Econometrica Supplementary Material

SUPPLEMENT TO "GENERALIZED BELIEF OPERATOR AND ROBUSTNESS IN BINARY-ACTION SUPERMODULAR GAMES"

(*Econometrica*, Vol. 88, No. 2, March 2020, 693–726)

DAISUKE OYAMA
Faculty of Economics, University of Tokyo

SATORU TAKAHASHI
Department of Economics, National University of Singapore

APPENDIX B

B.1. *Proof of Proposition 5*

THIS SECTION PROVIDES the proof of the "only if" part of Proposition 5; the "if" part follows from Basteck, Daniëls, and Heinemann (2010, Theorem 11).

We first review the framework of global games as well as some known results thereof according to Frankel, Morris, and Pauzner (2003) (FMP, henceforth) and Basteck, Daniëls, and Heinemann (2013) (BDH, henceforth), with an extension of allowing correlation in noise terms among players as in the original setup of Carlsson and van Damme (1993). In the (binary-action) global game $G(\nu)$ with the player set I and the action set $A_i = \{0, 1\}$ for each $i \in I$, a state of the world θ is drawn from the real line according to a continuous density ϕ with connected support and determines the payoffs $u_i(a, \theta)$ of the players. Each player $i \in I$ observes a noisy signal $x_i = \theta + \nu\varepsilon_i$, where $(\varepsilon_1, \dots, \varepsilon_{|I|})$ is a noise profile that is distributed independently of the state θ according to a continuous joint density ψ with support contained in $[-\frac{1}{2}, \frac{1}{2}]^{|I|}$, and $\nu > 0$ is a scale parameter. As in FMP, on the payoff functions the assumptions of Strategic complementarities (A1), Dominance regions (A2), State monotonicity (A3), and Payoff continuity (A4) are imposed. FMP show that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$.

BDH provide a useful method of identifying the global game selections via "simplified" global games. Given a complete information game $\mathbf{f}$ and a noise distribution ψ, the *lower ψ-global game elaboration* $\underline{e}(\mathbf{f}, \psi)$ of $\mathbf{f}$ is defined as follows: The state θ is drawn according to the uniform distribution over an interval containing $[-|A| - 1, |A| + 1] \subset \mathbb{R}$. Each player observes a noisy signal $x_i = \theta + \varepsilon_i$, where the noise profile $(\varepsilon_1, \dots, \varepsilon_{|I|})$ is distributed independently of θ according to ψ. Each player i's payoff function $\tilde{u}_i$ depends directly on the signal x_i and is given by

$$\tilde{u}_i(\mathbf{1}_{S\cup\{i\}}, x_i) = \begin{cases} f_i(S) & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0 \end{cases}$$

and $\tilde{u}_i(\mathbf{1}_S, x_i) = 0$ for $S \in \mathcal{I}_{-i}$. The upper ψ-global game elaboration $\overline{e}(\mathbf{f}, \psi)$ of $\mathbf{f}$ is defined symmetrically. Let $\overline{a}(\mathbf{f}, \psi)$ (resp., $\underline{a}(\mathbf{f}, \psi)$) denote the action profile in $\mathbf{f}$ that is played by the largest (resp., smallest) equilibrium strategy profile of $\underline{e}(\mathbf{f}, \psi)$ (resp., $\overline{e}(\mathbf{f}, \psi)$) at the signal profile $(x_i)_{i\in I}$ such that $x_i = |A|$ (resp., $x_i = -|A|$) for all $i \in I$.

Daisuke Oyama: oyama@e.u-tokyo.ac.jp
Satoru Takahashi: ecsst@nus.edu.sg

 https://doi.org/10.3982/ECTA17237

LEMMA B.1—BDH, Theorem 1: *For a given ψ, suppose that the global game $G(\nu)$ with noise distribution ψ embeds the complete information game* $\mathbf{f}$ *at* θ^*, *that is, that* $u_i(\mathbf{1}_{S\cup\{i\}}, \theta^*) - u_i(\mathbf{1}_S, \theta^*) = f_i(S)$ *for all* $i \in I$ *and* $S \in \mathcal{I}_{-i}$, *and let* $\underline{s}$ (*resp.*, $\overline{s}$) *be the left* (*resp., right*) *continuous version of its limit equilibrium strategy profile. Then* $\underline{s}(\theta^*) = \underline{a}(\mathbf{f}, \psi)$ *and* $\overline{s}(\theta^*) = \overline{a}(\mathbf{f}, \psi)$.

This result in particular implies that the action profiles $\underline{s}(\theta^*)$ and $\overline{s}(\theta^*)$ depend only on the noise distribution ψ of the global game that embeds $\mathbf{f}$, independent of the prior distribution ϕ and the payoff functions u_i.

Thus, the action profile $\underline{a}(\mathbf{f}, \psi)$ (resp., $\overline{a}(\mathbf{f}, \psi)$) is called the smallest (resp., largest) ψ-global game selection in $\mathbf{f}$. When $\underline{a}(\mathbf{f}, \psi) = \overline{a}(\mathbf{f}, \psi)$, it is called the ψ-global game selection in $\mathbf{f}$. An action profile a^* is a *noise-independent global game selection* in $\mathbf{f}$ if $a^* = \underline{a}(\mathbf{f}, \psi) = \overline{a}(\mathbf{f}, \psi)$ for all noise distributions ψ. By definition, if $a^* \neq \underline{a}(\mathbf{f}, \psi)$ or $a^* \neq \overline{a}(\mathbf{f}, \psi)$ for some ψ, then a^* is not a noise-independent global game selection in $\mathbf{f}$.

Now let us prove the "only if" part of Proposition 5.

PROOF OF THE "ONLY IF" PART OF PROPOSITION 5: Suppose that $\mathbf{1}_{S^*}$ is not robust in $\mathbf{f}$. In light of the result of BDH (Theorem 1), in order to show that $\mathbf{1}_{S^*}$ is not a noise-independent global game selection in $\mathbf{f}$, it suffices to construct a density function ψ for the noise terms such that $\overline{a}(\mathbf{f}, \psi) \neq \mathbf{1}_{S^*}$.

By Corollary A.1(1), S^* is not a monotone potential maximizer in $\mathbf{f}$. By Lemma A.1, either S^* is not a monotone potential maximizer in $\mathbf{f}^-$, or $\emptyset$ is not a monotone potential maximizer in $\mathbf{f}^+$. We assume the former by symmetry. By genericity, we assume that $\mathbf{f}^-$ satisfies condition (4.2), i.e., that there exists $\mu^- \in \Delta^*(\Gamma^-)$ such that

$$\sum_{S\in\mathcal{I}^-_{-i}} \mu^-\big(\{\gamma \in \Gamma^-_i \mid S^-(i, \gamma) = S\}\big) f^-_i(S) < 0 \tag{B.1}$$

holds for all $i \in I^-(\mu^-)$, where all the notations with a superscript "$-$" are understood with the players in S^*. We want to transform such a μ^- into a continuous density function on $[-\frac{1}{2}, \frac{1}{2}]^{|I|}$. First, a permutation $\gamma = (i_1, \dots, i_{|I^-(\mu^-)|})$ of $I^-(\mu^-)$ is drawn according to μ^-. The discrete noise profile $(\xi_i)_{i\in I}$ is then given by

$$\xi_i = \begin{cases} \ell & i = i_\ell \text{ for some } \ell \in \{1, \dots, |I^-(\mu^-)|\}, \\ |I^-(\mu^-)| + 1 & \text{if } i \in S^* \setminus I^-(\mu^-), \\ 0 & \text{if } i \in I \setminus S^*. \end{cases}$$

To make the noises continuous, add to $(\xi_i)_{i\in I}$ i.i.d. continuous random variables $(\zeta_i)_{i\in I}$ independent of $(\xi_i)_{i\in I}$, where each ζ_i has a density function $3 - 9|z|$ with support $[-\frac{1}{3}, \frac{1}{3}]$. Then map $(\xi_i + \zeta_i)_{i\in I}$ into $[-\frac{1}{2}, \frac{1}{2}]^{|I|}$ by letting $\varepsilon_i = \frac{1}{2}(\xi_i + \zeta_i)/(|I^-(\mu^-)| + \frac{4}{3})$ for each $i \in I$. Note that, by construction, we have $x_j \geq x_i$ if and only if $\xi_j > \xi_i$ for $i \in I^-(\mu^-)$ and $j \in I \setminus \{i\}$. Finally, let ψ be the density function of $(\varepsilon_i)_{i\in I}$, which is continuous and whose support is contained in $[-\frac{1}{2}, \frac{1}{2}]^{|I|}$.

With this noise distribution ψ, consider the lower ψ-global game elaboration $\underline{e}(\mathbf{f}, \psi)$, and let $\overline{s} = (\overline{s}_i)_{i\in I}$ be the largest equilibrium strategy profile of $\underline{e}(\mathbf{f}, \psi)$, which must be weakly increasing in the signals. We want to show that $\overline{s}(|A|) \neq \mathbf{1}_{S^*}$. Assume to the contrary that $\overline{s}(|A|) = \mathbf{1}_{S^*}$, so that each player $i \in S^*$ plays the threshold strategy with threshold $\overline{x}_i \in [0, |A|]$ which plays action 1 if and only if $x_i \geq \overline{x}_i$. At the threshold $\overline{x}_i$, player $i \in S^*$

must be indifferent between the two actions by the continuity of the expected payoffs in the signal x_i, which follows from the continuity of the prior density and the noise density ψ. Let $i \in I^-(\mu^-)$ be a player that has the smallest threshold among $I^-(\mu^-)$, and let $\hat{s}_{-i}$ be the strategy profile such that:

- for $j \in I^-(\mu^-) \setminus \{i\}$, $\hat{s}_j(x_j) = 1$ if and only if $x_j \geq \overline{x}_i$,
- for $j \in S^* \setminus I^-(\mu^-)$, $\hat{s}_j(x_j) = 1$ for all x_j, and
- for $j \in I \setminus S^*$, $\hat{s}_j(x_j) = 0$ for all x_j.

Note that $\overline{s}_{-i} \leq \hat{s}_{-i}$, i.e., $\overline{s}_j(x_j) \leq \hat{s}_j(x_j)$ for all $j \in I \setminus \{i\}$ and all x_j.

Against $\hat{s}_{-i}$, the expected payoff to action 1 for player i upon observing the signal $\overline{x}_i$ is written as follows:

$$
\begin{aligned}
&\mathbb{E}\big[\tilde{u}_i\big(\big(1, \hat{s}_{-i}(\cdot)\big), x_i\big) | x_i = \overline{x}_i\big] \\
&\quad = \mathbb{E}\big[f_i\big(\big\{j \in I^-\big(\mu^-\big) \setminus \{i\} \mid x_j \geq x_i\big\} \cup \big(S^* \setminus I^-\big(\mu^-\big)\big)\big) | x_i = \overline{x}_i\big] \\
&\quad = \mathbb{E}\big[f_i^-\big(\big\{j \in S^* \setminus \{i\} \mid \xi_j > \xi_i\big\}\big) | x_i = \overline{x}_i\big] \\
&\quad = \mathbb{E}\big[f_i^-\big(\big\{j \in S^* \setminus \{i\} \mid \xi_j > \xi_i\big\}\big)\big] \\
&\quad = \sum_{S \in \mathcal{I}_{-i}^-} \mu^-\big(\big\{\gamma \in \Gamma_i^- \mid S^-(i, \gamma) = S\big\}\big) f_i^-(S) < 0,
\end{aligned}
$$

where the second last equality holds due to the uniform prior, and the strict inequality by condition (B.1). Since $\overline{s}_{-i} \leq \hat{s}_{-i}$ and f_i is weakly increasing, it follows that

$$
\mathbb{E}\big[\tilde{u}_i\big(\big(1, \overline{s}_{-i}(\cdot)\big), x_i\big) | x_i = \overline{x}_i\big] \leq \mathbb{E}\big[\tilde{u}_i\big(\big(1, \hat{s}_{-i}(\cdot)\big), x_i\big) | x_i = \overline{x}_i\big] < 0.
$$

Thus, the player i has a strict incentive to play action 0 for the signal $\overline{x}_i$, which is a contradiction. Hence, we have shown that $\mathbf{1}_{S^*}$ is not a noise-independent global game selection in $\mathbf{f}$. *Q.E.D.*

REFERENCES

BASTECK, C., T. R. DANIËLS, AND F. HEINEMANN (2010): "Characterising Equilibrium Selection in Global Games With Strategic Complementarities," SFB 649 Discussion Paper 2010-008, Humboldt University. [1]

——— (2013): "Characterising Equilibrium Selection in Global Games With Strategic Complementarities," *Journal of Economic Theory*, 148, 2620–2637. [1]

CARLSSON, H., AND E. VAN DAMME (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61, 989–1018. [1]

FRANKEL, D. M., S. MORRIS, AND A. PAUZNER (2003): "Equilibrium Selection in Global Games With Strategic Complementarities," *Journal of Economic Theory*, 108, 1–44. [1]

Co-editor Joel Sobel handled this manuscript.

Manuscript received 17 April, 2019; final version accepted 4 November, 2019; available online 18 November, 2019.