Econometrica Supplementary Material

SUPPLEMENT TO "EXTREME POINTS AND MAJORIZATION: ECONOMIC APPLICATIONS"

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S.1. *Schur-Convex Functions and Functionals*

CONSIDER X_F AND X_G TO BE UNIFORM, discrete random variables, each taking n values $x_F = (x_F^1, \dots, x_F^n)$ and $x_G = (x_G^1, \dots, x_G^n)$, respectively. Then

$$x_F \prec_{\mathrm{dm}} x_G \quad \Leftrightarrow \quad F^{-1} \prec G^{-1} \quad \Leftrightarrow \quad G \prec F,$$

where $\prec_{\mathrm{dm}}$ denotes the classical discrete majorization relation due to Hardy, Littlewood, and Polya. Thus, discrete majorization is equivalent to the present majorization relation applied to quantile functions. A function $V : \mathbb{R}^n \to \mathbb{R}$ is *Schur-convex* (*concave*) if $V(\mathbf{x}) \geq V(\mathbf{y})$ ($V(\mathbf{x}) \leq V(\mathbf{y})$) whenever $\mathbf{x} \succ_{\mathrm{dm}} \mathbf{y}$. If V is a symmetric function, and if all its partial derivatives exist, then the *Schur–Ostrovski criterion* says that V is *Schur-convex (concave)* if and only if

$$(x_i - x_j)\left(\frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial x_j}\right) \geq (\leq) 0 \quad \text{for all } x.$$

It is useful to have a similar characterization for continuous majorization. Chan, Proschan, and Sethuraman (1987) showed that a law-invariant,[1] Gâteaux-differentiable functional $V : L^1(0,1) \to \mathbb{R}$ respects the majorization relation on $L^1(0,1)$, if and only if its *Gâteaux-derivatives* in specially defined directions are non-positive. The considered directions are of the form

$$h = \lambda_1 \mathbf{1}_{(a,b)} + \lambda_2 \mathbf{1}_{(c,d)}$$

with $0 \leq a < b < c < d \leq 1$ and $\lambda_1 \geq 0 \geq \lambda_2$ such that $\lambda_1(b-a) + \lambda_2(d-c) = 0$. Note that the function h takes at most two values that are different from zero, and is decreasing on $[a,b] \cup [c,d]$. Moreover, $\int_0^1 h(t)\,\mathrm{d}t = 0$.

This result also yields a simple intuition for the Fan–Lorentz theorem in the case where K is differentiable. Consider a monotonic f and note that, for any direction h,

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[1]This means that the functional is constant over the equivalence class of functions with the same non-decreasing rearrangement. This replaces the symmetry in the discrete formulation.

the Gâteaux-derivative of the functional $V(f)=\int_0^1 K(f(t),t)\,\mathrm{d}t$ is given by

$$\delta V(f,h)=\frac{d}{d\varepsilon}\int_0^1 K\bigl(f(t)+\varepsilon h(t),t\bigr)\,\mathrm{d}t\,\bigg|_{\varepsilon=0}=\int_0^1 K_f\bigl(f(t),t\bigr)h(t)\,\mathrm{d}t,$$

where the last equality follows by interchanging the order of differentiation and integration.[2] The Fan–Lorentz conditions imply together that

$$\frac{\mathrm{d}K_f}{\mathrm{d}t}=f_t\cdot K_{ff}+K_{ft}\geq 0.$$

For a direction h such that $\int_0^1 h(t)\,\mathrm{d}t=0$, and such that h is a decreasing two-step function as defined above, we obtain that

$$\delta V(f,h)=\int_0^1 K_f\bigl(f(t),t\bigr)h(t)\,\mathrm{d}t\leq 0.$$

Hence, the Fan–Lorentz functional $V(f)=\int_0^1 K(f(t),t)\,\mathrm{d}t$ is Schur-concave by the result of Chan, Proschan, and Sethuraman (1987).

S.2. *Decision-Making Under Uncertainty*

We briefly illustrate here how our insights can be applied in order to understand how agents with non-expected utility preferences choose among risky prospects.

S.2.1. *Rank-Dependent Utility and Choquet Capacities*

Quiggin (1982) and Yaari (1987) axiomatically derived utility functionals with rank-dependent assessments of probabilities of the form[3]

$$U(F)=\int_0^1 v(t)\,\mathrm{d}(g\circ F)(t),$$

where F is the distribution of a random variable on the interval $[0,1]$, $v:[0,1]\to R$ is continuous, strictly increasing, and bounded, and where $g:[0,1]\to[0,1]$ is strictly increasing, continuous, and onto. The function v represents a transformation of monetary payoffs, while the function g represents a transformation of probabilities.[4]

The case $g(x)=x$ yields the classical von Neumann–Morgenstern expected utility model where risk aversion is equivalent to v being concave. The case $v(x)=x$ yields Yaari's (1987) dual utility theory, where risk aversion is equivalent to g being concave. Because of the possible interactions between v and g, it is not clear what properties yield risk aversion in the general rank-dependent model. Using integration by parts, we can

[2]This is allowed since K is convex in f.

[3]Their theory is a bit more general (e.g., it allows a more general domain for the functions v and F). We keep here a framework that is compatible with the rest of the paper.

[4]For the sake of brevity, we assume below that both g and v are twice differentiable. Since the Fan–Lorentz result does not require differentiability, the observations below generalize.

also write

$$U(F)=\int_0^1 v(t)\,\mathrm{d}(g\circ F)(t)=v(1)-\int_0^1 v'(t)(g\circ F)(t)\,\mathrm{d}t$$
$$=v(1)+\int_0^1 K\bigl(F(t),t\bigr)\,\mathrm{d}t,$$

where

$$K(F,t)=-v'(t)(g\circ F),$$

and where we used $g(0)=0$ and $g(1)=1$. Then

$$\frac{\partial^2 K(F,t)}{\partial F\partial t}=-g'\bigl(F(t)\bigr)v''(t)\geq 0$$

for all t if and only if v is concave. Similarly,

$$\frac{\partial^2 K(F,t)}{\partial^2 F}=-g''\bigl(F(t)\bigr)v'(t)\geq 0$$

for all t if and only if g is concave.

Hence, the Fan–Lorentz conditions are satisfied if and only if $v''\leq 0$ and $g''\leq 0$. As a consequence, the utility functional $U=\int_0^1 v(t)\,\mathrm{d}(g\circ F)(t)$ is Schur-concave, and the agent whose preferences are represented by U is *risk averse*, exactly as under standard expected utility.[5]

Another important strand of the literature on non-expected utility considers ambiguity aversion. The main tool is the *Choquet integral* with respect to a (convex) *capacity* (this is unrelated to the Choquet representation used above!). Analogously to the derivations above, it can be shown that the Choquet integral yields a Schur-concave functional if and only if it is computed with respect to a convex capacity.

S.2.2. *A Portfolio Choice Problem*

Dybvig (1988) studied a simplified version of the following problem:

$$\min_X \mathbb{E}[XY]$$
$$\text{s.t. } X\geq_{cv} Z,$$

where Y and Z are given random variables. Y represents here the distribution of a pricing function over the states of the world, and the goal is to choose, given Y, the cheapest contingent claim X that is less risky than a given claim Z. To make the problem well-defined, Y needs to be essentially bounded and X, Z must be integrable. Recalling that

$$X\geq_{cv} Z \quad\Leftrightarrow\quad F_X\succ F_Z \quad\Leftrightarrow\quad F_X^{-1}\prec F_Z^{-1},$$

[5]The equivalence between the concavity of the functions v and g and risk aversion has been pointed out by Hong, Karni, and Safra (1987), who built on Machina (1982).

we obtain that

$$\mathbb{E}[XY] \geq \int_0^1 F_Y^{-1}(1-t)F_X^{-1}(t)\,\mathrm{d}t \geq \int_0^1 F_Y^{-1}(1-t)F_Z^{-1}(t)\,\mathrm{d}t,$$

where the first inequality follows by the rearrangement inequality of Hardy, Littlewood, and Polya (1929) (the anti-assortative part!), and where the second inequality follows by the Fan–Lorentz theorem.

By choosing a random variable X that has the same distribution as Z and that is anti-comonotonic with Y,[6] the lower bound $\int_0^1 F_Y^{-1}(1-t)F_Z^{-1}(t)\,\mathrm{d}t$ is attained, and hence such a choice solves the portfolio choice problem.[7]

If $Y' \leq_{cv} Y$, we obtain by the Fan–Lorentz inequality (now applied to the functional with argument F_Y^{-1}) that

$$\sup_{X \succ_{cv} Z} \mathbb{E}[XY] = \int_0^1 F_Y^{-1}(1-t)F_Z^{-1}(t)\,\mathrm{d}t \geq \int_0^1 F_{Y'}^{-1}(1-t)F_Z^{-1}(t)\,\mathrm{d}t = \sup_{X \succ_{cv} Z} \mathbb{E}\big[XY'\big].$$

In other words, a decision maker that becomes more informed (in the Blackwell sense) will bear a lower cost.

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[6] This can always be done if the underlying probability space is non-atomic. A random vector (X, Y) is anti-comonotonic if there exists a random variable W and non-decreasing functions h_1, h_2 such that $(X, Y) =^{\text{dist}} (h_1(W), -h_2(W))$.

[7] For more details on this problem, see Dana (2005) and the literature cited there. It does not use the Fan–Lorentz inequality.