/continuation decisions and just after the decisions' realizations—are important for the equilibrium analysis. Although there are no separate entry and survival stages, we use notation analogous to the main text and denote the *pre-decision* and *post-decision* values as $\tilde{v}_E$ and $\tilde{v}_S$. Figure S1 shows the points where these value functions apply.

An equilibrium strategy and its associated values satisfy

$$\tilde{v}_E(n, y) = \max_{a\in[0,1]} a\mathbb{E}_{\tilde{a}_S,\tilde{a}_E}\big[\tilde{v}_S\big(N', y\big)|N = n, Y = y\big], \tag{S2}$$

$$\tilde{v}_S\big(n', y\big) = \rho\mathbb{E}\big[\pi\big(n', Y'\big) + \tilde{v}_E\big(n', Y'\big)|Y = y\big], \tag{S3}$$

$$\tilde{a}_E(n + 1, y) \in \arg\max_{a\in[0,1]} a\big(\mathbb{E}_{\tilde{a}_S}\big[\tilde{v}_S\big(N', y\big)|N = n, Y = y\big] - \varphi(n + 1, y)\big), \tag{S4}$$

$$\tilde{a}_S(n, y) \in \arg\max_{a\in[0,1]} a\mathbb{E}_{\tilde{a}_S,\tilde{a}_E}\big[\tilde{v}_S\big(N', y\big)|N = n, Y = y\big]. \tag{S5}$$

The expectation operators condition on the deciding firm choosing to be active in the next period and on all other firms using the entry or exit rule in the operator's subscript. Theorem S1 ensures that (S2)–(S5) are not only necessary but also sufficient conditions for a symmetric Markov-perfect equilibrium.

S2.3. *Equilibrium Multiplicity and Refinement*

There can be more than one symmetric Markov-perfect equilibrium in this model. Unlike the model in the main text, it involves simultaneous moves by firms with heterogeneous payoffs (incumbents on the one hand and a potential entrant on the other). We demonstrate how this might lead to equilibrium multiplicity with a numerical example.

EXAMPLE S1: Suppose that the aggregate state Y follows a deterministic first-order Markov process such that $Y_1 = 3$, $Y_2 = 2$, and $Y_t = 0$ for all $t \geq 3$. Specify the per period profit as

$$\pi(n, y) = y/n - 1.5,$$

and assume that $\rho > 0$ and $\varphi(n, y) = \rho/4$. In this game, any firm serving the market in period 3 or beyond will earn a negative payoff. Therefore, in equilibrium, no firms will enter and all incumbents will exit in period 2 (demand state 2) and beyond (demand state 0).

In period 1 (demand state 3), equilibrium play is not so trivial. Suppose that one incumbent is active at the beginning of period 1. The static game (the analogue of Corollary 1's static game in the main text) between the potential entrant and the incumbent has two symmetric equilibria in pure strategies—either the potential entrant enters and the incumbent exits, or the incumbent stays active and the potential entrant stays out. The implied equilibrium entry and survival rules satisfy (1) ($\tilde{a}_E(2,3)=1, \tilde{a}_S(1,3)=0$) and (2) ($\tilde{a}_E(2,3)=0, \tilde{a}_S(1,3)=1$). There is also a mixed-strategy equilibrium, which implies ($\tilde{a}_E(2,3)=0.5, \tilde{a}_S(1,3)=0.25$). These are all symmetric equilibria because all players use the same strategy, even though that strategy's actions depend nontrivially on whether the player is a potential entrant or an incumbent at a particular node of the game tree.

Since an entrant always needs to pay a sunk cost to become active, its expected payoff from becoming active is strictly lower than that of an incumbent from remaining active. If an equilibrium strategy dictated that the potential entrant enters while an incumbent chooses to exit, that incumbent could make a side payment to the potential entrant in return for focusing on an alternative equilibrium strategy in which their roles were reversed. With this in mind, we focus on equilibria in which entry only occurs when all incumbents choose sure continuation. We label these "natural."

DEFINITION S4—Natural Markov-Perfect Equilibrium: A *natural* Markov-perfect equilibrium is a symmetric Markov-perfect equilibrium $(\tilde{a}_E, \tilde{a}_S)$ such that for all $(n, y) \in \mathbb{N} \times \mathcal{Y}$, $\tilde{a}_E(n+1, y) > 0$ implies $\tilde{a}_S(n, y) = 1$.

An analogue of the main text's Lemma 1 establishes that we can again restrict the equilibrium analysis to states (n, y) in $\{1, \dots, \check{n}\} \times \mathcal{Y}$.

LEMMA 1*—Bounded Number of Firms: *In a natural Markov-perfect equilibrium*, $\tilde{a}_E(n, y) = 0$ *and* $\tilde{a}_S(n, y) < 1$ *for all* $n > \check{n}$ *and* $y \in \mathcal{Y}$.

PROOF OF LEMMA 1*: First, we prove that $\tilde{a}_S(n, y) < 1$ for all $y \in \mathcal{Y}$ and $n > \check{n}$. Consider a period $t^\star$ subgame with $N_{t^\star} = n > \check{n}$ incumbent firms and demand state $Y_{t^\star} = y$. Define the random time τ as the first period weakly after $t^\star$ in which the firms choose exit with positive probability, with $\tau \equiv \infty$ if they never do:

$$\tau \equiv \min\bigl(\bigl\{t \geq t^\star : \tilde{a}_S(N_t, Y_t) < 1\bigr\} \cup \{\infty\}\bigr).$$

Suppose that $\tilde{a}_S(n, y) = 1$, so $\tau > t^\star$. By definition, exit occurs only in or after period τ, so we know that $N_t \geq n$ for $t \in \{t^\star + 1, \dots, \tau\}$. (As in the main text, we take $\tilde{a}_S(\cdot) = 1$ to dictate sure survival, not merely almost-sure survival.) Since $n > \check{n}$, this together with Assumption A2 implies that $\pi(N_t, Y_t) < 0$ for $t \in \{t^\star + 1, \dots, \tau\}$. If $\tau = \infty$, then the incumbent firms receive an infinite sequence of strictly negative payoffs. If instead $\tau < \infty$, then the incumbent firms receive a finite sequence of strictly negative payoffs followed by the pre-decision value from playing the period τ subgame $\tilde{v}_E(N_\tau, Y_\tau)$, which equals zero by (S2), (S5), and the definition of τ. Therefore, the period $t^\star$ post-decision value satisfies $\tilde{v}_S(n, y) < 0$. Since a period $t^\star$ incumbent firm can raise its payoff to zero by choosing certain exit, the supposition that $\tilde{a}_S(n, y) = 1$ must be incorrect.

Next, we will prove that $\tilde{a}_E(n, y) = 0$ for all $y \in \mathcal{Y}$ and $n > \check{n}$. Suppose that $\tilde{a}_E(n, y) > 0$ for some $n > \check{n}$. Since we are considering a natural equilibrium, we know that

$\tilde{a}_S(n-1, y) = 1$. Therefore, the entrant pays $\varphi(n, y) > 0$ to enter and receives $\pi(n, Y') < 0$ from production in the next period. We have already proven that this firm's pre-decision continuation value in the next period equals zero, so the firm earns a strictly negative payoff from entry. Since it could improve its payoff by choosing to surely not enter ($\tilde{a}_E(n, y) = 0$), the supposition that $\tilde{a}_E(n, y) > 0$ must be incorrect. *Q.E.D.*

As in the main text, we hereafter restrict attention to equilibria in strategies that *default to inactivity*. A strategy that defaults to inactivity requires a potential entrant that is indifferent between continuing with all n incumbents (as it would if it continued in any natural equilibrium) or not entering to stay out:

$$\tilde{v}_S(n+1, y) = \varphi(n+1, y) \quad \Rightarrow \quad \tilde{a}_E(n+1, y) = 0.$$

Similarly, such a strategy requires an incumbent firm that is indifferent between exit and continuing with any combination of the current incumbents, and that weakly prefers exit over continuing with all potentially active firms, to exit:

$$\tilde{v}_S(1, y) = \cdots = \tilde{v}_S(n, y) = 0 \quad \text{and} \quad \tilde{v}_S(n+1, y) \leq 0 \quad \Rightarrow \quad \tilde{a}_S(n, y) = 0.$$

S2.4. *Equilibrium Analysis*

The natural equilibrium refinement together with the default-to-inactivity restriction on equilibrium strategies (which, as in the main text, we will keep implicit in what follows) give us the following lemma.

LEMMA S2: *The entry strategy in a natural Markov-perfect equilibrium is pure*: $\tilde{a}_E(n, y) \in \{0, 1\}$ *for all* $n \in \{1, \ldots, \check{n}\}$ *and* $y \in \mathcal{Y}$.

PROOF: Since a natural equilibrium requires $\tilde{a}_S(n-1, y) = 1$ when $\tilde{a}_E(n, y) > 0$ for all $n > 1$, equilibrium condition (S4) can be rewritten as

$$\tilde{a}_E(n, y) \in \arg\max_{a \in [0,1]} a\big(\tilde{v}_S(n, y) - \varphi(n, y)\big).$$

This condition holds for all $n \geq 1$, because there are no other firms' survival decisions to consider in the trivial case that $n = 1$. Therefore, $\tilde{a}_E(n, y) = 1$ if $\tilde{v}_S(n, y) > \varphi(n, y)$ and $\tilde{a}_E(n, y) = 0$ if $\tilde{v}_S(n, y) < \varphi(n, y)$. When $\tilde{v}_S(n, y) = \varphi(n, y)$, the default-to-inactivity restriction requires $\tilde{a}_E(n, y) = 0$. *Q.E.D.*

LEMMA $2^\star$—Monotone Equilibrium Payoffs: *In a natural Markov-perfect equilibrium*, $\tilde{v}_S(n', y)$ *weakly decreases with* n' *for all* $y \in \mathcal{Y}$.

The proof of Lemma $2^\star$ is identical to that of Lemma 2 in the main text, except for minor changes in terminology and state variables related to the change in timing. We repeat it here for completeness only.

PROOF OF LEMMA $2^\star$: It suffices to prove that $\tilde{v}_S(n', y) \geq \tilde{v}_S(n'+1, y)$ for all $n' \geq 1$ and demand states $y \in \mathcal{Y}$. To this end, consider a subgame beginning immediately after period $t^\star$'s simultaneous continuation and entry outcomes are realized with $N_{t^\star+1} = n'$ and $Y_{t^\star} = y$. We call this the *original* subgame. Now consider a second period $t^\star$ subgame starting at the same point but with one additional firm. We refer to this as the *perturbed* subgame and use N_t^+ to denote the number of firms serving the market during period t within it. Finally, define the random time τ^+ as the first period weakly after $t^\star+1$ in which firms in the perturbed subgame choose exit with positive probability, with $\tau^+ \equiv \infty$ if they never do:

$$\tau^+ \equiv \min\left(\left\{t \geq t^\star + 1 : \tilde{a}_S\left(N_{E,t}^+, Y_t\right) < 1\right\} \cup \{\infty\}\right).$$

There is no exit before period τ^+ in the perturbed subgame and the period τ^+ pre-decision continuation value equals zero. Therefore, we can write

$$\tilde{v}_S\left(n'+1, y\right) = \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=t^\star+1}^{T} \rho^{t-t^\star} \mathbb{1}\left[t \leq \tau^+\right] \pi\left(N_t^+, Y_t\right) \middle| Y_{t^\star} = y\right].$$

Since τ^+ is a consequence of equilibrium choices, we know that $\tilde{v}_S(n'+1, y) > -\infty$.

Now consider an incumbent firm in the original subgame that (possibly) deviates after the period $t^\star$ survival stage by choosing to survive for sure as long as $t < \tau^+$ and to exit for sure if $t = \tau^+$. Let $\bar{N}_t$ denote the number of firms serving the market during period t in the original subgame *with this deviation*. Since the original strategy was part of a subgame-perfect equilibrium, $\tilde{v}_S(n', y)$ exceeds the expected payoff from following this deviating strategy. That is,

$$\tilde{v}_S\left(n', y\right) \geq \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=t^\star+1}^{T} \rho^{t-t^\star} \mathbb{1}\left[t \leq \tau^+\right] \pi(\bar{N}_t, Y_t) \middle| Y_{t^\star} = y\right].$$

To show that the limit on the right-hand side is well defined, note that $\bar{N}_t \leq N_t^+$ for all $t \leq \tau^+$. Otherwise, the two subgames would have potential entrants in the same states making different entry choices. This would violate either the presumption that the equilibrium strategy is Markov or that it defaults to inactivity. This and Assumption A3 imply that $\pi(\bar{N}_t, Y_t) \geq \pi(N_t^+, Y_t)$ for all $t = t^\star + 1, \ldots, \tau^+$. Combining this with $\tilde{v}_S(n'+1, y) > -\infty$ gives the desired result.

Because the difference of two convergent sequences' limits equals the limit of the sequences' difference, we can write

$$\begin{aligned} &\tilde{v}_S\left(n', y\right) - \tilde{v}_S\left(n'+1, y\right) \\ &\quad\geq \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=t^\star+1}^{T} \rho^{t-t^\star} \mathbb{1}\left[t \leq \tau^+\right]\left(\pi(\bar{N}_t, Y_t) - \pi\left(N_t^+, Y_t\right)\right) \middle| Y_{t^\star} = y\right]. \end{aligned}$$

Each term in the partial sum on the right-hand side is nonnegative, so we conclude that $\tilde{v}_S(n', y) - \tilde{v}_S(n'+1, y) \geq 0$. *Q.E.D.*

COROLLARY 1*: *Fix $n \in \{1, \ldots, \check{n}\}$ and $y \in \mathcal{Y}$, let $\tilde{v}_S$ be the post-decision value function associated with a natural Markov-perfect equilibrium that defaults to inactivity, and suppose that $\tilde{v}_S(n', y) \neq 0$ for at least one $n' \in \{1, \ldots, n\}$ and that $\tilde{v}_S(n+1, y) \neq \varphi(n+1, y)$. Consider the one-shot game in which n incumbent firms and one potential entrant simultaneously choose between activity and inactivity. This game has a symmetric Nash equilibrium—possibly in mixed strategies—in which the potential entrant's chosen probability of entry does not exceed the incumbents' chosen probability of survival. Furthermore, there is only one symmetric Nash equilibrium with this property. In it, the entry strategy is pure.*

PROOF OF COROLLARY 1⋆: Corollary 1⋆'s one-shot game falls into one of four mutually exclusive cases:

- $\tilde{v}_S(1, y) \leq 0$. Lemma 2⋆ implies that $\tilde{v}_S(n', y) \leq 0$ for all $n' > 1$. Therefore, exiting for sure (setting $\tilde{a}_S(n, y) = 0$) is a weakly dominant strategy for an incumbent. Furthermore, since $\tilde{v}_S(n', y) \neq 0$ for at least one $n' \in \{1, \dots, n\}$, we know that $\tilde{v}_S(n+1, y) \leq \tilde{v}_S(n, y) < 0$. Therefore, exiting for sure is also an incumbent's unique best response to any positive symmetric continuation probability and any entry probability. Since $\tilde{v}_S(n + 1, y) < 0 < \varphi(n + 1, y)$, surely not entering (setting $\tilde{a}_E(n + 1, y) = 0$) is a strictly dominant strategy for a potential entrant. Therefore, there is only one symmetric equilibrium, in which all incumbents exit for sure and the potential entrant stays out for sure.
- $\tilde{v}_S(1, y) > 0$ and $\tilde{v}_S(n, y) < 0$. To construct the equilibrium of interest, set $\tilde{a}_E(n + 1, y) = 0$. No symmetric equilibrium exists with this entry choice and a pure continuation strategy, because an incumbent's best response to all other incumbent firms continuing for sure is to exit for sure, and vice versa. In a mixed strategy equilibrium, incumbent firms must be indifferent between continuation and exit. By the intermediate value theorem, there is some $a \in (0, 1)$ that solves

$$\sum_{n'=1}^{n} \binom{n-1}{n'-1} a^{n'-1}(1-a)^{n-n'} \tilde{v}_S(n', y) = 0. \tag{S6}$$

Lemma 2⋆ guarantees that the left-hand side of (S6) weakly decreases with a, and the subcase's conditions strengthen that conclusion so that the left-hand side of (S6) strictly decreases with a. So, there is only one value of a that solves (S6). The pair $\tilde{a}_S(n, y) = a$ and $\tilde{a}_E(n + 1, y) = 0$ form one Nash equilibrium of this game.

To see that this is the only equilibrium in which $\tilde{a}_E(n + 1, y) \leq \tilde{a}_S(n, y)$, suppose to the contrary that there exists another such equilibrium in which $\tilde{a}_E(n + 1, y) > 0$. Lemma 2⋆ and the conditions of this case guarantee that $\tilde{v}_S(n, y) < 0$ and $\tilde{v}_S(n + 1, y) < 0$, so $\tilde{a}_S(n, y) < 1$. Since $\tilde{a}_S(n, y) \geq \tilde{a}_E(n + 1, y) > 0$ by assumption, we therefore know that both the potential entrant and the incumbents are indifferent between continuation and exit. The payoff to a potential entrant is strictly below the payoff to an incumbent in any outcomes of the game in which both are active, because only the potential entrant pays a positive entry cost. With Lemma 2⋆, this implies that both can only be indifferent if the potential entrant is more likely to end up with fewer competitors, $\tilde{a}_S(n, y) < \tilde{a}_E(n+1, y)$. This violates the restriction that $\tilde{a}_S(n, y) \geq \tilde{a}_E(n + 1, y)$.

- $\tilde{v}_S(n, y) \geq 0$ and $\tilde{v}_S(n + 1, y) < \varphi(n + 1, y)$. To construct the equilibrium of interest, set $\tilde{a}_E(n + 1, y) = 0$. Since $\tilde{v}_S(n, y) \geq 0$, pairing this with $\tilde{a}_S(n, y) = 1$ forms one Nash equilibrium. To show that this is the only equilibrium in which $\tilde{a}_E(n + 1, y) \leq \tilde{a}_S(n, y)$, we look at two cases.
  - First, suppose that there exists another such equilibrium in which $\tilde{a}_E(n+1, y) = 0$ and $\tilde{a}_S(n, y) \in [0, 1)$. By assumption, there exists an $n' \leq n$ such that $\tilde{v}_S(n', y) \neq 0$. Lemma 2⋆ and the supposition that $\tilde{v}_S(n, y) \geq 0$ together imply that $\tilde{v}_S(n^\star, y) > 0$ for all $n^\star \leq n'$. Therefore, the payoff to continuing with probability a when all other incumbents continue with probability $\tilde{a}_S(n, y) \in [0, 1)$ is strictly increasing in a, so continuing with any probability greater than $\tilde{a}_S(n, y)$ increases a firm's profit. Therefore, the original value of $\tilde{a}_S(n, y) \in [0, 1)$ paired with $\tilde{a}_E(n + 1, y) = 0$ cannot have formed an equilibrium.
  - Second, suppose that there exists another such equilibrium in which $\tilde{a}_E(n+1, y) > 0$. If $\tilde{a}_S(n, y) = 1$, then the payoff to the potential entrant would equal $\tilde{a}_E(n + 1, y)(\tilde{v}_S(n + 1, y) - \varphi(n + 1, y)) < 0$ in this equilibrium. Hence, the potential entrant could profitably deviate to $\tilde{a}_E(n + 1, y) = 0$, so there cannot be an equilibrium as supposed with

$\tilde{a}_S(n, y) = 1$. If instead $\tilde{a}_S(n, y) < 1$, then both the potential entrant and all incumbents are indifferent between activity and inactivity. With Lemma 2*, this implies that both can only be indifferent if the potential entrant is more likely to end up with fewer competitors, so $\tilde{a}_S(n, y) < \tilde{a}_E(n+1, y)$. This violates the restriction that $\tilde{a}_S(n, y) \geq \tilde{a}_E(n+1, y)$. Therefore, there cannot be an equilibrium as supposed with $\tilde{a}_S(n, y) < 1$ either.
Therefore, $\tilde{a}_E(n+1, y) = 0$ and $\tilde{a}_S(n, y) = 1$ form the only symmetric equilibrium in which $\tilde{a}_E(n, y) \leq \tilde{a}_S(n, y)$.

- $\tilde{v}_S(n+1, y) > \varphi(n+1, y)$. Lemma 2* and the case's precondition together give us that $\tilde{v}_S(n', y) > 0$ for all $n' \in \{1, \dots, n+1\}$, so sure continuation (setting $\tilde{a}_S(n, y) = 1$) is a dominant strategy for each incumbent. For the potential entrant, since $\tilde{v}_S(n+1, y) > \varphi(n+1, y)$, entering for sure (setting $\tilde{a}_E(n, y) = 1$) is a strictly dominant strategy. Therefore, there is a unique Nash equilibrium. In it, all incumbents choose sure continuation and the potential entrant chooses sure entry.

This establishes the equilibrium existence and uniqueness asserted by Corollary 1*. *Q.E.D.*

COROLLARY 2*: *If $\tilde{v}_E$ and $\tilde{v}_S$ are the pre-decision and post-decision value functions associated with a natural Markov-perfect equilibrium and $\tilde{a}_E$ is that equilibrium's entry rule, then*

$$\tilde{v}_E(n, y) = \max\{0, \tilde{a}_E(n+1, y)\tilde{v}_S(n+1, y) + \big(1 - \tilde{a}_E(n+1, y)\big)\tilde{v}_S(n, y)\}.$$

PROOF OF COROLLARY 2*: There are two cases to consider:

- $\tilde{v}_S(1, y) = \tilde{v}_S(2, y) = \cdots = \tilde{v}_S(n, y) = 0$. Lemma 2* implies that $\tilde{v}_S(n+1, y) \leq 0$, so setting $\tilde{a}_E(n+1, y) = 0$ is a dominant strategy for the potential entrant. Therefore, the value of certain continuation for any incumbent equals 0 and the value functions and entry rule satisfy the stated equality.
- There exists an $n' \in \{1, \dots, n\}$ such that $\tilde{v}_S(n', y) \neq 0$. This case contains two subcases:
  - $\tilde{v}_S(n+1, y) \neq \varphi(n+1, y)$. Corollary 1* applies. Its proof demonstrates that

$$\begin{aligned}&\mathbb{E}_{\tilde{a}_E, \tilde{a}_S}\big[\tilde{v}_S\big(N', y\big)|N = n, Y = y\big] \\ &\quad = \tilde{a}_E(n+1, y)\tilde{v}_S(n+1, y) + \big(1 - \tilde{a}_E(n+1, y)\big)\tilde{v}_S(n, y)\end{aligned}$$

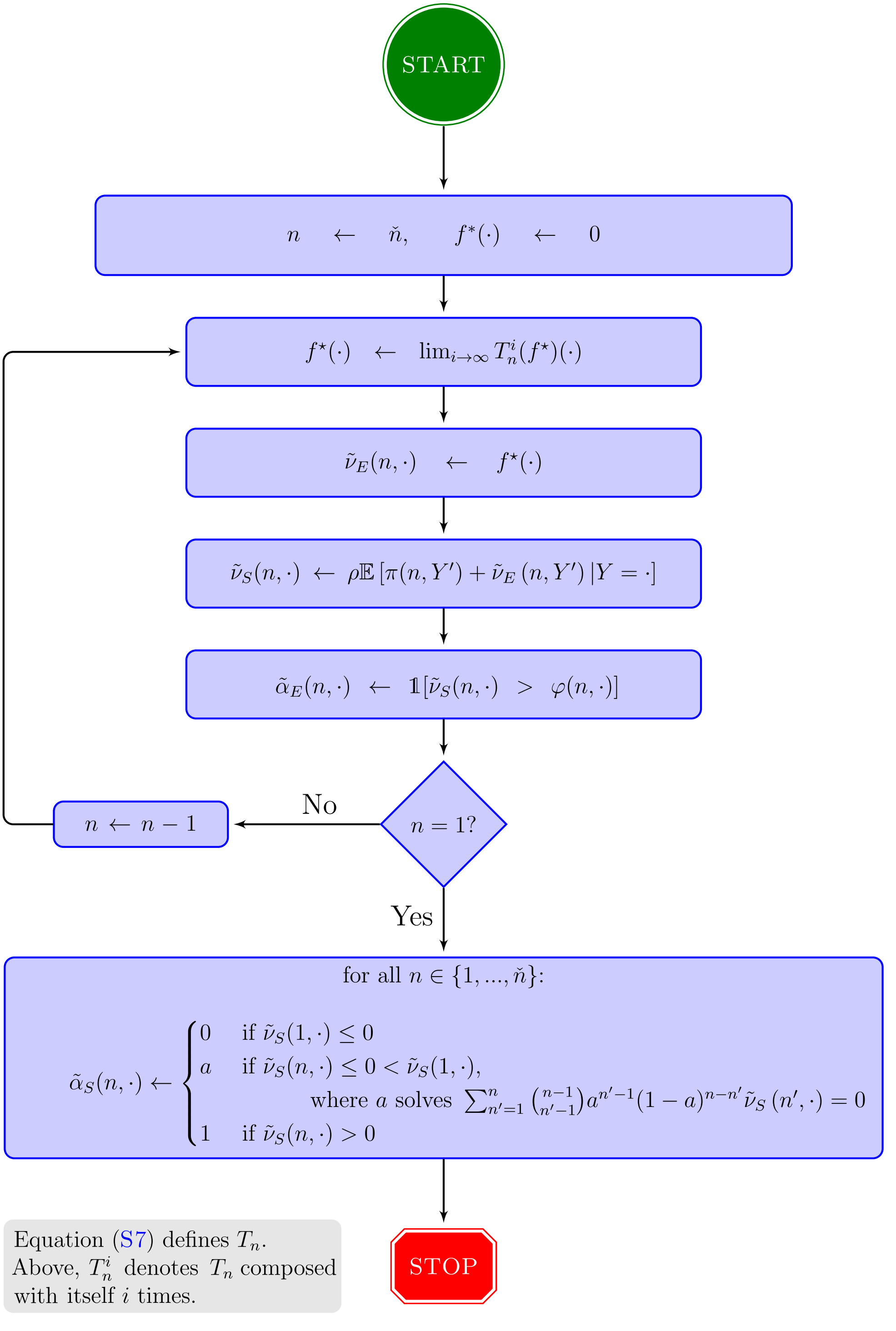
if $\tilde{v}_S(n, y) \geq 0$ (in which case $\tilde{a}_S(n, y) = 1$) and

$$\mathbb{E}_{\tilde{a}_E, \tilde{a}_S}\big[\tilde{v}_S\big(N', y\big)|N = n, Y = y\big] = 0$$

if $\tilde{v}_S(n, y) < 0$ (in which case $\tilde{a}_S(n, y) < 1$). Thus, the value functions and entry rule satisfy the stated equality.

  - $\tilde{v}_S(n+1, y) = \varphi(n+1, y)$. Lemma 2* guarantees that $\tilde{v}_S(n', y) > 0$ for all $n' \in \{1, \dots, n+1\}$, so sure continuation ($\tilde{a}_S(n, y) = 1$) is a dominant strategy for each incumbent. Since the equilibrium defaults to inactivity, $\tilde{a}_E(n+1, y) = 0$. Thus, the value functions and entry rule satisfy the stated equality. *Q.E.D.*

Just as with the model of the main text, we constructively demonstrate equilibrium existence and uniqueness. We present the algorithm for equilibrium calculation in Algorithm S1. It begins by initializing the number of firms under consideration (n) to $\check{n}$ and both the candidate equilibrium entry rule $\tilde{\alpha}_E : \mathbb{N} \times \mathcal{Y} \to \{0, 1\}$ and a dummy function $f^\star : \mathcal{Y} \to [0, \frac{\rho\check{\pi}}{1-\rho}]$ to zero. We will denote the candidate equilibrium pre-decision and post-decision value functions by $\tilde{\nu}_E$ and $\tilde{\nu}_S$, respectively. The algorithm then enters its main



ALGORITHM S1.—Equilibrium calculation for the alternative model.

loop, which begins by using Bellman equation iteration (on the dummy function $f^\star$) to solve a dynamic programming problem. The relevant Bellman operator is

$$\begin{aligned} T_n\bigl(f^\star\bigr)(y) = \max\bigl\{0, \bigl(1-\tilde{\alpha}_E(n+1,y)\bigr)\rho\mathbb{E}\bigl[\pi\bigl(n,Y'\bigr)+f^\star\bigl(Y'\bigr)|Y=y\bigr] \\ + \tilde{\alpha}_E(n+1,y)\rho\mathbb{E}\bigl[\pi\bigl(n+1,Y'\bigr)+\tilde{\nu}_E\bigl(\bigl(n+1,Y'\bigr),Y'\bigr)|Y=y\bigr]\bigr\}. \end{aligned} \tag{S7}$$

The next two steps assign the fixed point of T_n stored in $f^\star(\cdot)$ to $\tilde{\nu}_E(n,\cdot)$ and use (S3) to construct $\tilde{\nu}_S(n,\cdot)$. In the loop's final step, $\tilde{\alpha}_E(n,y)$ is set to $\mathbb{1}[\tilde{\nu}_S(n,y) > \varphi(n,y)]$. If the current value of n exceeds 1, it is decremented and the algorithm returns to the top of the main loop. If instead n equals 1, then the algorithm proceeds to its final task of setting the candidate equilibrium survival rule, which closely follows its calculation by Algorithm 1 in the main text. Algorithm S1 only computes candidate post-entry and post-survival values and a candidate survival rule on $\{1,\dots,\check{n}\}\times\mathcal{Y}$. As noted in the main text, it is straightforward to extend them to the full state space $\mathbb{N}\times\mathcal{Y}$. Because this extension is of little applied interest, we keep it implicit and simply refer to Algorithm S1 as computing a candidate equilibrium.

The appropriately modified version of Theorem 1 states that the candidate equilibrium strategies and payoffs arising from Algorithm S1 correspond to the unique natural Markov-perfect equilibrium.

THEOREM 1*—Equilibrium Existence and Uniqueness: *There exists a unique natural Markov-perfect equilibrium. The equilibrium strategy and corresponding equilibrium payoffs are those computed by Algorithm S1.*

PROOF OF THEOREM 1*: The proof is divided into three parts. First, we show that the candidate continuation values from Algorithm S1 satisfy the monotonicity requirements of Lemma 2*. Second, we use this to demonstrate that the candidate strategy indeed forms an equilibrium. Third, we establish equilibrium uniqueness.

Fix $n\in\{1,2,\dots,\check{n}-1\}$ and suppose that we know that $\tilde{\nu}_E(n+1,\cdot)\geq\cdots\geq\tilde{\nu}_E(\check{n},\cdot)$. This immediately implies that $\tilde{\alpha}_E(n+1,\cdot)\geq\cdots\geq\tilde{\alpha}_E(\check{n},\cdot)$. Evaluating T_n at $f^\star(\cdot)=\tilde{\nu}_E(n+1,\cdot)$ gives

$$\begin{aligned} T_n\bigl(f^\star\bigr)(y) &= \max\bigl\{0, \bigl(1-\tilde{\alpha}_E(n+1,y)\bigr)\rho\mathbb{E}\bigl[\pi\bigl(n,Y'\bigr)+f^\star\bigl(Y'\bigr)|Y=y\bigr] \\ &\quad + \tilde{\alpha}_E(n+1,y)\rho\mathbb{E}\bigl[\pi\bigl(n+1,Y'\bigr)+\tilde{\nu}_E\bigl(n+1,Y'\bigr)|Y=y\bigr]\bigr\} \\ &\geq \max\bigl\{0, \bigl(1-\tilde{\alpha}_E(n+1,y)\bigr)\rho\mathbb{E}\bigl[\pi\bigl(n+1,Y'\bigr)+f^\star\bigl(Y'\bigr)|Y=y\bigr] \qquad \text{(S8)} \\ &\quad + \tilde{\alpha}_E(n+1,y)\rho\mathbb{E}\bigl[\pi\bigl(n+2,Y'\bigr)+\tilde{\nu}_E\bigl(n+2,Y'\bigr)|Y=y\bigr]\bigr\} \\ &\geq \max\bigl\{0, \bigl(1-\tilde{\alpha}_E(n+2,y)\bigr)\rho\mathbb{E}\bigl[\pi\bigl(n+1,Y'\bigr)+f^\star\bigl(Y'\bigr)|Y=y\bigr] \qquad \text{(S9)} \\ &\quad + \tilde{\alpha}_E(n+2,y)\rho\mathbb{E}\bigl[\pi\bigl(n+2,Y'\bigr)+\tilde{\nu}_E\bigl(n+2,Y'\bigr)|Y=y\bigr]\bigr\} \\ &= \tilde{\nu}_E(n+1,y) \end{aligned}$$

for any $y\in\mathcal{Y}$. (For the case with $n=\check{n}-1$, we define $\tilde{\nu}_E(\check{n}+1,\cdot)=\tilde{\alpha}_E(\check{n}+1,\cdot)=0$.) The inequality in (S8) follows from Assumption A3 and the presumption that $\tilde{\nu}_E(n+1,\cdot)\geq\tilde{\nu}_E(n+2,\cdot)$. The inequality in (S9) follows from $\tilde{\alpha}_E(n+2,y)\geq\tilde{\alpha}_E(n+1,y)$ and

$$\mathbb{E}\bigl[\pi\bigl(n+1,Y'\bigr)+f^\star\bigl(Y'\bigr)|Y=y\bigr]\geq\mathbb{E}\bigl[\pi\bigl(n+2,Y'\bigr)+\tilde{\nu}_E\bigl(n+2,Y'\bigr)|Y=y\bigr].$$

The final equality follows from the equivalence of $\tilde{v}_E(n+1, Y')$ with $f^\star(Y')$. The operator T_n is a monotone contraction mapping, so $T_n(f^\star)(n, \cdot) \geq \tilde{v}_E(n+1, \cdot)$ implies that $\tilde{v}_E(n, \cdot) \geq \tilde{v}_E(n+1, \cdot)$. Recursively applying this argument for n decreasing from $\check{n}-1$ to 1 proves that $\tilde{v}_E(1, \cdot) \geq \tilde{v}_E(2, \cdot) \geq \cdots \geq \tilde{v}_E(\check{n}, \cdot)$ and $\tilde{\alpha}_E(1, \cdot) \geq \tilde{\alpha}_E(2, \cdot) \geq \cdots \geq \tilde{\alpha}_E(\check{n}, \cdot)$. With Assumption A3, this monotonicity implies that

$$\tilde{v}_S(n', \cdot) = \rho \mathbb{E}\big[\pi(n', Y') + \tilde{v}_E(n', Y') | Y = \cdot\big]$$

weakly decreases with n'. This is the desired monotonicity result.

For the second part, first note that the algorithm always sets $\tilde{\alpha}_E(n+1, y) = \mathbb{1}\{\tilde{v}_S(n+1, y) > \varphi(n+1, y)\}$. Next, consider the requirements of (S4) and (S5). For states (n, y) such that $\tilde{v}_S(n, y) = \cdots = \tilde{v}_S(1, y) = 0$, the monotonicity of $\tilde{v}_S(n^\star, y)$ in $n^\star$ ensures that $\tilde{v}_S(n+1, y) \leq 0$. Therefore, (S4) requires $\tilde{\alpha}_E(n+1, y) = 0$, which is indeed the case. Given this, (S5) imposes only the trivial requirement that $\tilde{\alpha}_S(n, y) \in [0, 1]$. Algorithm S1's selection of $\tilde{\alpha}_S(n, y) = 0$ satisfies this. If instead $\tilde{v}_S(n+1, y) = \varphi(n+1, y)$, then the monotonicity of $\tilde{v}_S(n^\star, y)$ in $n^\star$ implies that $\tilde{v}_S(n^\star, y) > 0$ for $n^\star = 1, \ldots, n$. Algorithm S1 sets $\tilde{\alpha}_E(n+1, y) = 0$ and $\tilde{\alpha}_S(n, y) = 1$ for these states, which satisfies both (S4) and (S5). For all other states (n, y), Algorithm S1 sets $\tilde{\alpha}_E(n+1, y)$ and $\tilde{\alpha}_S(n, y)$ to the Nash equilibrium strategies from Corollary 1⋆'s game with n incumbents and payoffs $\tilde{v}_S(n', y)$ from continuation with $n' = 1, \ldots, n+1$ firms, which satisfies both (S4) and (S5). Equation (S2) requires $\tilde{v}_E(n, y)$ to equal the expected payoff from this game to the potential entrant, which is true by construction. Similarly, Algorithm S1 sets $\tilde{v}_S(n, y)$ so that it and $\tilde{v}_E(n, y)$ satisfy (S3) automatically. We conclude that Algorithm S1's candidate strategy indeed forms an equilibrium.

The remainder of this proof demonstrates equilibrium uniqueness. Corollary 2⋆ implies that any equilibrium $\tilde{v}_E(\check{n}, \cdot)$ equals the unique fixed point of $T_{\check{n}}$, $\tilde{v}_E(\check{n}, \cdot)$. Given this result, Equation (S3) implies that $\tilde{v}_S(\check{n}, \cdot) = \tilde{v}_S(\check{n}, \cdot)$ in any equilibrium. In turn, this result and Lemma 2⋆ together imply that $\tilde{v}_S(1, \cdot) \geq \tilde{v}_S(2, \cdot) \geq \cdots \geq \tilde{v}_S(\check{n}-1, \cdot) \geq \tilde{v}_S(\check{n}, \cdot)$. So when $\tilde{v}_S(\check{n}, y) > \varphi(\check{n}, y)$, a potential entrant's unique payoff-maximizing choice is sure entry. If instead $\tilde{v}_S(\check{n}, y) = \varphi(\check{n}, y)$, then the restriction that the equilibrium strategy defaults to inactivity requires the potential entrant to surely not enter. Finally, if $\tilde{v}_S(\check{n}, y) < \varphi(\check{n}, y)$, then the restriction that the equilibrium is natural also requires the potential entrant to surely not enter. Algorithm S1's setting of $\tilde{\alpha}_E(\check{n}, y) = \mathbb{1}[\tilde{v}_S(\check{n}, y) > \varphi(\check{n}, y)]$ is the only choice for this entry rule that satisfies these restrictions.

Next, repeat the following argument for n decreasing from $\check{n}-1$ to 1. For given n, suppose that we have determined that $\tilde{v}_E(n^\star, \cdot) = \tilde{v}_E(n^\star, \cdot)$ and $\tilde{a}_E(n^\star, \cdot) = \tilde{\alpha}_E(n^\star, \cdot)$ for $n^\star = n+1, \ldots, \check{n}$. Corollary 2⋆ implies that any equilibrium $\tilde{v}_E(n, \cdot)$ equals the unique fixed point of T_n, $\tilde{v}_E(n, \cdot)$. Given this result, Equation (S3) implies that $\tilde{v}_S(n, \cdot) = \tilde{v}_S(n, \cdot)$ in any equilibrium. In turn, this result and Lemma 2⋆ together imply that if $n > 1$, then $\tilde{v}_S(1, \cdot) \geq \cdots \geq \tilde{v}_S(n, \cdot)$. So when $\tilde{v}_S(n, y) > \varphi(n, y)$, a potential entrant's unique payoff-maximizing choice is sure entry. If instead $\tilde{v}_S(n, y) = \varphi(n, y)$, then the restriction that the equilibrium strategy defaults to inactivity requires the potential entrant to surely not enter. Finally, if $\tilde{v}_S(n, y) < \varphi(n, y)$, then the restriction that the equilibrium is natural also requires the potential entrant to surely not enter. Algorithm S1's setting of $\tilde{\alpha}_E(n, y) = \mathbb{1}[\tilde{v}_S(n, y) > \varphi(n, y)]$ is the only choice for this entry rule that satisfies these restrictions.

The completion of this recursion establishes that $\tilde{v}_E = \tilde{v}_E$, $\tilde{v}_S = \tilde{v}_S$, and $\tilde{a}_E = \tilde{\alpha}_E$ in any equilibrium. Applying Corollary 1⋆ then determines that $\tilde{a}_S(n, y) = \tilde{\alpha}_S(n, y)$ for any (n, y) such that there exists an $n' \in \{1, \ldots, n\}$ with $\tilde{v}_S(n', y) \neq 0$ and $\tilde{v}_S(n+1, y) \neq \varphi(n+1, y)$. States that do not satisfy its preconditions fall into two cases. If $\tilde{v}_S(1, y) = \tilde{v}_S(2, y) = \cdots = \tilde{v}_S(n, y) = 0$, then the monotonicity of $\tilde{v}_S(n^\star, y)$ in $n^\star$ guarantees that $\tilde{v}_S(n+1, y) \leq 0$.

Algorithm S1 sets $\tilde{\alpha}_S(n, y) = 0$ for these states, which the assumption that the equilibrium strategy defaults to inactivity requires. If instead $\tilde{v}_S(n+1, y) = \varphi(n+1, y)$, then monotonicity of $\tilde{v}_S(n^\star, y)$ in $n^\star$ guarantees that $\tilde{v}_S(1, y) \geq \tilde{v}_S(2, y) \geq \cdots \geq \tilde{v}_S(n+1, y) > 0$. Given these values from survival, sure continuation is a dominant strategy for any incumbent. Algorithm S1 indeed sets $\tilde{\alpha}_S(n, y)$ to 1, as required. We conclude that the equilibrium strategy computed by Algorithm S1 is the only natural Markov-perfect equilibrium strategy that defaults to inactivity. *Q.E.D.*

REFERENCES

ABBRING, J. H., J. R. CAMPBELL, J. TILLY, AND N. YANG (2017): "Very Simple Markov-Perfect Industry Dynamics: Empirics," Discussion Paper 2017-021, CentER, Tilburg. [1]

——— (2018): "Very Simple Markov-Perfect Industry Dynamics: Theory," *Econometrica*, 86, 721–735. [1]

FUDENBERG, D., AND J. TIROLE (1991): *Game Theory*. Cambridge: MIT Press. [1,3]

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