# SUPPLEMENT TO "NON-CLAIRVOYANT DYNAMIC MECHANISM DESIGN"

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## APPENDIX A: PROOFS FROM SECTION 3

### A.1. *Stronger Properties of Bank Account Mechanisms*

IN ADDITION TO (DIC) AND (EPIR), bank account mechanisms satisfy stronger versions of both IC and IR, which we describe below. The mechanism is per-period incentive compatible, that is, the buyer's utility in each given period is maximized by reporting truthfully in that period,

$$\theta_t \in \arg\max_{\hat{\theta}_t} u_t\big(\theta_t; \theta^{t-1}, \hat{\theta}_t\big) \quad \forall t, \theta^t \in \Theta^t, \tag{PPIC}$$

and the expected continuation utility is independent of the type reported, that is,

$$U_t\big(\theta^{t-1}, \hat{\theta}_t\big) \text{ is independent of } \hat{\theta}_t \quad \forall t, \theta^{t-1} \in \Theta^{t-1}, \hat{\theta}_t \in \Theta. \tag{INDU}$$

It is straightforward from the definition of (DIC) that (PPIC) and (INDU) imply (DIC).

The mechanism also satisfies a stronger version of (EPIR): It is ex post individually rational for every prefix and every realization of the random variables,

$$\sum_{\tau=1}^{t} u_\tau\big(\theta_\tau; \theta^\tau\big) \geq 0 \quad \forall t, \theta^t \in \Theta^t. \tag{PFIR}$$

Moreover, each individual period is individually rational in expectation:

$$\mathbb{E}_{\theta_t}\big[u_t\big(\theta_t; \theta^t\big)\big] \geq 0 \quad \forall t, \theta^{t-1} \in \Theta^{t-1}. \tag{P$\mathbb{E}$IR}$$

The fact that bank account mechanisms also satisfy (PPIC), (INDU), (PFIR), and (P$\mathbb{E}$IR) follows directly from the proof of Lemma 3.2.

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## A.2. *Proof of Lemma 3.2*

PROOF OF LEMMA 3.2: First, we prove that conditions (IC) and (BI) imply that the mechanism satisfies (DIC). By definition,

$$u_t\big(\theta_t; \hat{\theta}^{t-1}, \hat{\theta}_t\big) = \theta_t \cdot x_t\big(\hat{\theta}^{t-1}, \hat{\theta}_t\big) - p_t\big(\hat{\theta}^{t-1}, \hat{\theta}_t\big)$$
$$= \theta_t \cdot x_t^B\big(\hat{\theta}_t, b_{t-1}\big(\hat{\theta}^{t-1}\big)\big) - p_t^B\big(\hat{\theta}_t, b_{t-1}\big(\hat{\theta}^{t-1}\big)\big).$$

Combined with (IC), we have

$$u_t\big(\theta_t; \hat{\theta}^{t-1}, \hat{\theta}_t\big) = \theta_t \cdot x_t^B\big(\hat{\theta}_t, b_{t-1}\big(\hat{\theta}^{t-1}\big)\big) - p_t^B\big(\hat{\theta}_t, b_{t-1}\big(\hat{\theta}^{t-1}\big)\big)$$
$$\leq \theta_t \cdot x_t^B\big(\theta_t, b_{t-1}\big(\hat{\theta}^{t-1}\big)\big) - p_t^B\big(\theta_t, b_{t-1}\big(\hat{\theta}^{t-1}\big)\big) = u_t\big(\theta_t; \hat{\theta}^{t-1}, \theta_t\big). \qquad \text{(A.1)}$$

By (BI), $\mathbb{E}_{\theta_\tau}[u_\tau(\theta_\tau; \hat{\theta}^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_\tau)]$ is constant with respect to $b_{\tau-1} = b_{\tau-1}(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_{\tau-1})$ and hence also constant in $\hat{\theta}_t$, namely,

$$\mathbb{E}_{\theta_\tau}\big[u_\tau\big(\theta_\tau; \hat{\theta}^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_\tau\big)\big] = \mathbb{E}_{\theta_\tau}\big[u_\tau\big(\theta_\tau; \hat{\theta}^{t-1}, \theta_t, \theta_{t+1}, \ldots, \theta_\tau\big)\big].$$

Therefore,

$$\mathbb{E}_{\theta_{t+1},\ldots,\theta_T}\left[\sum_{\tau=t+1}^T u_\tau\big(\theta_\tau; \hat{\theta}^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_\tau\big)\right]$$
$$= \mathbb{E}_{\theta_{t+1},\ldots,\theta_T}\left[\sum_{\tau=t+1}^T u_\tau\big(\theta_\tau; \hat{\theta}^{t-1}, \theta_t, \theta_{t+1}, \ldots, \theta_\tau\big)\right]. \qquad \text{(A.2)}$$

Adding (A.1) and (A.2) together, we have

$$u_t\big(\theta_t; \hat{\theta}^{t-1}, \hat{\theta}_t\big) + \mathbb{E}_{\theta_{t+1},\ldots,\theta_T}\left[\sum_{\tau=t+1}^T u_\tau\big(\theta_\tau; \hat{\theta}^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_\tau\big)\right]$$
$$\leq u_t\big(\theta_t; \hat{\theta}^{t-1}, \theta_t\big) + \mathbb{E}_{\theta_{t+1},\ldots,\theta_T}\left[\sum_{\tau=t+1}^T u_\tau\big(\theta_\tau; \hat{\theta}^{t-1}, \theta_t, \theta_{t+1}, \ldots, \theta_\tau\big)\right],$$

which directly implies (DIC).

We then show that (BU) implies (EPIR). Summing up (BU) for $t = 1$ to $T$, we have

$$\sum_{t=1}^T b_t \leq \sum_{t=1}^T \big(b_{t-1} + \theta_t \cdot x_t^B(\theta_t, b_{t-1}) - p_t^B(\theta_t, b_{t-1})\big),$$
$$\implies \sum_{t=1}^T \big(\theta_t \cdot x_t^B(\theta_t, b_{t-1}) - p_t^B(\theta_t, b_{t-1})\big) \geq b_T - b_0 \geq 0. \qquad \text{(A.3)}$$

Again, by definition,

$$u_t\big(\theta_t; \theta^t\big) = \theta_t \cdot x_t\big(\theta^t\big) - p_t\big(\theta^t\big) = \theta_t \cdot x_t^B(\theta_t, b_{t-1}) - p_t^B(\theta_t, b_{t-1}).$$

Condition (EPIR) is then implied by (A.3), that is,

$$\sum_{t=1}^{T} u_t(\theta_t; \theta^t) = \sum_{t=1}^{T} (\theta_t \cdot x_t^B(\theta_t, b_{t-1}) - p_t^B(\theta_t, b_{t-1})) \geq 0.$$

Q.E.D.

### A.3. *Proof of Lemma 3.3*

The first step of the proof is a symmetrization lemma. Central to this lemma is the concept of partially realized utility, which measures the expected utility of an agent conditioned on some prefix of the type vector:

$$\bar{U}_t(\theta^t) = \sum_{\tau=1}^{t} u_t(\theta_\tau; \theta^\tau) + U_t(\theta^t).$$

In addition, the dynamic mechanism after symmetrization will satisfy the *payment-frontloading* and *symmetry* properties.

DEFINITION A.1—Payment Frontloading: A dynamic mechanism is payment frontloading if

$$u_t(\theta^t) = 0 \quad \text{for } t < T \quad \text{and} \quad u_T(\theta^T) \geq 0 \quad \forall \theta^T. \tag{PF}$$

The property is a stronger version of (EPIR).

DEFINITION A.2—Symmetry Condition: A dynamic mechanism satisfies the symmetry condition if for every $t < s$ and $\theta^t, \theta'^t \in \Theta^t$,

$$\text{if } \bar{U}_t(\theta^t) = \bar{U}_t(\theta'^t), \text{ then}$$
$$x_s(\theta^t, \theta_{t+1}, \ldots, \theta_s) = x_s(\theta'^t, \theta_{t+1}, \ldots, \theta_s) \tag{SYMM}$$
$$\text{and} \quad p_s(\theta^t, \theta_{t+1}, \ldots, \theta_s) = p_s(\theta'^t, \theta_{t+1}, \ldots, \theta_s).$$

LEMMA A.3—Payment Frontloading: *For any mechanism* $(x_t, p_t)$ *satisfying* (DIC) *and* (EPIR), *there is a mechanism also satisfying* (DIC) *and* (EPIR) *with the same allocation and ex post revenue such that the agent is charged her full surplus in each period except the last*.

LEMMA A.4—Symmetrization: *Any dynamic mechanism satisfying* (DIC) *and* (PF) *can be transformed into a mechanism* $(x_t, p_t)_t$ *with at least the same welfare and at least the same revenue as the original dynamic mechanism, satisfying*: (i) (DIC), (ii) (PF), *and* (iii) (SYMM).

At first glance, our symmetrization lemma resembles the promised utility framework of Thomas and Worrall (1990), which can be viewed as a symmetrization of the mechanism with respect to the continuation utilities $U_t$. Their result can be viewed as an application of the principle of optimality of the theory of dynamic programming (Bertsekas (2000)), which describes the structure of an optimal solution that can be obtained by solving an infinite-size dynamic program. The symmetrization obtained in Thomas and Worrall (1990) is insufficient for our needs. Our solution is to transform the optimization program to a different space and apply the principle of optimality to the transformed program.

Next, we prove the frontloading and symmetrization lemmas leading to the proof of Lemma 3.3.

PROOF OF LEMMA A.3: Given a mechanism $(x_t, p_t)$ that satisfies (DIC) and (EPIR), define mechanism $(x_t, \tilde{p}_t)$ such that $\tilde{p}_t(\theta^t) = \theta_t \cdot x_t(\theta^t)$ for $t < T$ and

$$\tilde{p}_T(\theta^T) = \sum_{t=1}^{T} p_t(\theta^t) - \sum_{t=1}^{T-1} \theta_t \cdot x_t(\theta^t).$$

The mechanism clearly has the same revenue as the original, since for any $\theta^T$, we have $\sum_{t=1}^{T} p_t(\theta^t) = \sum_{t=1}^{T} \tilde{p}_t(\theta^t)$. Since the ex post allocation and ex post revenue are the same in the two mechanisms for every $\theta^T$, the ex post utility should also be the same. In particular, it should always be nonnegative and therefore (EPIR) holds. Since (DIC) can be formulated in terms of ex post utilities, it also holds after the transformation. _Q.E.D._

One important property that we will use heavily is that since $u_t(\theta^t) = 0$ for all $t < T$, the continuation utility $U_t$ and the partially realized utility $\bar{U}_t$ become synonymous.

PROOF OF LEMMA A.4: By Lemma A.3, we can assume that $(x_t, p_t)$ is a payment-frontloading mechanism. Let us first define property (SYM$_t$):

$$\begin{aligned}
&\text{if } \bar{U}_t(\theta^t) = \bar{U}_t(\theta'^t), \text{ then } \forall s \geq t, \theta^s \in \Theta^s, \theta'^t \in \Theta^t, \\
&x_s(\theta^t, \theta_{t+1}, \ldots, \theta_s) = x_s(\theta'^t, \theta_{t+1}, \ldots, \theta_s) \qquad \text{(SYM}_t\text{)} \\
&\text{and} \quad p_s(\theta^t, \theta_{t+1}, \ldots, \theta_s) = p_s(\theta'^t, \theta_{t+1}, \ldots, \theta_s).
\end{aligned}$$

We will show that (SYM$_t$) works for all $t$ by induction. Precisely, we show that if $(x_t, p_t)$ is payment-frontloading satisfying (SYM$_t$) for $t \leq \tau - 1$, we can transform it into a payment-frontloading mechanism with at least the same revenue such that (SYM$_t$) holds for all $t \leq \tau$.

For the inductive step, partition the set of all possible type vectors $\theta^\tau$ into classes with the same partially realized utility, that is,

$$S_\tau(z) = \{\theta^\tau | \bar{U}_\tau(\theta^\tau) = z\}.$$

For each $z$, choose $\theta^{*\tau}(z) \in S_\tau(z)$ maximizing the expected welfare of future periods

$$W_\tau(\theta^\tau) = \mathbb{E}_{\theta_{\tau+1}, \ldots, \theta_T}\left[\sum_{t=\tau+1}^{T} \theta_t \cdot x_t(\theta^\tau, \theta_{\tau+1}, \ldots, \theta_t)\right].$$

We define mechanism $(\tilde{x}, \tilde{p})$ such that $\tilde{x}_t = x_t$ and $\tilde{p}_t = p_t$ for $t \leq \tau$. For $t > \tau$, we have

$$\begin{aligned}
\tilde{x}_t(\theta^t) &= x_t(\tilde{\theta}^\tau, \theta_{\tau+1}, \ldots, \theta_t), \quad \text{where } \tilde{\theta}^\tau = \theta^{*\tau}(\bar{U}_\tau(\theta^\tau)), \\
\tilde{p}_t(\theta^t) &= p_t(\tilde{\theta}^\tau, \theta_{\tau+1}, \ldots, \theta_t), \quad \text{where } \tilde{\theta}^\tau = \theta^{*\tau}(\bar{U}_\tau(\theta^\tau)).
\end{aligned}$$

We argue that $(\tilde{x}_t, \tilde{p}_t)$ has the desired properties:
- It is still a payment-frontloading mechanism, since the allocation and payments from each type vector of length $t$ are replaced by the allocation and payments of another type vector of length $t$, so the agent still has zero utility in all steps except the last.

- It is still (EPIR). Let $\tilde{u}_t(\theta^t)$ be the period utility of mechanism $(\tilde{x}_t, \tilde{p}_t)$. Since it is still payment-frontloading, $\tilde{u}_t(\theta^t) = 0$ for all $t < T$. Therefore, it is enough to argue that $\tilde{u}_T(\theta^T) \geq 0$. By the transformation, there is another type vector $\theta'^{T-1}$ such that

$$\tilde{u}_T(\theta^T) = \theta_t \cdot \tilde{x}_t(\theta^t) - \tilde{p}_t(\theta^t) = \theta_t \cdot x_t(\theta'^{t-1}, \theta_t) - p_t(\theta'^{t-1}, \theta_t) \geq 0,$$

  since the original mechanism is also (EPIR) and payment-frontloading.
- It is still (DIC). For $t > \tau$, the (DIC) condition follows directly from the fact that the original mechanism is (DIC). For $t = \tau$, we use the fact that

$$\theta_\tau = \arg\max_{\hat{\theta}_\tau} u_t(\theta_\tau; \hat{\theta}^\tau) + U_t(\hat{\theta}^\tau) = \arg\max_{\hat{\theta}_\tau} \tilde{u}_t(\theta_\tau; \hat{\theta}^\tau) + \tilde{U}_t(\hat{\theta}^\tau),$$

  where $\tilde{U}_t$ is the continuation utility of the transformed mechanism. This expression holds since (i) we did not change the period utility of period $\tau$, (ii) we were careful to change the mechanism to preserve partially realized utilities, and (iii) since the mechanism was a payment-frontloading mechanism, the partially realized utilities coincide with continuation utilities, so we are also preserving continuation utilities. Finally, for $t < \tau$, we can use the same argument. Since the continuation utilities of period $\tau$ are preserved and the period utilities between periods $t$ and $\tau$ are preserved, the continuation utility of period $t$ is also preserved.
- Condition ($\text{SYM}_t$) holds for $t = \tau$. This condition holds by design.
- Condition ($\text{SYM}_t$) holds for $t \leq \tau - 1$: The condition is clearly true for $s \leq \tau$. For $s < \tau \leq u$, consider two type vectors $\theta'^s$ and $\theta''^s$ with the same continuation utility in the original mechanism. They must also have the same continuation utility in the new mechanism, since we argue that the continuation utilities are preserved for $t \leq \tau$. By the induction hypothesis, the allocation and payments must be the same in the original mechanism for $(\theta'^s, \theta_{s+1}, \ldots, \theta_u)$ and $(\theta''^s, \theta_{s+1}, \ldots, \theta_u)$ for any type vector $\theta_{s+1}, \ldots, \theta_u$. Therefore, $U_\tau(\theta'^s, \theta_{s+1}, \ldots, \theta_\tau) = U_\tau(\theta''^s, \theta_{s+1}, \ldots, \theta_\tau) := z$, which means that both types are in the class, that is, $(\theta'^s, \theta_{s+1}, \ldots, \theta_\tau), (\theta''^s, \theta_{s+1}, \ldots, \theta_\tau) \in S_\tau(z)$. Therefore,

$$\tilde{x}_u(\theta'^s, \theta_{s+1}, \ldots, \theta_u) = x_u(\theta^{*\tau}(z), \theta_{\tau+1}, \ldots, \theta_u) = \tilde{x}_u(\theta''^s, \theta_{s+1}, \ldots, \theta_u),$$
$$\tilde{p}_u(\theta'^s, \theta_{s+1}, \ldots, \theta_u) = p_u(\theta^{*\tau}(z), \theta_{\tau+1}, \ldots, \theta_u) = \tilde{p}_u(\theta''^s, \theta_{s+1}, \ldots, \theta_u).$$

- The expected welfare does not decrease, since we always replace a suffix of the mechanism by one with at least the same expected welfare:

$$\text{SW} = \mathbb{E}_{\theta^\tau}\left[\sum_{t=1}^\tau \theta_t \cdot x_t(\theta^t) + W_\tau(\theta^\tau)\right] \leq \mathbb{E}_{\theta^\tau}\left[\sum_{t=1}^\tau \theta_t \cdot x_t(\theta^t) + W_\tau(\theta^{*\tau}(\bar{U}_\tau(\theta^\tau)))\right]$$

$$= \mathbb{E}_{\theta^\tau}\left[\sum_{t=1}^\tau \theta_t \cdot \tilde{x}_t(\theta^t) + \tilde{W}_\tau(\theta^\tau)\right] = \widetilde{\text{SW}}.$$

- The expected revenue does not decrease, since expected revenue is the difference between expected welfare and expected utility, and we argue that welfare does not decrease, and the expected utility is the same. *Q.E.D.*

PROOF OF LEMMA 3.3: A direct consequence of Lemma A.4 is that we can write $x_t = x_t(\theta_t, \bar{U}_{t-1})$ and $p_t = p_t(\theta_t, \bar{U}_{t-1})$. Additionally, $\bar{U}_t = \bar{U}_t(\theta_t, \bar{U}_{t-1})$ because by the payment-frontloading property,

$$\bar{U}_t = \mathbb{E}_{\theta_{t+1}, \ldots, \theta_T}\left[\sum_{s=t+1}^{T} \theta_s \cdot x_s(\theta_t, \ldots, \theta_s, \bar{U}_{t-1}) - p_s(\theta_t, \ldots, \theta_s, \bar{U}_{t-1}) \Big| \theta_t\right].$$

This allows us to define a bank account mechanism as follows. First, we define the balance

$$b_t^B(\theta^t) = \bar{U}_t(\theta^t) - \mu_t \quad \text{for } t < T \quad \text{and} \quad b_T^B(\theta^T) = -\mu_T = 0,$$

where $\mu_t = \min_{\theta^t} \bar{U}_t(\theta^t)$ for $t < T$ and $\mu_T = 0$. Note that by Jensen's inequality, $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_T = 0$. The allocation is the same as the original mechanism $x_t^B(\theta^t) = x_t(\theta^t)$ and payments are computed as

$$p_t^B(\theta^t) = p_t(\theta^t) - b_t^B(\theta^t) + b_{t-1}^B(\theta^{t-1}).$$

Since there is a one-to-one mapping between $\bar{U}_t$ and $b_t^B$, allocations, payments, and bank account updates can be computed from the previous state of the bank accounts, that is, $x_t^B(\theta^t) = x_t^B(\theta_t, b_{t-1}) = x_t^B(\theta_t, b_{t-1}^B(\theta^{t-1}))$, and the same holds for payments $p_t^B$. Note that we set the payments and the balance such that

$$\theta_t \cdot x_t - p_t + b_t^B = \bar{U}_t - \mu_t. \tag{$\diamond$}$$

This is true because for $t < T$, the per-period utility $\theta_t \cdot x_t - p_t$ is zero since the mechanism is payment frontloading; for $t = T$, $b_T^B = -\mu_T = 0$, and $\theta_T \cdot x_T - p_T = \bar{U}_T$, since by the payment-frontloading property, the agent has nonzero utility only in the last period.

We will use this fact to check that the mechanism is a valid bank account mechanism. First, note that by design, $b_t^B(\theta^t)$ is always nonnegative and $b_0^B = 0$. It remains to check conditions (IC), (BI), and (BU).

Condition (IC) follows from the definition of $p_t^B$ and the fact that the original mechanism is (DIC), since the maximization problem in (IC) becomes the same optimization in (DIC) with an additional constant term. For $t = T$, this is trivial since $b_T^B(\theta^t) = 0$. For $t < T$, we have

$$u_t^B(\theta_t; \theta^{t-1}, \hat{\theta}_t) = \theta_t \cdot x_t^B(\theta^{t-1}, \hat{\theta}_t) - p_t^B(\theta^{t-1}, \hat{\theta}_t)$$
$$= \theta_t \cdot x_t(\hat{\theta}_t, \bar{U}_t) - p_t(\theta^{t-1}, \hat{\theta}_t) + \bar{U}_t(\theta^{t-1}, \hat{\theta}_t) - (\bar{U}_{t-1}(\theta^{t-1}) + \mu_t - \mu_{t-1}),$$

since the term $\bar{U}_{t-1}(\theta^{t-1}) + \mu_t - \mu_{t-1}$ is a constant in $\hat{\theta}_t$ and $\bar{U}_t(\theta^{t-1}, \hat{\theta}_t) = U_t(\theta^{t-1}, \hat{\theta}_t)$ by the payment-frontloading property. To check condition (BI), we apply equation ($\diamond$):

$$\mathbb{E}_{\theta_t}[\theta_t \cdot x_t^B - p_t^B] = \mathbb{E}_{\theta_t}[\theta_t \cdot x_t - p_t + b_t^B - b_{t-1}^B] = \mathbb{E}_{\theta_t}[\bar{U}_t - \mu_t - (\bar{U}_{t-1} - \mu_{t-1})] = \mu_{t-1} - \mu_t \geq 0.$$

This establishes (BI), since the outcome is a constant that depends solely on $t$ and not on the value of $b_t^B$. For condition (BU), we again apply equation ($\diamond$),

$$b_{t-1}^B + \theta_t \cdot x_t - p_t^B = b_{t-1}^B + \theta_t \cdot x_t - p_t + b_t^B - b_{t-1}^B = \bar{U}_t - \mu_t \geq b_t^B,$$

where the last inequality holds with equality for all $t < T$. *Q.E.D.*

## APPENDIX B: DIFFERENT NOTIONS OF IC AND IR

### B.1. *Stronger IR Notions*

The main body of the paper focuses on satisfying (DIC) and (EPIR) and the main design goals. In Lemma 3.2, we argue that bank account mechanisms satisfy even stronger notions. There are various variations over those notions that we can satisfy by slightly changing the mechanism. For example, Lemma 3.2 implies that we satisfy the following notion of expected IR continuation:

$$\mathbb{E}_{\theta_t,\dots,\theta_T}\left[\sum_{\tau=t}^{T} u_\tau\big(\theta_\tau; \theta^\tau\big)\Big|\theta^{t-1}\right] \geq 0 \quad \forall t, \theta^{t-1} \in \Theta^{t-1}.$$

The reader might ask whether it is possible to satisfy the same notion ex post with respect to the $t$th type $\theta_t$. In other words, can we satisfy the notion

$$\mathbb{E}_{\theta_{t+1},\dots,\theta_T}\left[\sum_{\tau=t}^{T} u_\tau\big(\theta_\tau; \theta^\tau\big)\Big|\theta^t\right] \geq 0 \quad \forall t, \theta^t \in \Theta^t?$$

Note that they only differ in the conditioning of the expectations. This can be achieved by any bank account mechanism by changing the payment rule to

$$\hat{p}_t(\theta_t, b) = p_t(\theta_t, b) + b_t(\theta_t, b) - b \quad \text{for } t < T,$$
$$\hat{p}_T(\theta_T, b) = p_T(\theta_T, b) - b.$$

The reader can verify that all properties studied are preserved under this notion. In fact, condition (BU) implies that the previous transformation satisfies the even stronger notion of ex post per-period IR. That is, under the $\hat{p}_t$ payment rules, the mechanism satisfies for all realization of types $\theta^t$ and all periods $t$,

$$u_t\big(\theta_t; \theta^t\big) \geq 0. \tag{PPIR}$$

This transformation almost preserves non-clairvoyance. If the original mechanism was non-clairvoyant, the new mechanism is what we call *quasi-non-clairvoyant*. A quasi-non-clairvoyant mechanism is the one that needs to be told when the last period is in that period so that it can tailor its allocation and payment to the fact that we are in the last period. This is exactly what is required to implement the previous transformation.

We know that there is a mechanism that is per-period individually rational, dynamic incentive compatible, and quasi-non-clairvoyant. Can we obtain the previous combination with actual non-clairvoyance instead of quasi-non-clairvoyance? The answer is unfortunately no.

LEMMA B.1: *Any revenue that can be obtained by a non-clairvoyant mechanism that satisfies* (DIC) *and* (PPIR) *can also be obtained by running a static, individually rational, and incentive compatible auction in each period.*

PROOF: The proof follows directly from Lemma C.1 in the following section, which states that a non-clairvoyant (DIC) mechanism must also satisfy per-period IC. *Q.E.D.*

## B.2. *Stronger IC Notions*

Similarly, we can ask the same question about IC. Can we achieve even stronger notions of IC? For example, can we achieve a version of (DIC) that holds for every realization of types in future periods instead of in expectation over future periods? We call such a mechanism super dynamic incentive compatible:

$$\theta_t = \underset{\hat{\theta}_t}{\arg\max} \sum_{\tau=t}^{T} u_\tau(\theta_\tau; \hat{\theta}^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_\tau) \quad \forall \hat{\theta}^{t-1}, \theta_{t+1}, \ldots, \theta_T. \qquad \text{(sDIC)}$$

Unfortunately, this notion is too strong, as shown in Lemma B.2, which is restated here for convenience.

LEMMA B.2: *Any revenue that can be obtained in a mechanism satisfying* (sDIC) *and* (EPIR) *can be obtained by running a static individually rational and incentive compatible mechanism in each period.*

PROOF: Consider the single-period mechanism with allocation defined by $\hat{x}(\hat{\theta}) = x_1(\hat{\theta})$. By the (sDIC) property, for every $\theta_2, \ldots, \theta_T$, the payment rule $\hat{p}(\hat{\theta}) = p_1(\hat{\theta}) - \sum_{t=2}^{T} u_t(\theta_t; \hat{\theta}, \theta_2, \ldots, \theta_t)$ implements $\hat{x}$. Since the payment rule $\hat{p}$ is determined from $\hat{x}$ up to a constant, the term $\sum_{t=2}^{T} u_t(\theta_t; \hat{\theta}, \theta_2, \ldots, \theta_t)$ must be decomposable into a term that depends only on $\theta_1$ and a term that depends on $\theta_2, \ldots, \theta_T$. Consider

$$\sum_{t=2}^{T} u_t(\theta_t; \hat{\theta}, \theta_2, \ldots, \theta_t) = \alpha(\theta_1) + \beta(\theta_2, \ldots, \theta_T).$$

Since $u^T = u_1(\theta_1; \theta_1) + \alpha(\theta_1) + \beta(\theta_2, \ldots, \theta_T)$ is nonnegative for every type profile, we can adjust $\alpha$ and $\beta$ such that $u_1(\theta_1; \theta_1) + \alpha(\theta_1) \geq 0$ for every $\theta_1$ and $\beta(\theta_2, \ldots, \theta_T) \geq 0$ for every $\theta_2, \ldots, \theta_T$. We can then define the following mechanism.
- For the first period, allocate according to $x_1(\theta_1)$ and charge $p_1(\theta_1) - \alpha(\theta_1)$.
- For all remaining periods, allocate according to $\mathbb{E}_{\theta_1}[x_t(\theta_1, \theta_2, \ldots, \theta_t)]$ and charge $\mathbb{E}_{\theta_1}[p_t(\theta_1, \theta_2, \ldots, \theta_t)]$ with an additional payment of $\mathbb{E}_{\theta_1}[\alpha(\theta_1)]$ in the last period only.

We obtain a mechanism that is single-period incentive compatible and individually rational for the first period and a mechanism satisfying (sDIC) and (EPIR) for periods 2 to $T$. Note that the revenue remains the same.

By induction, we can find a mechanism that runs a static auction in each period and has the same revenue as the original mechanism.                    *Q.E.D.*

## APPENDIX C: PROOF OF THE NON-CLAIRVOYANCE GAP THEOREMS

### C.1. *Characterization of Non-Clairvoyant Mechanisms*

We start by proving the following strong property of non-clairvoyant mechanisms.

LEMMA C.1: *If $x_t(F^t, \theta^t)$, $p_t(F^t, \theta^t)$ represent a non-clairvoyant mechanism satisfying* (DIC) *and $U_{t,T}(F^T, \theta^t)$ for $t < T$ is the continuation utility of the corresponding clairvoyant*

*mechanism*,

$$U_{t,T}\big(F^T, \theta^t\big) = \mathbb{E}_{\theta_{t+1},\ldots,\theta_T \sim F_{t+1},\ldots,F_T} \left[ \sum_{s=t+1}^{T} \theta_s \cdot x_s\big(F^s, \theta^s\big) - p_s\big(F^s, \theta^s\big) \right],$$

*then $U_{t,T}(F^T, \theta^t)$ does not depend on $\theta^t$, that is, $U_{t,T}(F^T, \theta^t) = U_{t,T}(F^T, \theta'^t)$.*

PROOF: Fix $F^T$ and $\theta^t$. First, we show that $U_{t,T}(F^T, \theta^{t-1}, \hat{\theta}_t)$ does not depend on $\hat{\theta}_t$. Define the single-period mechanism for a buyer with valuation $\hat{\theta}_t \sim F_t$ that allocates according to $\hat{x}_t(\hat{\theta}_t) = x_t(F^t, \theta^{t-1}, \hat{\theta}_t)$ and charges payments according to $\hat{p}(\hat{\theta}_t) = p_t(F^t, \theta^{t-1}, \hat{\theta}_t) - U_{t,T}(F^T, \theta^{t-1}, \hat{\theta}_t)$. By the fact that the dynamic mechanism satisfies (DIC), this mechanism must be incentive compatible, so the payment rule is uniquely defined by the allocation rule up to a constant. Define an alternative payment rule $p'_t(\hat{\theta}_t) = p_t(F^t, \theta^{t-1}, \hat{\theta}_t)$. The mechanism defined by $\hat{x}_t$, $p'$ must also be incentive compatible since the clairvoyant mechanism corresponding to the prior distribution sequence $F^t$ is also (DIC). Since these are two single-period incentive compatible mechanisms with the same allocation rule, the payment rule must differ by a constant. Thus, the difference $U_{t,T}(F^T, \theta^{t-1}, \hat{\theta}_t)$ cannot depend on $\hat{\theta}_t$.

We use induction to show that $U_{t,T}(F^T, \theta^{t-1}, \hat{\theta}_t)$ does not depend on $\theta_{t-1}$. Since we know that $U_{t,T}$ does not depend on $\theta_t$, we indicate it by writing $U_{t,T}(F^T, \theta^{t-1})$. By definition,

$$U_{t-1,T}\big(F^T, \theta^{t-2}\big) = U_{t-1,t}\big(F^t, \theta^{t-2}\big) + \mathbb{E}_{\theta_t \sim F_t}\big[U_{t,T}\big(F^T, \theta^{t-1}\big)\big].$$

Since the last term does not depend on $\theta_t$, we can remove the expectation:

$$U_{t,T}\big(F^T, \theta^{t-1}\big) = U_{t-1,T}\big(F^T, \theta^{t-2}\big) - U_{t-1,t}\big(F^t, \theta^{t-2}\big).$$

Hence, $U_{t,T}(F^T, \theta^{t-1})$ does not depend on $\theta_{t-1}$. Repeating the same argument, we can show that $U_{t,T}$ depends only on the distributions $F^T$. Q.E.D.

To prove Lemma 5.2, we first prove a symmetrization lemma in the style of Lemma A.4. There are some important differences: Instead of the partially realized utility used in Lemma A.4, we will use the utility observed to date, which is a quantity we have access to in non-clairvoyant mechanisms since it only involves the past. The second major difference is that it will not involve payment frontloading, since we have no access to the future. The reason that we can do without those is the stronger property satisfied by the non-clairvoyant mechanism described in Lemma C.1.

LEMMA C.2—Non-Clairvoyant Symmetrization: *Given a non-clairvoyant dynamic mechanism $x_t(F^t, \theta^t)$, $p_t(F^t, \theta^t)$, there is a non-clairvoyant mechanism $\tilde{x}_t(F^t, \theta^t)$, $\tilde{p}_t(F^t, \theta^t)$ with the same revenue for each sequence of prior distributions, that is, for each $F^T$,*

$$\mathbb{E}_{\theta^t \sim F^t} \left[ \sum_{t=1}^{T} p_t\big(F^t, \theta^t\big) \right] = \mathbb{E}_{\theta^t \sim F^t} \left[ \sum_{t=1}^{T} \tilde{p}_t\big(F^t, \theta^t\big) \right],$$

*satisfying the following symmetry property: If $\sum_{s=1}^{t} \tilde{u}_s(F^s, \theta^s) = \sum_{s=1}^{t} \tilde{u}_s(F^s, \theta'^s)$, then*

$$\tilde{x}_{t'}\big(F^t, F_{t+1}, \ldots, F_{t'}, \theta^t, \theta_{t+1}, \ldots, \theta_{t'}\big) = \tilde{x}_{t'}\big(F^t, F_{t+1}, \ldots, F_{t'}, \theta'^t, \theta_{t+1}, \ldots, \theta_{t'}\big),$$

$$\tilde{p}_{t'}\big(F^t, F_{t+1}, \ldots, F_{t'}, \theta^t, \theta_{t+1}, \ldots, \theta_{t'}\big) = \tilde{p}_{t'}\big(F^t, F_{t+1}, \ldots, F_{t'}, \theta'^t, \theta_{t+1}, \ldots, \theta_{t'}\big).$$

PROOF: To prevent notations from becoming too verbose, we define $u^t(F^t, \theta^t) = \sum_{s=1}^{t} u_s(F^s, \theta^s)$. Assume that the symmetric property holds for $t < \tau$. We will construct a mechanism for which the symmetric property holds for any $t \leq \tau$. Define $\tilde{x}_t$ and $\tilde{p}_t$ as follows. For $t \leq \tau$, let $\tilde{x}_t = x_t$ and $\tilde{p}_t = p_t$. For $t > \tau$, define

$$\tilde{x}_t(F^t, \theta^t) = \mathbb{E}_{\theta'^\tau \sim F^\tau}\big[x_t(F^t, \theta'^\tau, \theta_{\tau+1}, \ldots, \theta_t)|u^\tau(F^\tau, \theta^\tau) = u^\tau(F^\tau, \theta'^\tau)\big],$$

$$\tilde{p}_t(F^t, \theta^t) = \mathbb{E}_{\theta'^\tau \sim F^\tau}\big[p_t(F^t, \theta'^\tau, \theta_{\tau+1}, \ldots, \theta_t)|u^\tau(F^\tau, \theta^\tau) = u^\tau(F^\tau, \theta'^\tau)\big].$$

In other words, we replace the allocation and payments in periods $t > \tau$ by the expected allocation and payments for types $\theta'^\tau, \theta_{\tau+1}, \ldots, \theta_t$ such that the total utility accrued by the buyer in periods $1, \ldots, \tau$ is the same as for $\theta^\tau$. We argue that this mechanism still has the desired properties:

- It is still non-clairvoyant: This is clear by construction since at period $t$, the mechanism is only a function of $F^t$ and $\theta^t$. Note that it is crucial that we symmetrize using a quantity that we can measure with information available at period $t$.
- It is still (EPIR). To check this property, let $\tilde{u}_t$ be the utility under the new mechanism; then, if $E$ is the event that $u^\tau(F^\tau, \theta^\tau) = u^\tau(F^\tau, \theta'^\tau)$,

$$\tilde{u}^T(F^T, \theta^T) = u^\tau(F^\tau, \theta^\tau) + \mathbb{E}_{\theta'^\tau}\left[\sum_{s=\tau+1}^{T} u_s(F^s, \theta'^\tau, \theta_{\tau+1}, \ldots, \theta_s)\Big| E\right]$$

$$= \mathbb{E}_{\theta'^\tau}\left[u^\tau(F^\tau, \theta'^\tau) + \sum_{s=\tau+1}^{T} u_s(F^s, \theta'^\tau, \theta_{\tau+1}, \ldots, \theta_s)\Big| E\right] \geq 0.$$

- It is still (DIC). The (DIC) condition holds for $t > \tau$ since at that point the mechanism is simply a distribution of mechanisms satisfying the (DIC) condition. For $t \leq \tau$, we will use Lemma C.1 to argue that the expression in the maximization problem remains the same. In the following expression, we omit $F^t$ for clarity of presentation:

$$\tilde{u}_t(\theta^t) + \tilde{U}_t(\theta^t) = u_t(\theta^t) + \mathbb{E}\left[\sum_{s=t+1}^{\tau} u_s(\theta^s)\right] + \mathbb{E}_{\theta'^t}[U_\tau(\theta'^\tau)|E(\theta^\tau)],$$

where $E(\theta^\tau)$ is the event determining the set of $\theta'^\tau$ on which we will condition. This event is a function of $\theta^\tau$. However, by Lemma C.1, $U_\tau$ is a constant, so the expectation and the event we are conditioning on are irrelevant; therefore, we have

$$\tilde{u}_t(\theta^t) + \tilde{U}_t(\theta^t) = u_t(\theta^t) + \mathbb{E}\left[\sum_{s=t+1}^{\tau} u_s(\theta^s)\right] + U_\tau(\theta'^\tau) = u_t(\theta^t) + U_t(\theta^t).$$

- The symmetry condition holds for $t = \tau$ by design.
- The symmetry condition holds for $t < \tau$ using an argument analogous to that used in Lemma C.1.

- The expected revenue is the same for the following reasons (again, we omit $F^T$):

$$\mathbb{E}_{\theta^T}\left[\sum_{t=1}^{T}\tilde{p}_t(\theta^t)\right]$$

$$= \mathbb{E}_{\theta^\tau}\left[\sum_{t=1}^{\tau}p_t(\theta^\tau)\right] + \mathbb{E}_{\theta^T}\mathbb{E}_{\theta'^\tau}\left[\sum_{t=\tau+1}^{T}p_t(\theta'^\tau, \theta_{\tau+1}, \ldots, \theta_T)\Big|u^\tau(\theta^\tau) = u^\tau(\theta'^\tau)\right].$$

which equals the original revenue since the distributions of $\theta^\tau$ and $\theta'^\tau$ are the same.

*Q.E.D.*

The symmetrization condition is the main ingredient to show that all non-clairvoyant mechanisms are bank account mechanisms. The reader is invited to contrast how much simpler this proof is than the proof of its clairvoyant counterpart. In a sense, Lemma C.1 already provides us with most of the proof.

PROOF OF LEMMA 5.2: Assume that $x_t$ and $p_t$ satisfy the conditions in the non-clairvoyant symmetrization lemma (Lemma C.1). Define the bank balance as $b_t(F^t, \theta^t) = \sum_{s=1}^{t}u_t(F^t, \theta^t)$. From symmetrization, it is clear that $x_t$, $p_t$ can be written as a bank account mechanism. The (BI) condition follows directly from Lemma C.1. With the current definition of bank accounts, condition (BU) becomes trivial: The first inequality follows from (EPIR) and the second holds with equality. *Q.E.D.*

## C.2. *Lower Bound for Non-Clairvoyant Mechanisms*

In this section, we will prove Theorem 5.1. To do so, let us initially define two distributions defined by their cumulative density functions and parameterized by a constant $\mu > 0$ to be defined later:

$$F_1(\theta) = (1 - e^{-\mu^2})\frac{\theta\mu}{\theta\mu + 1} \quad \text{for } \theta \leq e^{\mu^2} \quad \text{and} \quad F_1(\theta) = 1 \quad \text{otherwise,}$$

$$F_2(\theta) = \left[1 - \frac{\epsilon}{\theta}\right]^+.$$

We will consider two scenarios: In the first, there is a single item with distribution $F_1$, and in the second, there are two items, the first with distribution $F_1$ and the second with distribution $F_2$. It is instructive to start by computing the optimal clairvoyant dynamic mechanism in each of the settings. By Lemma 3.3, we can restrict our attention to bank account mechanisms.

SCENARIO 1—One item with distribution $F_1$: Since there is only one period, the optimal mechanism is Myerson's auction. For the single-buyer case, it can be described as the posted-price mechanism at $\rho$ maximizing $\rho(1 - F_1(\rho))$, which is $\rho = e^{\mu^2}$, and the revenue is

$$\text{REV}^*(F_1) = \rho(1 - F_1(\rho)) = 1 + \frac{1}{\mu} + O(e^{-\mu^2}).$$

SCENARIO 2—Two items with distributions $F_1$ and $F_2$: Since the optimal mechanism can be described as a bank account mechanism, assume that $x_t, p_t$ is the optimal bank account mechanism. By condition (BU), the state of the bank account at the end of period 1 is at most $u_1$, which is at most $e^{\mu^2}$. The mechanism in the second period can be described as spending some amount that is at most the balance from the account and running an incentive compatible and individually rational mechanism. Since the distribution $F_2$ is such that $\rho(1 - F_2(\rho)) = \epsilon$ for all $\rho$ (i.e., it is an equal-revenue distribution), the revenue obtained from the second period is at most $b_1 + \epsilon \leq u_1 + \epsilon$. Therefore, the total revenue is at most the welfare of the first period plus $\epsilon$. In other words, an upper bound to optimal revenue is $\mathbb{E}_{\theta_1 \sim F_1}[\theta_1] + \epsilon$.

We present a mechanism that achieves that much revenue. In the first period, the item is given for free to the buyer and we add her value of the item to her bank account. In the second period, we first spend the entire balance of the bank account and then post a price $p(b_1)$ satisfying condition (BI). Regardless of the price we post, the revenue will be $b_1 + \epsilon$. Therefore, the expected revenue of this mechanism is

$$\text{REV}^*(F_1, F_2) = \mathbb{E}_{\theta_1 \sim F_1}[\theta_1] + \epsilon = 1 + \mu + \epsilon + O\big(\mu e^{-\mu^2}\big).$$

*Comparison of the Two Scenarios.*   We note that depending on whether there will be a second item, two entirely different approaches are taken for the first item. If there is no second item, we allocate the second item with very low probability and charge a very high price if it is allocated. If there is a second item, we always allocate the first item and charge nothing for it. A non-clairvoyant mechanism must attempt to balance those two extremes: It needs to allocate the first item such that if there is no second item, the revenue is sufficient compared to the optimal single-item auction. However, it also needs to ensure that the bank balance after the first period is large enough to allow for greater freedom in allocating the second item.

*Non-Clairvoyant Mechanism.*   Consider a non-clairvoyant mechanism and let $x_1(F_1, \theta_1)$, $p_1(F_1, \theta_1)$ be the auction for the first item with distribution $F_1$. This auction must be incentive compatible and individually rational, so it must be a distribution over posted-price mechanisms, say a random posted price $\rho \sim G$. Therefore,

$$\text{REV}^M(F_1) = \mathbb{E}_{\rho \sim G}\big[\rho\big(1 - F_1(\rho)\big)\big],$$

and since every non-clairvoyant mechanism can be written as a bank account mechanism (Lemma 5.2), we can use the same argument as in Scenario 2 above to argue that

$$\text{REV}^M(F_1, F_2) \leq \epsilon + \mathbb{E}_{\rho \sim G}\big[\mathbb{E}[\theta_1 \cdot \mathbf{1}_{\theta_1 \geq \rho}]\big].$$

We are ready to prove the lower bound theorem.

PROOF OF THEOREM 5.1: Assume that the non-clairvoyant mechanism is an $\alpha$-approximation to the clairvoyant benchmark, and consider the setup with $F_1$ and $F_2$ de-

scribed in this section. Then

$$
\frac{2}{\alpha} = 2\min\left(\frac{\text{REV}^M(F_1)}{\text{REV}^*(F_1)}, \frac{\text{REV}^M(F_1, F_2)}{\text{REV}^*(F_1, F_2)}\right) \le \frac{\text{REV}^M(F_1)}{\text{REV}^*(F_1)} + \frac{\text{REV}^M(F_1, F_2)}{\text{REV}^*(F_1, F_2)}
$$

$$
\le \mathbb{E}_{\rho \sim G}\big[\beta(\rho)\big] \le \max_{\rho}\big[\beta(\rho)\big],
$$

$$
\text{where } \beta(\rho) = \frac{\rho\big(1 - F_1(\rho)\big)}{\text{REV}^*(F_1)} + \frac{\epsilon + \mathbb{E}[\theta_1 \cdot \mathbf{1}_{\theta_1 \ge \rho}]}{\text{REV}^*(F_1, F_2)}.
$$

The remainder of the proof is calculus heavy[1], and involves explicitly substituting the values of those expressions and evaluating the maximum of $\beta(\rho)$. Taking the limit as $\mu \to \infty$ will provide us the desired bound.

Denote $r_1 = 1/\text{REV}^*(F_1)$ and $r_{12} = 1/\text{REV}^*(F_1, F_2)$. Then

$$
\beta(\rho) = r_1 \rho\big(1 - F_1(\rho)\big) + r_{12}\left(\epsilon + \int_{\rho}^{e^{\mu^2}} \theta \, dF_1(\theta)\right).
$$

Taking the derivative of $\beta$,

$$
\beta'(\rho) = r_1\big(1 - F_1(\rho) - \rho F_1'(\rho)\big) - r_{12}\rho F_1'(\rho)
$$

$$
= r_1 - \big(1 - e^{\mu^2}\big)r_1 \frac{\rho\mu}{\rho\mu + 1} - (r_1 + r_{12})\big(1 - e^{\mu^2}\big)\frac{\rho\mu}{(\rho\mu + 1)^2}.
$$

Denoting $\zeta = 1 - e^{\mu^2}$ and letting $\beta'(\rho) = 0$,

$$
r_1(1 - \zeta)(\rho\mu + 1)^2 - r_{12}\zeta(\rho\mu + 1) + (r_1 + r_{12})\zeta = 0
$$

$$
\implies \quad \rho\mu + 1 = \frac{r_{12}\zeta}{2r_1(1 - \zeta)}\left(1 \pm \sqrt{1 - \frac{4r_1(1 - \zeta)}{r_{12}\zeta}\left(1 + \frac{r_1}{r_{12}}\right)}\right).
$$

Since

$$
\frac{r_1}{r_{12}} = \frac{\text{REV}^*(F_1, F_2)}{\text{REV}^*(F_1)} = \frac{1 + \mu + \epsilon + O\big(\mu e^{-\mu^2}\big)}{1 + 1/\mu + O\big(e^{-\mu^2}\big)} = \mu + \epsilon + o(1),
$$

$\frac{4r_1(1 - \zeta)}{r_{12}\zeta}(1 + \frac{r_1}{r_{12}}) \approx 4\mu^2 e^{-\mu^2} \ll 1$. Hence, $\beta'(\rho) = 0$ has two roots. Because $\beta'(0) = r_1 > 0$, the local maximum of $\beta(\rho)$ is reached at the smaller root:

$$
\rho^*\mu + 1 = \frac{r_{12}\zeta}{2r_1(1 - \zeta)}\left(1 - \left(1 - \frac{4r_1(1 - \zeta)}{r_{12}\zeta}\left(1 + \frac{r_1}{r_{12}}\right) + o\big(e^{-\mu^2}\big)\right)\right)
$$

$$
= 1 + \mu + \epsilon + o(1)
$$

$$
\implies \quad \rho^* = 1 + o(1).
$$

---

[1]We will omit some less important calculation details and Taylor expansion will be repeatedly used.

Therefore, the maximum value is reached at either $\rho^*$ or $e^{\mu^2}$, and for sufficiently large $\mu$,

$$\max_{\rho} \beta(\rho) = \max\big(\beta(\rho^*), \beta(e^{\mu^2})\big) = 1 + 1/\mu + o(1/\mu) \leq 1 + 2/\mu.$$

Hence, the lower bound of $\alpha$ is obtained, which is 2 as $\mu \to \infty$:

$$\alpha \geq \frac{2}{\max\limits_{\rho} \beta(\rho)} \geq \frac{2}{1 + 2/\mu}. \qquad\qquad Q.E.D.$$

## APPENDIX D: DETAILS OMITTED FROM MULTIPLE BUYERS

### D.1. *Multiple-Buyer Dynamic Mechanism Design*

We start by extending the concepts in the paper to multiple buyers. Consider a set $N$ of $n$ agents who participate in the mechanism for $T$ periods. For each agent $i \in N$ and each $t \in \{1, \ldots, T\}$, the type $\theta_t^i$ of agent $i$ in period $t$ is drawn independently from a distribution $F_t^i$. When we omit the superscript $i$, we refer to the vector of types $\theta_t = (\theta_t^1, \ldots, \theta_t^n)$. As is usual in mechanism design, we refer to $\theta_t^{-i}$ as the vector of types of all agents except $i$. Agent $i$ has a value $v^i : \Theta \times [0, 1] \to \mathbb{R}_+$. A dynamic mechanism corresponds to pairs of maps,

- outcome, $x_t : \Theta^{tN} \times (\Delta\Theta)^{TN} \to [0, 1]$,
- payment, $p_t : \Theta^{tN} \times (\Delta\Theta)^{TN} \to \mathbb{R}$.

Similar to the single-buyer case, we can define the notion of continuation utility $U_t^i(\hat{\theta}^t; F^T)$ as the expected total utility of a buyer in periods $t + 1$ to $T$ if her history of reports up to period $t$ is $\hat{\theta}^t$ and all the buyers report truthfully from period $t + 1$ onward. This allows us to define the analogue of condition (DIC) for multiple buyers, which we call dynamic Bayesian incentive compatibility. We call it Bayesian since each buyer takes expectations over the behavior of all other buyers and assumes they bid truthfully. The condition can be written as

$$\theta_t^i = \arg\max_{\hat{\theta}_t^i} \mathbb{E}_{\theta_t^{-i}}\big[u_t^i\big(\theta_t^i; \hat{\theta}^{t-1}, (\theta_t^{-i}, \hat{\theta}_t^i); F^T\big) + U_t^i\big(\hat{\theta}^{t-1}, (\theta_t^{-i}, \hat{\theta}_t^i); F^T\big)\big] \tag{DBIC}$$

$$\forall i \in [n], t \in [T], \hat{\theta}^{t-1}, \theta_t^i \in \Theta^t,$$

Recall that while the condition (DIC) for a single buyer can be justified by the dynamic version of the revelation principle, no such equivalence can be obtained for multiple buyers. What we have here is ex post IC: It is optimal for a buyer to report her type truthfully as long as all the other buyers also do so. We refer to Athey and Segal (2013) or Pavan, Segal, and Toikka (2014) for a discussion of the relation between IC in dynamic settings and the revelation principle, as well as Mookherjee and Reichelstein (1992) and Bergemann and Morris (2005) for the comparison of dominant strategy implementation, ex post implementation, and Bayesian implementation.

The condition (EPIR) is generalized in the natural way. Every buyer derives nonnegative utility on every sample path if she is behaving truthfully.

The notion of non-clairvoyance again corresponds to the restriction that the allocation and payment functions at time $t$ must depend only on $(\theta^t, F^t)$, that is, cannot depend on distributional knowledge of future periods.

## D.2. *Multiple-Buyer Bank Account Mechanisms*

We define a bank account mechanism for $n$ buyers as follows.

- A static single-period mechanism $x_t^B(\theta_t, b)$, $p_t^B(\theta_t, b)$ parameterized by an $n$-dimensional bank balance $b \in \mathbb{R}_+^n$ that is single-period Bayesian incentive compatible, that is, satisfies the multiple-buyer version of (IC) and the multiple-buyer version of (BI):

$$\mathbb{E}_{\theta_t}\big[\theta_t^i \cdot x_t^B(\theta_t, b) - p_t^B(\theta_t, b)\big] \text{ is a nonnegative constant not depending on } b.$$

- A balance update policy $b_t^B(\theta_t, b)$ satisfying a multiple-buyer equivalent of condition (BU):

$$0 \le b_t^{B,i}(\theta_t, b) \le b^i + u_t^{B,i}(\theta_t, b) \quad \text{and} \quad b_0^i = 0.$$

As before, it is useful to define a notion of spend $s_t^i$ as

$$s_t^i(b_{t-1}) = \Big[-\min_{\theta_t^i} \mathbb{E}_{\theta_t^{-i}}\big[\theta_t^i \cdot x_t(\theta_t, b_{t-1}) - p_t^i(\theta_t, b_{t-1})\big]\Big]^+.$$

Both the clairvoyant (Lemma 3.3) and non-clairvoyant (Lemma 5.2) reductions still hold in the multiple-buyer setting with essentially the same proofs by adapting the notation.

## D.3. *Proof of Theorem 6.1*

In addition to the notion of the spend, it will also be useful to define an auxiliary notion called the deposit,

$$d_t^i(\theta_t, b_{t-1}) = b_t^i(\theta_t, b_{t-1}) - b_{t-1}^i + s_t^i(b_{t-1}),$$

so that we can describe the balance update policy in terms of the spend and deposit:

$$b_t^i = b_{t-1}^i + d_t^i - s_t^i.$$

In particular, if we write $p_t^i$ as $p_t^i = p'^i_t + s_t^i$ and $u_t^i = u'^i_t - s_t^i$, then the (BU) condition can be rewritten as

$$d_t^i \le u'^i_t.$$

PROOF OF THEOREM 6.1: Fix a time horizon $T$ and distributions $F_t^i$ for $t = 1, \dots, T$ and $i = 1, \dots, n$. Let $(x^*, p^*)$ be the optimal clairvoyant mechanism for this setting. By the multiple-buyer version of Lemma 3.3, we can write the bank account mechanism in terms of a spend policy $s_t^*$, a deposit policy $d_t^*$, and an incentive compatible and individually rational payment function $p'^*_t$ such that:

$$p_t^{*i} = p'^{*i}_t + s_t^{*i}, \qquad b_t^{*i} = b_{t-1}^{*i} - s_t^{*i} + d_t^{*i}.$$

Similarly, let $x_t$, $p'_t$, $s_t$, and $d_t$ describe the NONCLAIRVOYANTBALANCE mechanism where the spend term corresponds to the expected utility of the money-burning component.

*Step 1: Bounding $p'^*$ Using the Myerson Component.*    Our first observation is that since for each period $x_t^*$, $p_t'^*$ is individually rational and Bayesian incentive compatible, its revenue must be dominated by the Myerson auction: $\mathbb{E}_{\theta_t}[\sum_i p_t'^{*i}(\theta^t)] \leq \mathbb{E}_{\theta_t}[\sum_i p_t^{M,i}(\theta_t)]$. This already tells us that the revenue we obtain from selling the 1/5 fraction of each item using Myerson's auction dominates within a factor of 5 the $\mathbb{E}[\sum_{i,t} p_t'^{*i}]$ component of the revenue of the optimal clairvoyant mechanism.

*Step 2: Lower Bound on the Balance of the Non-Clairvoyant Mechanism.*    We are left to show that the remaining component $\mathbb{E}[\sum_{i,t} s_t^{*i}]$ of the revenue of the optimal clairvoyant mechanism is dominated by the combination of the second-price auction and the money-burning auction within a factor of 5. We will show by induction that for every fixed sequence of types and for all buyers $\theta^T$, the following invariant holds. Since the types for all buyers are fixed for all periods, we omit the type vectors in the notation

$$b_t^i + \sum_{\tau=1}^t s_\tau^i \geq \frac{2}{5}\left(b_t^{*i} + \sum_{\tau=1}^t s_\tau^{*i} - \sum_{\tau=1}^t \theta_\tau^{(2)} x_\tau^{*i}\right), \tag{D.1}$$

where $\theta_\tau^{(2)}$ is the second-highest type. This is true for $t=0$ since both balances are initially zero. Assume that it is valid for $t$. Then substituting the balance update formula $b_{t+1}^i = b_t^i - s_{t+1}^i + d_{t+1}^i$ for both the non-clairvoyant and the clairvoyant mechanism, we obtain

$$b_{t+1}^i + \sum_{\tau=1}^{t+1} s_\tau^i - d_{t+1}^i \geq \frac{2}{5}\left(b_{t+1}^{*i} + \sum_{\tau=1}^{t+1} s_\tau^{*i} - \sum_{\tau=1}^t \theta_\tau^{(2)} x_\tau^{*i} - d_{t+1}^{*i}\right).$$

By (BU), $d_{t+1}^{*i} \leq u_{t+1}'^{*i} \leq \theta_{t+1}^i x_{t+1}^{*i}$. If $i$ is not the agent with the highest type, then $\theta_{t+1}^i \leq \theta_{t+1}^{(2)}$, and this step is completed given the fact that $d_{t+1}^i \geq 0$ and $\theta_{t+1}^{(2)} x_{t+1}^{*i} \geq d_{t+1}^{*i}$. If $i$ is the agent with the highest type, then

$$d_{t+1}^i = \frac{2}{5}\left(\theta_{t+1}^i - \theta_{t+1}^{(2)}\right) \geq \frac{2}{5}\left(\theta_{t+1}^i - \theta_{t+1}^{(2)}\right) x_{t+1}^{*i} \geq \frac{2}{5}\left(d_{t+1}^{*i} - \theta_{t+1}^{(2)} x_{t+1}^{*i}\right),$$

since we only deposit for the top agent in the second-price auction mechanism. By substituting this bound, we obtain the invariant for $t+1$.

*Step 3: Charging Scheme for Spend.*    We will construct a charging scheme to re-attribute the spend in the non-clairvoyant mechanism such that it better resembles the spend of the optimal clairvoyant mechanism. For each fixed $\theta^T$, we will define a charging scheme $c_t^i \geq 0$ such that for each period $t$, we have $\sum_i c_t^i \leq \sum_i s_t^i$. We will do so in such a way that we can more easily compare $s_t^{*i}$ with $c_t^i$.

We know by (BI) that there is a solution to the money-burning problem in period $t$ with $\mathbb{E}[\tilde{u}_t^i] \geq s_t^{*i}$, since the clairvoyant mechanism with balance $b_{t-1}^*$ provides such a solution. Thus, by rescaling the mechanism, there must be a solution to the money-burning problem with constraints $\mathbb{E}[\tilde{u}_t^i] \leq \frac{5}{2} b_{t-1}^i$ such that $\mathbb{E}[\tilde{u}_t^i] = \min(s_t^{*i}, \frac{5}{2} b_{t-1}^i)$. In particular, it means

$$\sum_i s_t^i \geq \frac{2}{5} \sum_i \min\left(s_t^{*i}, \frac{5}{2} b_{t-1}^i\right).$$

This motivates us to define the charging scheme

$$c_t^i = \min\left(\frac{2}{5}s_t^{*i}, b_{t-1}^i\right).$$

Based on how we compute the charge, we divide the set of agents in each period into a set $A_t$ of agents ahead and a set $B_t$ of agents behind. We say that agent $i$ is behind ($i \in B_t$) if $b_{t-1}^i \le \frac{2}{5}s_t^{*i}$, and we say that $i$ is ahead ($i \in A_t$) otherwise. For $i \in B_t$, we can produce a good bound on the total spend using (D.1):

$$c_t^i = b_{t-1}^i \ge \frac{2}{5}\left(b_{t-1}^{*i} + \sum_{\tau=1}^{t-1} s_\tau^{*i} - \sum_{\tau=1}^{t-1} \theta_\tau^{(2)} x_\tau^{*i}\right) - \sum_{\tau=1}^{t-1} s_\tau^i.$$

Reorganizing the expression and using that $s_t^{*i} \le b_{t-1}^{*i}$, we obtain

$$c_t^i + \sum_{\tau=1}^{t-1} s_\tau^i + \frac{2}{5}\sum_{\tau=1}^{t-1} \theta_\tau^{(2)} x_\tau^{*i} \ge \frac{2}{5}\sum_{\tau=1}^{t} s_\tau^{*i}. \tag{D.2}$$

A similar bound applies to an ahead agent $i \in A_t$. Let $t'$ be the last period before $t$ where $i \in B_{t'}$. This is well defined since all agents are behind in period zero. Therefore, (D.2) holds for $t'$. Therefore, we can sum $\sum_{\tau=t'+1}^{t} c_\tau^i \ge \frac{2}{5}\sum_{\tau=t'+1}^{t} s_\tau^{*i}$ to that bound and obtain

$$\sum_{\tau=1}^{t'-1} s_\tau^i + \sum_{\tau=t'}^{t} c_\tau^i + \frac{2}{5}\sum_{\tau=1}^{t'-1} \theta_\tau^{(2)} x_\tau^{*i} \ge \frac{2}{5}\sum_{\tau=1}^{t} s_\tau^{*i}. \tag{D.3}$$

*Step 4: Bounding the Spend of the Non-Clairvoyant Mechanism.* If either $i \in B_t$ (D.2) or $i \in A_t$ (D.3), we can bound the spends as

$$\sum_{\tau=1}^{t} s_\tau^i + \sum_{\tau=1}^{t} c_\tau^i + \frac{2}{5}\sum_{\tau=1}^{t} \theta_\tau^{(2)} x_\tau^{*i} \ge \frac{2}{5}\sum_{\tau=1}^{t} s_\tau^{*i}.$$

Summing over all agents $i$ and using the fact that $\sum_i c_t^i \le \sum_i s_t^i$, we have

$$2\sum_i \sum_{\tau=1}^{T} s_\tau^i + \frac{2}{5}\sum_{\tau=1}^{t} \theta_\tau^{(2)} \ge \frac{2}{5}\sum_i \sum_{\tau=1}^{t} s_\tau^{*i}.$$

Dividing the expression by 2, we see that the sum of total spends of the non-clairvoyant mechanism together with the revenue obtained from the second-price auction component gives us a five-approximation to the total spend of the optimal clairvoyant mechanism.
                                                                                    *Q.E.D.*

## D.4. *Other Omitted Proofs From Section 6*

PROOF OF LEMMA 6.2: By the (BI) property, the expected utility in subsequent rounds is not a function of the current reported type, so it is enough to argue that the three components of the NONCLAIRVOYANTBALANCE mechanism are dominant strategy incentive compatible in the static sense. This is trivial to check for the second-price and Myerson components. For the money-burning auction, we refer the reader to Appendix E, where we discuss how to construct this component.                                                *Q.E.D.*

PROOF OF THEOREM 6.3: The proof is almost implied by the arguments we made in the proof of Theorem 6.1. Since there are only two periods in total, we have the following characteristics:

- The spend of the clairvoyant mechanism in the first period is zero: $\sum_i s_1^{*i} = 0$. Therefore, the non-clairvoyant mechanism does not lose any spend for not including money-burning auction in the first period.
- The total spend only depends on the balance from the first period ($b_1$). Therefore, the non-clairvoyant mechanism does not lose any spend for not including the second-price auction in the second period.
- The total spend only comes from the second period. Hence, the money-burning auction is optimal in the spend.

By combining these observations, we can conclude that for any type vector sequence $\theta^2$,

$$\theta_1^{(2)} + \sum_i s_1^i + s_2^i \geq \frac{1}{2} \sum_i s_1^{*i} + s_2^{*i}.$$

Combining this with the fact that the non-clairvoyant mechanism sells half of the item via Myerson's auction, we conclude that it is a non-clairvoyant two-approximation.    *Q.E.D.*

## APPENDIX E: IMPLEMENTATION OF NONCLAIRVOYANTBALANCE

Here we show that all three components of the NONCLAIRVOYANTBALANCE mechanism are simple auctions: Each of them corresponds to maximizing some notion of virtual values.

The first component of the NONCLAIRVOYANTBALANCE mechanism is a second-price auction that does not use any information about the distribution, and the virtual value is simply the buyer's value. The second component is the Myerson auction, which, of course, is a virtual value maximizer.

Most of our work will focus on arguing that the third component—the money-burning auction with utility constraints—has a simple format and can be implemented as a virtual value maximizer. In what follows, for ease of presentation, we focus on discrete distributions. Assume therefore that the space of valuation functions is a finite set of nonnegative numbers, that is, $\Theta = \{\theta_1, \ldots, \theta_K\} \subset \mathbb{R}_+$. As we focus on a single period, we ignore the subscript $t$. Instead, $\theta_j$ will refer to the $j$th value in support of the distribution. As before, let $n$ be the number of buyers. The distributions $F^i$ will be discrete distributions represented by a vector of $K$ nonnegative numbers $f^i(\theta_1), \ldots, f^i(\theta_K)$ summing to 1. We will also denote the cumulative density function of the distribution by $F^i(\theta) = \sum_{\theta_j \leq \theta} f^i(\theta_j)$.

### E.1. *Optimal Money Burning With Caps Is a Scaled Virtual Value Maximizer*

Since the optimal money-burning mechanism can be written as an optimization problem in the reduced form, it is possible to directly obtain an algorithm using the framework of Cai, Daskalakis, and Weinberg (2012a, 2012b). For the special case of money burning, an alternative solution goes through the techniques developed by Hartline and Roughgarden (2008). A black-box application of Cai, Daskalakis, and Weinberg (2012a, 2012b) guarantees that the auction is Bayesian incentive compatible. For Lemma 6.2, it will be useful to describe the auction via the virtual value technique of Hartline and Roughgarden (2008) to show that the optimal capped money-burning auction is dominant strategy incentive compatible. We discuss the construction below.

Without any caps on the utilities, the optimal money-burning auction is analyzed by Hartline and Roughgarden (2008) and shown to be a virtual value maximization for a different notion of virtual values known as *virtual values for utility*. As in the Myerson auction, the virtual values of Hartline and Roughgarden (2008) can be computed as a function of the distribution, and if not monotone, they must be ironed using the same procedure used to iron the Myersonian virtual values. While originally developed for continuous distributions, the exact approach[2] described by Elkind (2007) can be used to compute ironed virtual values for utility for all buyers. We can summarize their results as follows.

THEOREM E.1—Hartline and Roughgarden (2008): *Given distributions* $F^1, \ldots, F^n$ *of support* $\Theta = \{\theta_1, \ldots, \theta_K\}$, *there exist nondecreasing maps* $\vartheta^i : \Theta \to \mathbb{R}$ (*called ironed virtual values for utility*) *such that for any Bayesian incentive compatible and individually rational mechanism* $(x^i, p^i)$ *and for every agent* $i$,

$$\mathbb{E}_{\theta \sim F}\big[u^i(\theta)\big] = \mathbb{E}_{\theta \sim F}\big[\vartheta^i\big(\theta^i\big)x^i(\theta)\big].$$

*Moreover, the optimal mechanism* (*with or without utility caps*) *is such that the allocation and payments only depend on the virtual values* $\vartheta^i(\theta^i)$.

The proof of the theorem follows from combining Lemma 2.6, Lemma 2.8, and Theorem 2.9 in Hartline and Roughgarden (2008). For the "moreover" part, although their paper does not consider any sort of utility caps, the presence of caps does not affect any of their proofs.

From Theorem E.1, we can describe the optimal auction as a monotone allocation that depends only on virtual values. We abuse notation and use $f^i$ to denote the distribution on the virtual values, that is, $f^i(\bar{\vartheta}^i) = \sum_{\theta^i \in \Theta; \vartheta^i(\theta^i) = \bar{\vartheta}^i} f^i(\theta^i)$. We also define the allocation directly in terms of virtual values $x^i(\vartheta)$. We then describe the format of the optimal auction.

THEOREM E.2—Optimal Capped Money Burning: *The auction maximizing capped utility* $\sum_i \min(b^i, \mathbb{E}[u^i])$ *is parameterized by* $w^i, q^i$, *which chooses the agent with largest scaled virtual value* $w^i \vartheta^i$ (*subject to some tie-breaking rule*) *and allocates to this agent with probability* $q^i$.

PROOF: Using Theorem E.1, we can formulate the optimal money burning with caps problem as finding a monotone allocation function $x^i(\vartheta)$ defined on the virtual values maximizing $\mathbb{E}[\vartheta^i x^i(\vartheta)]$. We solve the problem

$$\max \sum_i \min\big(b^i, \mathbb{E}\big[\vartheta^i x^i(\vartheta)\big]\big) \quad \text{s.t. monotonicity}$$

---

[2]Given a discrete distribution described by $f(\theta_1), \ldots, f(\theta_K)$ nonnegative and summing to 1 with $\theta_1 < \theta_2 < \cdots < \theta_K$, Elkind (2007) defines a discrete notion of the Myersonian virtual value as $\varphi_j^i = \theta_j - (\theta_{j+1}^i - \theta_j^i)\frac{1 - F^i(\theta_j)}{f^i(\theta_j)}$. Those are then ironed by defining for each $i$ a set of $K$ two-dimensional points $(F(\theta_j^i), \sum_{j' \leq j} f_{j'}^i \varphi_{j'}^i)$, computing the lower convex hull, and defining the ironed virtual values as the slopes of segments of the convex hull corresponding to each point. The same exact computation can be done by replacing the original Myersonian notion of virtual values $\varphi_j^i$ with the definition of virtual values for utility $\vartheta_j^i = (\theta_{j+1}^i - \theta_j^i)\frac{1 - F^i(\theta_j)}{f^i(\theta_j)}$.

and rescale $x^i$ by multiplying it by a probability $q^i$ such that it obeys the constraints $\mathbb{E}[\sum_i \vartheta^i x^i(\vartheta)] \leq b^i$ while keeping the same objective value. In the following formulation, we relax the constraint that the allocation needs to be monotone and obtain the primal-dual pair

$$
\begin{aligned}
&\max_{x,u} \quad \sum_i u^i & &\min_{w,y,z} \quad \sum_i y^i b^i + \sum_\vartheta z(\vartheta)\\
&\text{s.t.} \quad u^i \leq \sum_\vartheta \vartheta^i x^i(\vartheta) f(\vartheta) \quad \forall i \quad (w^i) & &\text{s.t.} \quad z(\vartheta) \geq \vartheta^i f(\vartheta) w^i \quad \forall i, \vartheta \quad (x^i(\vartheta))\\
& & & \qquad\quad y^i + w^i \geq 1 \quad \forall i \qquad\qquad (u^i)\\
&\qquad\quad u^i \leq b^i \quad \forall i \qquad\qquad (y^i)\\
&\qquad\quad \sum_i x^i(\vartheta) \leq 1 \quad \forall \vartheta \qquad (z(\vartheta))\\
&\qquad\quad x^i(\vartheta) \geq 0 \quad \forall i, \vartheta & &\qquad\quad y^i, w^i, z(\vartheta) \geq 0 \quad \forall i, \vartheta
\end{aligned}
$$

Assume that we have an optimal primal-dual pair. Then, if for some profile of virtual values $\vartheta$, agent $i$ is allocated with nonzero probability, that is, $x^i(\vartheta) > 0$, then by complementary slackness, we must have for all $j \neq i$,

$$
\vartheta^i w^i f(\vartheta) = z(\vartheta) \geq \vartheta^j w^j f(\vartheta),
$$

where the equality follows from complementary slackness and the inequality follows from feasibility. This means that $i \in \arg\max_i \vartheta^i w^i$, except when $f(\vartheta) = 0$.[3]

We still need to argue that the item is always allocated in an optimal solution. We again use complementary slackness. If the item is not completely allocated for a profile $\vartheta$, we must have $z(\vartheta) = 0$ and, therefore, for all agents $i$,

$$
0 = z(\vartheta) \geq \vartheta^i w^i f(\vartheta) \geq 0,
$$

so $\vartheta^i w^i$ must be zero except when $f(\vartheta) = 0$.

Finally, observe that although we relaxed monotonicity in the program, the complementarity constraints imply that under any tie-breaking rule, the allocation is monotone. *Q.E.D.*

## APPENDIX F: ASYMPTOTICALLY OPTIMAL MULTIPLE-BUYER MECHANISM

We now extend the construction in Theorem 5.4 to multiple buyers using the assumption that the distribution of $\max_i v_t^i$ is in $\mathcal{F}_{\epsilon,\bar{v}}$ for all periods and $T \to \infty$. We will replace the *give for free* mechanism by a *second-price auction* and the *spend mechanism* by a *spend-throttled second-price auction*.

We use $x^{SP,i}$ and $p^{SP,i}$ to denote the allocation and payment of buyer $i$ in a (static) second-price auction. It is useful to define $\mu_t^i$ as the expected utility of agent $i$ under a second-price auction and define $W_t$ as the expected welfare of the second-price auction, that is,

$$
\mu_t^i = \mathbb{E}_{\theta_t \sim F_t}\big[\theta_t^i \cdot x_t^{i,SP} - p_t^{i,SP}\big], \qquad W_t = \mathbb{E}_{\theta_t \sim F_t}\bigg[\sum_i \theta_t^i \cdot x_t^{i,SP}\bigg].
$$

---

[3]Assuming that these properties ($x^i(\vartheta) > 0 \implies i \in \arg\max_i \vartheta^i w^i$ and $z(\vartheta) = 0 \implies \vartheta^i w^i = 0$) still hold when $f(\vartheta) = 0$ never changes the optimality or the feasibility of the solution.

The dynamic component will correspond to a throttled second-price auction. The item is allocated to the highest bidder (breaking ties arbitrarily) if her balance exceeds the expected utility of a second-price auction, that is, if $b_{t-1}^i \geq \mu_t^i$. If the highest bidder meets the minimum balance condition, she is allocated and charged as in a second-price auction plus an extra amount $\mu_t^i$. Otherwise, the allocation and payment to this agent is zero. Formally,

$$x_t^{i,\mathsf{ST}} = x_t^{i,\mathsf{SP}} \cdot \mathbf{1}\{b_{t-1}^i \geq \mu_t^i\}, \qquad p_t^{i,\mathsf{ST}} = \left(p_t^{i,\mathsf{SP}} + \mu_t^i\right) \cdot \mathbf{1}\{b_{t-1}^i \geq \mu_t^i\}.$$

The ASYMPOPTIMALMULTI mechanism will be a combination of those mechanisms with proportions given by weights $q_t^i$. In other words, the final mechanism is defined as

$$x_t^i = q_t^i \cdot x_t^{i,\mathsf{SP}} + \left(1 - q_t^i\right) \cdot x_t^{i,\mathsf{ST}}, \qquad p_t^i = q_t^i \cdot p_t^{i,\mathsf{SP}} + \left(1 - q_t^i\right) \cdot p_t^{i,\mathsf{ST}}.$$

We update the balance by simply adding the per-period utility to it:

$$b_t^i = b_{t-1}^i + \theta_t^i \cdot x_t^i - p_t^i.$$

To complete the description of the mechanism, we recursively define the weights as

$$q_t^i = \mathbf{1}\left\{\sum_{\tau=1}^{t-1} q_\tau^i \mu_\tau^i \leq W_t \sqrt{t} \ln t\right\}.$$

Note that at any given point, different agents may have different weights for $\mathsf{SP}$ and $\mathsf{ST}$. This is possible since the difference between the two mechanisms is whether the winner is throttled. The determination of the winner is the same for both mechanisms.

It is straightforward to check that ASYMPOPTIMALMULTI is a non-clairvoyant bank account mechanism. Below, we argue that it is also asymptotically optimal.

THEOREM F.1: *For any positive numbers $\epsilon < \bar{v}$, there is a constant $C_{\epsilon,\bar{v}}$ depending only on those parameters such that the revenue of the ASYMPOPTIMALMULTI mechanism is at least*

$$\mathrm{REV} \geq \sum_{t=1}^{T} \mathbb{E}\left[\max_i \theta_t^i\right] - n C_{\epsilon,\bar{v}} \cdot \sqrt{T} \ln T \tag{F.1}$$

*whenever the distribution of the highest value is in $\mathcal{F}_{\epsilon,\bar{v}}$ for all $t$. In particular, the optimal solution to* (MAXIMIN) *tends to 1 as $T \to \infty$ when the highest value distributions are restricted to $\mathcal{F}_{\epsilon,\bar{v}}$.*

The proof of Theorem F.1 follows a similar idea to that used in the proof of Theorem 5.4. In the initial periods, the second-price auction $\mathsf{SP}$ is used, allowing each buyer to accumulate a sufficient balance. After a sufficient number of rounds, the balance of an agent is large enough with high probability or the suboptimality of applying the second-price auction to her is sublinear. Whenever the balance is sufficient, the mechanism starts allocating according to mechanism $\mathsf{ST}$. For this auction, the allocation is efficient and the expected utility of the agent is zero and, hence, the seller is able to extract full surplus.

PROOF OF THEOREM F.1: We first bound the revenue of ASYMPOPTIMALMULTI,

$$\mathrm{REV} = \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E}\left[q_t^i \cdot p_t^{i,\mathsf{SP}} + \left(1 - q_t^i\right) \cdot p_t^{i,\mathsf{ST}}\right].$$

The first term can be bounded as

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbb{E}[q_t^i \cdot p_t^{i,\text{SP}}] = \sum_{t=1}^{T}\sum_{i=1}^{n} q_t^i(\mathbb{E}[\theta_t^i \cdot x_t^{i,\text{SP}}] - \mu_t^i),$$

and the second as

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\mathbb{E}[(1-q_t^i)\cdot p_t^{i,\text{ST}}] = \sum_{t=1}^{T}\sum_{i=1}^{n}(1-q_t^i)\mathbb{E}[\theta_t^i \cdot x_t^{i,\text{SP}}]\cdot\Pr[b_{t-1}^i \geq \mu_t^i].$$

Summing them up, we obtain

$$\text{REV} \geq \sum_{t=1}^{T}W_t - \sum_{t=1}^{T}\sum_{i=1}^{n} q_t^i\mu_t^i + (1-q_t^i)\cdot W_t \cdot \Pr[b_{t-1}^i < \mu_t^i]. \tag{F.2}$$

To conclude the proof, we argue that the last two terms grow sublinearly in $T$. It is straightforward to bound the first term from the definition of $q_t^i$ and the fact that $W_t \leq \bar{v}$ for all $t$ that

$$\sum_{t=1}^{T} q_t^i\mu_t^i \leq \bar{v} + \max_t W_t\sqrt{t}\ln t \leq \bar{v}(1 + \sqrt{T}\ln T).$$

To bound the remaining term, we observe that $\mathbb{E}[b_{t-1}^i] = \sum_{\tau=1}^{t-1} q_\tau^i\mu_\tau^i$. Hence, $q_t^i = 0$ occurs when $\mathbb{E}[b_{t-1}^i] > W_t\sqrt{t}\ln t$. When $q_t^i = 0$, for $t > \exp(4\bar{v}^2/\epsilon^2)$, we have

$$\mathbb{E}[b_{t-1}^i] > W_t\sqrt{t}\ln t \geq \epsilon\sqrt{t}\ln t > 2\bar{v}\sqrt{t\ln t} \geq \bar{v}\sqrt{t\ln t} + \bar{v}. \tag{F.3}$$

Similar to the proof of Theorem 5.4, we can define a martingale $\tilde{b}_t^i = b_t^i - \sum_{\tau=1}^{t} q_\tau^i\mu_\tau^i$. Since it has bounded variation $|\tilde{b}_t^i - \tilde{b}_{t-1}^i| < \bar{v}$, we can apply the Azuma–Hoeffding inequality (Azuma (1967)) to establish that

$$\Pr[b_t^i < \mathbb{E}[b_{t-1}^i] - y] = \Pr[\tilde{b}_t^i < -y] \leq \exp\left(-\frac{y^2}{2t\bar{v}^2}\right). \tag{F.4}$$

Therefore,

$$\begin{aligned}
\Pr[b_{t-1}^i < \mu_t^i | q_t^i = 0] &= \Pr[b_{t-1}^i < \mathbb{E}[b_{t-1}^i] - (\mathbb{E}[b_{t-1}^i] - \mu_t^i)|q_t^i = 0] \\
&\leq^{(\text{F.3})} \Pr[b_{t-1}^i < \mathbb{E}[b_{t-1}^i] - \bar{v}\sqrt{t\ln t}] \\
&\leq^{(\text{F.4})} \exp\left(-\frac{(\bar{v}\sqrt{t\ln t})^2}{2t\bar{v}^2}\right) = 1/\sqrt{t}.
\end{aligned}$$

This allows us to bound the second term in (F.2) as

$$\sum_{t=1}^{T}(1-q_t^i)\cdot W_t \cdot \Pr[b_{t-1}^i < \mu_t^i] \leq \bar{v}\cdot\exp\left(\frac{4\bar{v}^2}{\epsilon^2}\right) + 2\bar{v}\sqrt{T}.$$

$$Q.E.D.$$

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