Econometrica Supplementary Material

SUPPLEMENT TO "NETWORKS, PHILLIPS CURVES, AND MONETARY POLICY"

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This document collects alternative regression specifications for Section 7 of the paper, and detailed proofs that were omitted from the main text.

1. PHILLIPS CURVE REGRESSIONS

1.1. *Divine Coincidence versus Other Price Indices*

FIGURE 5 COMPARES THE WEIGHTING scheme in the divine coincidence index versus the PCE, which is the most common indicator for Phillips curve regressions and monetary policy, at an aggregated 21-sector level.

Consumer prices do not include wage inflation, which in the divine coincidence index has the highest weight of 18%. The PCE assigns the highest weight to health care, real estate, and nondurable goods. This sectors have large consumption shares, but relatively small total sale shares. By contrast, the divine coincidence index assigns a high weight to large intermediate good sectors with sticky prices, such as professional services, financial intermediation, and durable goods.

Figure 6 plots a time series of the divine coincidence index against CPI, PCE, their core versions, and the PPI.

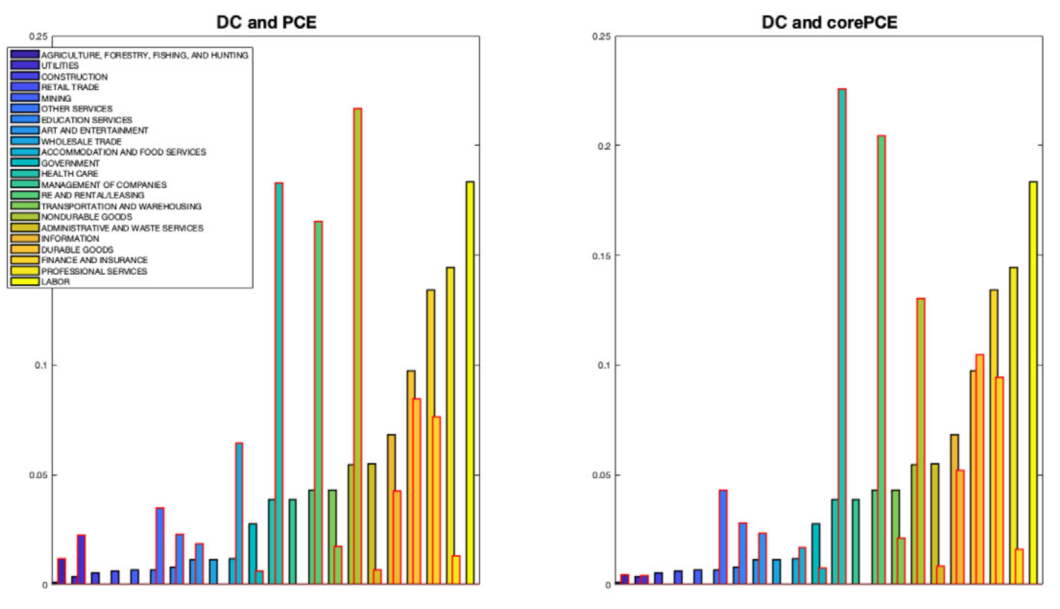

FIGURE 5.—DC and PCE weights (the bars with red borders correspond to the PCE).

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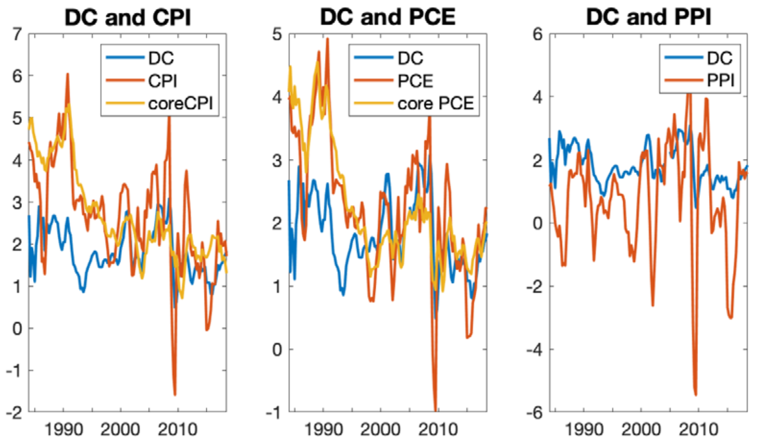

FIGURE 6.—Comparison of DC against consumer and producer prices (1984–2018).

Section 3.1 of this supplement provides additional detail about the sectoral price series used to construct the divine coincidence index.

1.2. *Regressions With Alternative Measures of the Output Gap*

Tables VI and VII replicate Table V in the main paper, replacing the unemployment gap with the CBO output gap and the unemployment rate as independent variables.
IX and X replicate Table VIII in the main paper, replacing the unemployment gap with the CBO output gap and the unemployment rate as independent variables.
Section 3.2 of this Supplemental Material reports estimates for all coefficients in the regression equation (37).

TABLE VI

REGRESSIONS OF YEARLY INFLATION ON THE CBO'S OUTPUT GAP, 1984–2018.

	–		Mich		SPF CPI		GMM
	Gap	R^2	Gap	R^2	Gap	R^2	Gap
DC	0.17 (0.02)	0.27	0.19 (0.02)	0.33	0.16 (0.02)	0.33	–0.13 (0.03)
CPI	0.26 (0.06)	0.11	0.29 (0.07)	0.13	0.25 (0.06)	0.13	0.07 (0.03)
coreCPI	0.06 (0.06)	0.01	0.13 (0.05)	0.2	0.04 (0.06)	0.15	–0.03 (0.01)
PCE	0.18 (0.06)	0.07	0.2 (0.06)	0.09	0.17 (0.06)	0.09	0.1 (0.02)
corePCE	0.05 (0.05)	0.01	0.1 (0.05)	0.13	0.04 (0.05)	0.1	0.03 (0.01)
PPI	0.79 (0.23)	0.08	0.61 (0.22)	0.16	0.82 (0.22)	0.12	0.9 (0.27)

TABLE VII

REGRESSIONS OF YEARLY INFLATION ON THE UNEMPLOYMENT RATE, 1984–2018.

	–		Mich		SPF CPI		GMM
	Rate	R^2	Rate	R^2	Rate	R^2	Rate
DC	−0.13 (0.03)	0.13	−0.15 (0.03)	0.18	−0.13 (0.03)	0.21	−0.11 (0.04)
CPI	−0.13 (0.07)	0.02	−0.17 (0.08)	0.04	−0.13 (0.07)	0.06	−0.08 (0.03)
coreCPI	0.01 (0.07)	0	−0.08 (0.06)	0.17	0.01 (0.06)	0.14	−0.04 (0.04)
PCE	−0.05 (0.06)	0	−0.07 (0.07)	0.02	−0.05 (0.06)	0.04	−0.04 (0.01)
corePCE	0.04 (0.06)	0	−0.02 (0.06)	0.11	0.04 (0.05)	0.1	−0.01 (0.01)
PPI	−0.38 (0.26)	0.02	−0.12 (0.25)	0.11	−0.38 (0.25)	0.04	−0.24 (0.16)

TABLE VIII

REGRESSION RESULTS FOR THE CBO UNEMPLOYMENT GAP, WITH CP SHOCK.

	DC	CPI	core CPI	PCE	core PCE
Cost-push	0.5627 (0.2345)	2.5545 (0.565)	0.4886 (0.4768)	2.3948 (0.4745)	1.1224 (0.4102)
Gap	−0.1921 (0.0374)	−0.1906 (0.0758)	−0.2175 (0.064)	−0.0783 (0.0637)	−0.0886 (0.0551)
Intercept	2.0842 (0.058)	3.2239 (0.1398)	2.8559 (0.118)	2.6509 (0.1174)	2.397 (0.1015)
R-squared	0.3317	0.2782	0.142	0.2558	0.1275

TABLE IX

REGRESSION RESULTS FOR THE CBO OUTPUT GAP, WITH CP SHOCK.

	DC	CPI	core CPI	PCE	core PCE
Cost-push	0.6059 (0.2604)	2.5472 (0.5964)	0.6387 (0.5145)	2.4715 (0.4983)	1.2896 (0.4333)
Gap	0.1241 (0.0332)	0.1363 (0.0682)	0.1176 (0.0588)	0.0369 (0.057)	0.0225 (0.0495)
Intercept	2.0936 (0.0633)	3.2425 (0.145)	2.8535 (0.1251)	2.6467 (0.1212)	2.3802 (0.1054)
R-squared	0.2458	0.2635	0.0852	0.2484	0.1086

TABLE X

REGRESSION RESULTS FOR THE UNEMPLOYMENT RATE, WITH CP SHOCK.

	DC	CPI	core CPI	PCE	core PCE
Cost-push	0.6321	2.8683	0.8598	2.6413	1.3999
	(0.2357)	(0.5706)	(0.4905)	(0.4722)	(0.4102)
Gap	−0.1880	−0.0954	−0.1038	0.006	0.0063
	(0.0374)	(0.0811)	(0.0697)	(0.0671)	(0.0583)
Intercept	2.0911	3.1954	2.8214	2.6213	2.3637
	(0.0594)	(0.1439)	(0.1237)	(0.1191)	(0.1034)
R-squared	0.309	0.2462	0.0706	0.2456	0.1071

2. PROOFS

2.1. *Basic Results From Section 3*

PROOF OF LEMMA 1: The flex-price equilibrium is efficient. Therefore, the equilibrium allocation can be derived as the solution of the following planning problem:

$$\max_{L,\{L_i,Q_i,\{X_{ij}\}\}} \frac{\mathcal{C}(\{Q_i\}_{i=1}^N)^{1-\gamma}}{1-\gamma} - \frac{L^{1+\varphi}}{1+\varphi}$$
$$\text{s.t.} \quad Q_i + \sum_j X_{ij} = A_i F_i(\{X_{ij}\}, L_i) \quad \forall i, \tag{38}$$
$$\sum_i L_i = L.$$

We defined aggregate real output as the consumption aggregate of the representative agent; therefore, the change in natural output is given by

$$y_{\text{nat}} = \sum_i \frac{\partial \log C^*}{\partial \log A_i} \log A_i,$$

where

$$C^* \equiv \mathcal{C}(\{Q_i^*\}_{i=1}^N)$$

is aggregate output under the optimal allocation.

The optimization problem in (38) can be solved in two steps: first, we choose $\{L_i, Q_i, \{X_{ij}\}\}$ for given L; then we choose the optimal L. Formally, solving problem (38) is equivalent to solving

$$\frac{C^*(L; A)^{1-\gamma}}{1-\gamma} = \max_{\{L_i,Q_i,\{X_{ij}\}\}} \frac{\mathcal{C}(\{Q_i\})^{1-\gamma}}{1-\gamma}$$
$$\text{s.t.} \quad Q_i + \sum_j X_{ij} = A_i F_i(\{X_{ij}\}, L_i) \quad \forall i, \tag{39}$$
$$\sum_i L_i = L,$$

and

$$\max_{L} \frac{C^*(L; A)^{1-\gamma}}{1-\gamma} - \frac{L^{1+\varphi}}{1+\varphi}. \tag{40}$$

The solution of (40) must satisfy

$$C^*(L; A)^{\gamma} L^{\varphi} = \frac{\partial C^*}{\partial L}. \tag{41}$$

Applying the envelope theorem to problem (39), we have

$$\frac{\partial C^*}{\partial L} = C^{*\gamma} \nu_L, \tag{42}$$

where ν_L is the Lagrange multiplier associated to the constraint $\sum_i L_i = L$. Hence, the first-order condition (41) becomes

$$L^{\varphi} = \nu_L$$

and so we have

$$\frac{\partial \log L}{\partial \log A_i} = \frac{1}{\varphi} \frac{\partial \log \nu_L}{\partial \log A_i}. \tag{43}$$

Applying again the envelope theorem to problem (39) then yields

$$\frac{\partial \log C^*}{\partial \log A_i} = C^{*\gamma} \left(\frac{\nu_L L}{\varphi C^*} \frac{\partial \log \nu_L}{\partial \log A_i} + \frac{\nu_i F_i(\{X_{ij}\}, L_i)}{C^*} \right). \tag{44}$$

We now rewrite the two terms on the right-hand side of equation (44). First, we show that

$$C^{*\gamma} \frac{\nu_i F_i(\{X_{ij}\}, L_i)}{C^*} = \lambda_i, \tag{45}$$

where λ_i is i's sales share in total GDP; second, we show that

$$C^{*\gamma} \frac{\nu_L L}{\varphi C^*} \frac{\partial \log \nu_L}{\partial \log A_i} = \frac{1}{\varphi} \lambda_i - \frac{\gamma}{\varphi} \frac{\partial \log C^*}{\partial \log A_i}. \tag{46}$$

Putting equations (45) and (46) together in turn implies

$$\frac{\partial \log C^*}{\partial \log A_i} = \frac{1+\varphi}{\gamma+\varphi} \lambda_i,$$

which is the result that we set out to demonstrate.

Let us first prove (45). To do this, we show that, in the competitive equilibrium, $C^{*\gamma}\nu_i$ is equal to the price of good i relative to the CPI. It then follows from the definition of the sales share λ_i that

$$C^{*\gamma} \frac{\nu_i F_i(\{X_{ij}\}, L_i)}{C^*} = \frac{P_i F_i(\{X_{ij}\}, L_i)}{P C^*} = \lambda_i.$$

Denote by $\mathcal{C}_i$ the partial derivative of the consumption aggregator $\mathcal{C}$ with respect to Q_i. From the FOCs of problem (39), we have that $\mathcal{C}_i = C^\gamma \nu_i$, and from consumer optimization in the competitive equilibrium, we have $\frac{\mathcal{C}_j}{\mathcal{C}_i} = \frac{P_j}{P_i}$. Thus,

$$\frac{\mathcal{C}_j}{\mathcal{C}_i} = \frac{\nu_j}{\nu_i} = \frac{P_j}{P_i}.$$

Using the fact that $\mathcal{C}$ is homogeneous of degree 1, and normalizing the CPI to 1 ($\sum_j \frac{P_j Q_j}{C} = 1$), we have

$$1 = \frac{\sum_j \mathcal{C}_j Q_j}{C} = \sum_j \frac{\mathcal{C}_j}{\mathcal{C}_i} Q_j \frac{\mathcal{C}_i}{C} = \sum_j \frac{P_j Q_j}{C} \frac{\mathcal{C}_i}{P_i} = \frac{\mathcal{C}_i}{P_i}.$$

Hence, $P_i = \mathcal{C}_i$, and the FOCs for (39) further imply $P_i = C^\gamma \nu_i$.

Let us now derive equation (46). From the FOCs of (39), it holds that $C^\gamma \nu_L = C^\gamma \nu_i A_i F_{iL} = P_i A_i F_{iL} = W \ \forall i$, where the last equality follows from firm optimization in the competitive equilibrium. Moreover, from the consumers' budget constraint we have that $C^{*\gamma} \frac{\nu_L L}{C^*} = \frac{WL}{C^*} = 1$. Thus,

$$C^{*\gamma} \frac{\nu_L L}{\varphi C^*} \frac{\partial \log \nu_L}{\partial \log A_i} = \frac{1}{\varphi}\left(\frac{\partial \log W}{\partial \log A_i} - \gamma \frac{\partial \log C^*}{\partial \log A_i}\right).$$

To conclude the proof, we need to show that

$$\frac{\partial \log W}{\partial \log A_i} = \lambda_i.$$

Using again the consumers' budget constraint, we have

$$\frac{\partial \log W}{\partial \log A_i} = \frac{\partial \log C^*}{\partial \log A_i} + \left(\frac{\partial \log C^*}{\partial \log L} - 1\right)\frac{\partial \log L}{\partial \log A_i} = \lambda_i,$$

where the second equality follows from equations (42), (43), and (44). *Q.E.D.*

PROOF OF LEMMA 3: Let us start by relating the output gap with the real wage gap ($\tilde{w} - \beta^T \tilde{\mathbf{p}}_t$), by log-linearizing the consumption-leisure tradeoff (3), and subtracting the natural variables:

$$\tilde{w} - \beta^T \tilde{\mathbf{p}}_t = \gamma \tilde{y} + \varphi \tilde{l}.$$

Moreover, following Remark 5, the output and employment gaps on the right-hand side coincide. Therefore, we have

$$\tilde{w} - \beta^T \tilde{\mathbf{p}}_t = (\gamma + \varphi)\tilde{y}. \tag{47}$$

Next, let us express the real wage gap on the left-hand side of (47) as a function of markups. To do so, move from the firms' cost minimization problem (5), to write the marginal cost gap as

$$\tilde{\mathbf{mc}}_t = \boldsymbol{\alpha} \tilde{w}_t + \Omega \tilde{\mathbf{p}}_t,$$

and relate sectoral log-markups $\boldsymbol{\mu}$ with price gaps and marginal cost gaps using the definition

$$\tilde{\mathbf{p}}_t = \tilde{\mathbf{mc}}_t + \boldsymbol{\mu}_t.$$

Together, the last two equations yield

$$\tilde{\mathbf{mc}}_t = \mathbf{1}\tilde{w}_t + (I - \Omega)^{-1}\Omega\boldsymbol{\mu}_t$$

and

$$\boldsymbol{\beta}^T\tilde{\mathbf{p}}_t = w_t + \boldsymbol{\lambda}^T\boldsymbol{\mu}_t.$$

Rearranging the last equation allows us to express the real wage gap in terms of markups:

$$\tilde{w}_t - \boldsymbol{\beta}^T\tilde{\mathbf{p}}_t = -\boldsymbol{\lambda}^T\boldsymbol{\mu}_t. \tag{48}$$

Plugging equation (48) into (47) yields the result. *Q.E.D.*

Derivation of the Labor Supply Curve

Log-linearizing equation (3) yields

$$w_t - \beta^T\mathbf{p}_t = \varphi l_t + \gamma y_t. \tag{49}$$

Remark 5 and Lemma 1 allow us to write (49) in gaps:

$$\begin{aligned}\pi_{wt} - \pi_t^C &= (\gamma + \varphi)\tilde{y}_t + \gamma y_{t,\text{nat}} + \varphi l_{t,\text{nat}} \\ &= (\gamma + \varphi)\tilde{y}_t + \boldsymbol{\lambda}^T d\log\mathbf{A}_t.\end{aligned}$$

Law of Motion for Sectoral Inflation Rates

Log-linearizing equation (7) yields the following expression for the optimal reset price:

$$p_{it}^* = \big(1 - \rho(1 - \delta_i)\big)\log mc_{it} + \rho(1 - \delta_i)\mathbb{E}p_{it+1}^*.$$

To express inflation in terms of the initial price and the optimal reset price, we log-linearize the evolution of sectoral price indexes:

$$\pi_{it} = \delta_i\big[p_{it}^* - p_{it-1}\big].$$

The firms' desired price change relative to the previous period can then be written as

$$p_{it}^* - p_{it-1} = \big(1 - \rho(1 - \delta_i)\big)(\log mc_{it} - \log mc_{it-1} - \mu_{it-1}) + \rho(1 - \delta_i)\left[\frac{1}{\delta_i}\mathbb{E}\pi_{it+1} + \pi_{it}\right].$$

Combining these three equations yields the result.

2.2. *Phillips Curves*

PROOF OF PROPOSITION 1: It remains to establish that the divine coincidence Phillips curve is the only one without endogenous cost-push shocks. To prove this, it is enough to show that the vector of weights which characterizes the divine coincidence index, $\boldsymbol{\lambda}^T(I - \Delta)\Delta^{-1}$, is the only element in the left ker of the matrix $\mathcal{V}$.

Consider then all vectors $\mathbf{x}$ such that $\mathbf{x}^T\mathcal{V}=\mathbf{0}$. Note that

$$\begin{aligned}&\mathbf{x}^T\mathcal{V}=\mathbf{0}\\ \iff\quad &\tilde{\mathbf{x}}^T\big[\boldsymbol{\alpha}\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]-\big(1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}\big)I\big]=\mathbf{0},\end{aligned}\tag{50}$$

where $\tilde{\mathbf{x}}^T\equiv\mathbf{x}^T\Delta(I-\Omega\Delta)^{-1}$.

To prove uniqueness, we need to show that all vectors $\tilde{\mathbf{x}}$ satisfying (50) are proportional to $\boldsymbol{\lambda}^T(I-\Delta)(I-\Omega\Delta)^{-1}$.

From (50), we have the relation

$$\big(1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}\big)\tilde{x}_j=\tilde{\mathbf{x}}^T\boldsymbol{\alpha}\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]_j\quad\forall j.\tag{51}$$

The product $\tilde{\mathbf{x}}^T\boldsymbol{\alpha}$ is a scalar, and we must have $\tilde{\mathbf{x}}^T\boldsymbol{\alpha}\neq0$ (otherwise we would get $\tilde{\mathbf{x}}^T=\mathbf{0}$). Therefore, (51) implies the condition[1]

$$\frac{\tilde{\mathbf{x}}_i}{\tilde{\mathbf{x}}_j}=\frac{\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]_i}{\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]_j}.$$

To complete the proof, it is easy to verify that

$$\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}=\boldsymbol{\lambda}^T(I-\Delta)(I-\Omega\Delta)^{-1}.\qquad Q.E.D.$$

PROOF OF PROPOSITION 2: It remains to establish that $\mathcal{V}\mathbf{1}=\mathbf{0}$, $I-\mathcal{V}$ is invertible, and $(I-\mathcal{V})_{ij}\in[0,1]$. Note that the first statement is equivalent to $\mathcal{V}(I-\Omega)^{-1}\boldsymbol{\alpha}=\mathbf{0}$. We have

$$\mathcal{V}(I-\Omega)^{-1}=\Delta(I-\Omega\Delta)^{-1}\bigg[\boldsymbol{\alpha}\frac{\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}}{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}-I\bigg].$$

Remark 3 implies

$$\mathcal{V}(I-\Omega)^{-1}\boldsymbol{\alpha}=\Delta(I-\Omega\Delta)^{-1}\bigg[\boldsymbol{\alpha}\frac{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}-\boldsymbol{\alpha}\bigg]=0.$$

We then prove that $\boldsymbol{\alpha}$ is the only element of $\ker(\mathcal{V}(I-\Omega)^{-1})$. Since the term $\Delta(I-\Omega\Delta)^{-1}$ is invertible, for every vector $\mathbf{x}\neq0$ such that $\mathcal{V}(I-\Omega)^{-1}\mathbf{x}=\mathbf{0}$ it must hold that

$$\begin{aligned}&\alpha\frac{\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]\mathbf{x}}{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}=x\\ \iff\quad&\alpha_i\frac{\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]\mathbf{x}}{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}=x_i\quad\forall i\end{aligned}\tag{52}$$

with

$$\frac{\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]\mathbf{x}}{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}\in\mathbb{R}\neq0$$

[1]The ratio on the right-hand side is well defined. In fact, Remark 7 implies

$$\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\big]_j>\big[\boldsymbol{\lambda}^T-\boldsymbol{\beta}^T(I-\Omega)^{-1}\big]_j=0\quad\forall j.$$

(otherwise we would have $\mathbf{x} = \mathbf{0}$). Equation (52) then implies

$$\frac{\alpha_i}{\alpha_j} = \frac{x_i}{x_j} \quad \forall i, j.$$

Therefore, $\mathbf{x}$ is proportional to the vector of labor shares $\boldsymbol{\alpha}$.

We now show that, as long as no sector has fully flexible prices ($\delta_i < 1\ \forall i$), the matrix $I - \mathcal{V}$ is invertible. Let us rewrite

$$I - \mathcal{V} = \left(I + \mathcal{B}\boldsymbol{\beta}^T\right)(I - \Delta\Omega)^{-1}(I - \Delta). \tag{53}$$

The matrix $(I + \mathcal{B}\boldsymbol{\beta}^T)$ has eigenvalues 1 (and all vectors orthogonal to $\boldsymbol{\beta}$ are corresponding eigenvectors) and $\frac{1}{1-\boldsymbol{\beta}^T\Delta(I-\Omega\Delta)^{-1}\boldsymbol{\alpha}}$, with corresponding eigenvector $\mathcal{B}$. Therefore, it is invertible. The matrix $(I - \Delta\Omega)^{-1}(I - \Delta)$ is invertible because we assumed that no sector has fully rigid or fully flexible prices. Thus, $I - \mathcal{V}$ is invertible. Also note that $I - \mathcal{V}$ has all positive elements, because all factors in (53) have positive elements. *Q.E.D.*

2.3. *Welfare and Optimal Policy*

PROOF OF PROPOSITION 3: The proof relies on a second-order approximation of the scaled difference between the realized value of any variable Z and its efficient value Z^*:

$$\frac{Z - Z^*}{Z} \simeq \log\left(\frac{Z}{Z^*}\right) + \frac{1}{2}\log\left(\frac{Z}{Z^*}\right)^2.$$

The unapproximated log difference is denoted by

$$\hat{z} \equiv \log\left(\frac{Z}{Z^*}\right).$$

Lemma 2 proves that $d\log y = d\log L$ to a first order. Therefore, we must have

$$\hat{y} = \underbrace{\hat{l}}_{\text{first order}} - \underbrace{d}_{\text{second order}} + \text{higher order terms},$$

where $\hat{l}$ is the log change in labor supply relative to the efficient equilibrium, and d is a second-order TFP loss. The main part of the proof consists in deriving d. Once we have an expression for d, we can approximate the utility function around the efficient outcome as

$$\begin{aligned}\frac{U - U^*}{U_c C} &\simeq \hat{y} + \frac{1}{2}\hat{y}^2 + \frac{1}{2}\frac{U_{cc}C}{U_c}\hat{y}^2 + \frac{U_l L}{U_c C}\left(\hat{l} + \frac{1}{2}\frac{U_{ll}N}{U_l}\hat{l}^2\right) \\ &= \hat{y} + \frac{1-\gamma}{2}\hat{y}^2 - \left(\hat{l} + \frac{1+\varphi}{2}\hat{l}^2\right) \\ &= \hat{y} + \frac{1-\gamma}{2}\hat{y}^2 - \left(\hat{y} + d + \frac{1+\varphi}{2}\hat{y}^2\right) \\ &= -\frac{\gamma+\varphi}{2}\tilde{y}^2 - d.\end{aligned}$$

The last equality holds because, to the second order, $\hat{y}^2 = \tilde{y}^2$ and $d^2 = \hat{y}d = 0$.

Let us then derive the second-order TFP loss d. Equation (54) relates aggregate output per unit labor with real wages and the labor share $\Lambda \equiv \frac{wL}{P^C Y}$:

$$\frac{Y}{L} = \frac{1}{\Lambda}\frac{W}{P^C}. \tag{54}$$

In log-deviations from steady state, equation (54) becomes

$$\hat{y} = \hat{w} - \hat{p}^C - \hat{\Lambda} + \hat{l}. \tag{55}$$

Following the same steps as in the proof of Lemma 3, a first-order approximation for the change in real wages $\hat{w} - \hat{p}^C$ is given by

$$\hat{w} - \hat{p}^C \approx \tilde{\boldsymbol{\lambda}}^T(\boldsymbol{\xi} - \boldsymbol{\mu}), \tag{56}$$

where we used the following notation:

- $\boldsymbol{\xi}$ denotes sectoral TFP losses coming from misallocation of inputs across the firms within sector i. Its components are defined as

  $$\xi_i \equiv \frac{Y_i}{A_i F\big(\{X_{ij}\}, L_i\big)},$$

  where

  $$L_i = \int L_i(f)\,df$$

  and

  $$X_{ij} = \int X_{ij}(f)\,df$$

  are the aggregate labor and intermediate inputs hired by sector i, f indexes producers within sector i, and A_i is its exogenous productivity shifter. In the presence of price dispersion across producers within i, the input quantities $L_i(f)$ and $X_{ij}(f)$ are not constant across firms f,[2] which implies $Y_i < A_i F(\{X_{ij}\}, L_i)$ and $\xi_i < 1$.
- $\tilde{\boldsymbol{\lambda}} \equiv \boldsymbol{\beta}^T(I - \tilde{\Omega})^{-1}$ is the vector of cost-based Domar weights, and $\tilde{\Omega}_{ij} \equiv \frac{P_j X_{ij}}{MC_i Y_i}$ is the cost-based input-output matrix. At the efficient equilibrium, the cost-based input-output matrix coincides with the sales-based one, $\Omega_{ij} \equiv \frac{P_j X_{ij}}{P_i Y_i}$, defined in the main text.

Combining (56) with (55), we obtain

$$\hat{y} - \hat{l} \approx \tilde{\boldsymbol{\lambda}}^T(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}}) - \hat{\Lambda}. \tag{57}$$

In equation (57), $\boldsymbol{\xi}$ captures sector-level TFP losses due to misallocation within sectors, while the term $-\tilde{\boldsymbol{\lambda}}^T\hat{\boldsymbol{\mu}} - \hat{\Lambda}$ captures an aggregate TFP loss coming from misallocation across sectors. We know from Lemma 2 that both are zero to a first order. We then compute a second-order approximation of $-\tilde{\boldsymbol{\lambda}}^T\hat{\boldsymbol{\mu}} - \hat{\Lambda}$ as function of sectoral markups, and a

[2] Note that the ratios $\frac{L_i}{X_{ij}}$ and $\frac{X_{ij}}{X_{ik}}$ instead are constant across firms f.

second-order approximation of $\boldsymbol{\xi}$ as a function of inflation rates. Note that the second-order terms in $\boldsymbol{\mu} d\log\boldsymbol{\xi}$ vanish around the efficient steady state.

Let us first derive Λ from the consumers' budget constraint. Recall that the budget constraint is given by

$$PC = wL + \Pi - T,$$

where Π are aggregate profits and T is the lump-sum tax used to finance input subsidies. Dividing both sides by PY, we find

$$1 = \Lambda + \frac{\Pi - T}{PY} = \Lambda + \sum_i \lambda_i \left(1 - \frac{1}{m_i}\right), \tag{58}$$

where m_i is the markup of sector i (in levels):

$$m_i \equiv \frac{P_i}{MC_i}.$$

A first-order approximation of equation (58) yields

$$d\log\Lambda = -\frac{1}{\Lambda}\left(\sum_i d\lambda_i\left(1 - \frac{1}{m_i}\right) + \sum_i \lambda_i \frac{\mu_i}{m_i}\right).$$

Therefore,

$$-\lambda^T\hat{\mu} - \hat{\Lambda} \approx \sum_i \left(\frac{1}{\Lambda m_i} - 1\right)\lambda_i\mu_i + \sum_i \frac{d\lambda_i}{\Lambda}\left(1 - \frac{1}{m_i}\right), \tag{59}$$

which is zero around the efficient steady state ($m_i = 1 \forall i$, $\Lambda = 1$). Deriving equation (59) a second time around the efficient steady state yields

$$-\lambda^T\hat{\mu} - \hat{\Lambda} \approx \sum_i d\lambda_i\mu_i - \sum_i \lambda_i\mu_i^2 - \left(\sum_i \lambda_i\mu_i\right)^2 + \sum_i (d\lambda_i - d\tilde{\lambda}_i)\mu_i. \tag{60}$$

We now derive the change in sales shares $d\boldsymbol{\lambda}$. From the definition of $\boldsymbol{\lambda}$, we have

$$d\boldsymbol{\lambda} = \left[d\boldsymbol{\beta}^T + \boldsymbol{\lambda}^T\, d\Omega\right](I - \Omega)^{-1}.$$

Solving for changes in consumption and input shares yields

$$\begin{aligned}\sum_i d\lambda_i\mu_i = {} & \sum_i \sum_{t,l} \beta_t\beta_l(1 - \sigma_{tl})(d\log p_l - d\log p_t)(I - \Omega)^{-1}_{ti}\mu_i \\ & + \sum_i \sum_t \lambda_t \sum_{k,l} \omega_{tk}\omega_{tl}\left(1 - \theta^t_{kl}\right)(d\log p_l - d\log p_k)(I - \Omega)^{-1}_{ki}\mu_i \\ & + \sum_{i,,j} \lambda_i\left[\Omega(I - \Omega)^{-1}\right]_{ij}\mu_i\mu_j. \end{aligned} \tag{61}$$

Solving for changes in relative prices in (61), we obtain

$$\begin{aligned}\sum_i d\lambda_i\mu_i = &-\sum_{t,l}\beta_t\beta_l(1-\sigma_{tl})(I-\Omega)^{-1}_{ti}\sum_j\big((I-\Omega)^{-1}_{lj}-(I-\Omega)^{-1}_{tj}\big)(\mu_j-\log A_j)\mu_i\\ &-\sum_i\sum_t\lambda_t\sum_{j,k}\sum_l\omega_{tl}\omega_{tk}\big(1-\theta^t_{kl}\big)(I-\Omega)^{-1}_{ki}\\ &\times\sum_j\big((I-\Omega)^{-1}_{lj}-(I-\Omega)^{-1}_{kj}\big)(\mu_j-\log A_j)\mu_i\\ &+\sum_{i,j}\lambda_i\big[\Omega(I-\Omega)^{-1}\big]_{ij}\mu_i\mu_j. \end{aligned} \tag{62}$$

To this we also need to add the cross-partial from the interaction between productivity and markups, which comes from the Hulten term in $\hat{y}-\hat{l}$.[3] This is given by

$$\sum_{i,j}\frac{d\tilde{\lambda}_i}{d\log m_j}\log A_i\mu_j.$$

As we can see from equation (62), around an efficient equilibrium it holds that

$$\sum_{i,j}\frac{d\tilde{\lambda}_i}{d\log m_j}\log A_i\mu_j=-\sum_{i,j}\frac{d\lambda_i}{d\log a_j}\log A_j\mu_i.$$

Moreover, we have

$$\frac{d\lambda_i-d\tilde{\lambda}_i}{d\log A_j}=0.$$

Therefore, all terms in $\log A_i\mu_j$ cancel out, and we are left with second derivatives with respect to markups only.

The terms in $\mu_i\mu_j$ that are independent of substitution elasticities in the first two lines of (62) sum up to

$$\Big(\sum_i\lambda_i\mu_i\Big)^2+\sum_i\lambda_i\mu_i^2.$$

Moreover, using the relation

$$d\omega_{ij}=d\tilde{\omega}_{ij}-\omega_{ij}\mu_i,$$

it is easy to verify that

$$\sum_i(d\lambda_i-d\tilde{\lambda}_i)\mu_i=-\sum_{i,,j}\lambda_i\big[\Omega(I-\Omega)^{-1}\big]_{ij}\mu_i\mu_j,$$

[3]The Hulten term is

$$y-l=\tilde{\lambda}^T a.$$

This term cancels out to a first order when looking at deviations from the efficient equilibrium. However, the second-order cross-partial does not cancel out.

which cancels out with the last term in (62). Therefore, all terms in $\mu_i \mu_j$ that are independent of substitution elasticities cancel out.

Plugging (62) into (60), the remaining terms yield

$$
\begin{aligned}
-\boldsymbol{\lambda}^T \hat{\boldsymbol{\mu}} - \hat{\Lambda} \approx\ & \frac{1}{2} \sum_{i,j} \sum_{t,l} \beta_t \beta_l \sigma_{tl} \big((I-\Omega)^{-1}_{ti} - (I-\Omega)^{-1}_{li} \big) \big((I-\Omega)^{-1}_{tj} - (I-\Omega)^{-1}_{lj} \big) \mu_i \mu_j \\
& + \frac{1}{2} \sum_{i,j} \sum_{t} \lambda_t \sum_{k,l} \bigg[\sum_{l} \omega_{tl} \omega_{tk} \theta^t_{kl} \big((I-\Omega)^{-1}_{ki} - (I-\Omega)^{-1}_{li} \big) \\
& \times \big((I-\Omega)^{-1}_{kj} - (I-\Omega)^{-1}_{lj} \big) \bigg] \mu_i \mu_j. \qquad (63)
\end{aligned}
$$

Equation (63) can further be written in terms of the substitution operators Φ introduced in Definition 4, which yields the quadratic function

$$
\begin{aligned}
-\boldsymbol{\lambda}^T \hat{\boldsymbol{\mu}} - \hat{\Lambda} & \approx \frac{1}{2} \sum_{i,j} \bigg[\Phi_C \big((I-\Omega)^{-1}_{(i)}, (I-\Omega)^{-1}_{(j)} \big) \\
& \qquad - \sum_t \lambda_t \Phi^{\mu}_t \big((I-\Omega)^{-1}_{(i)}, (I-\Omega)^{-1}_{(j)} \big) \bigg] \mu_i \mu_j \\
& = \frac{1}{2} \sum_{i,j} \mathcal{L}^{\text{across}}_{ij} \mu_i \mu_j. \qquad (64)
\end{aligned}
$$

To obtain the expression in the paper, just replace

$$
\boldsymbol{\mu} = (I-\Omega)\mathbf{p} - \boldsymbol{\alpha} w
$$

and note that $\mathcal{L}^{\text{across}}_{ij} \boldsymbol{\alpha} = \mathbf{0}$. This concludes the derivation of the aggregate TFP loss from across-sector misallocation.

The derivation of within-sector TFP losses $\boldsymbol{\xi}$ is the same as in the traditional one-sector model (Gali (2015),Woodford (2003)), just replicated sector by sector. Index by f the different varieties of product i and note that, given the CES assumption, sectoral output can be written as

$$
Y_i = A_i F\big(\{x_{ij}\}, L_i\big) \frac{p_i^{-\epsilon_i}}{\int p_{if}^{-\epsilon_i}\, df}, \qquad (65)
$$

where x_{ij} and L_i are defined above. The productivity wedges ξ therefore are given by

$$
\xi_i = \log \left(\frac{p_i^{-\epsilon_i}}{\int p_{it}^{-\epsilon_i}\, dt} \right).
$$

To a first order, ξ_i can be approximated as

$$d\xi_i = \epsilon_i \left[\frac{\int p_{if}^{-\epsilon_i} d\log p_{if}\, df}{\int p_{if}^{-\epsilon_i}\, df} - \frac{\int p_{if}^{1-\epsilon_i} d\log p_{if}\, df}{\int p_{if}^{1-\epsilon_i}\, df} \right]. \tag{66}$$

Given the Calvo assumption, around the efficient steady state we have that

$$\frac{\int p_{if}^{-\epsilon_i} d\log p_{if}\, df}{\int p_{if}^{-\epsilon_i}\, df} = \frac{\int p_{if}^{1-\epsilon_i} d\log p_{if}\, df}{\int p_{if}^{1-\epsilon_i}\, df} = \delta d\log mc_i,$$

so that $d\xi_i = 0$.

The second-order approximation $d^2\xi_i$ is obtained by deriving (66) a second time with respect to $\{d\log p_{it}\}$. We have

$$d^2\xi_i = \epsilon_i \left[\int (\log p_{if} - \log p_i)^2\, df - \left(\int (\log p_{if} - \log p_i)\, df \right)^2 \right].$$

The following lemma shows how to write the present discounted sum of within-sector losses as a function of sectoral inflation rates.

LEMMA 5: *It holds that*

$$\sum_{s\geq 0} \rho^s \xi_{it+s} \approx \epsilon_i \frac{1-\hat{\delta}_i}{\hat{\delta}_i} \sum_{s\geq 0} \rho^s \pi_{it+s}^2. \tag{67}$$

Define

$$D_{it} \equiv \int (\log p_{ift+s} - \log p_{it+s})^2\, df - \left(\int (\log p_{ift+s} - \log p_{it+s})\, df \right)^2.$$

Given the Calvo assumption, in each sector i the fraction δ_i of firms who adjust prices set

$$\log p_{ift} - \log p_{it} = (1-\delta_i)\big(\log p_{it}^* - \log p_{it-1}\big) = \frac{1-\delta_i}{\delta_i} \pi_{it}.$$

For the remaining fraction $(1-\delta_i)$ of non-adjusting firms, we have

$$\begin{aligned} \log p_{ift} - \log p_{it} &= (-\delta_i)\big(\log p_{it}^* - \log p_{it-1}\big) + (\log p_{ift-1} - \log p_{it-1}) \\ &= (\log p_{ift-1} - \log p_{it-1}) - \pi_{it}. \end{aligned}$$

It follows that, around a steady state where $\log p_{ift} - \log p_{it} = 0\ \forall f$,

$$D_{it} = (1-\delta_i)\left(\frac{1}{\delta_i} \pi_{it}^2 + D_{it-1} \right).$$

This in turn implies the desired result:

$$\sum_s \rho^s D_{it+s} = \sum_s \rho^s \frac{1-\delta_i}{\delta_i}\pi_{is}^2\left(\sum_{\tau\geq s}\big(\rho(1-\delta_i)\big)^{\tau-s}\right) = \frac{1-\hat{\delta}_i}{\hat{\delta}_i}\sum_s \rho^s \pi_{is}^2.$$

Together, equations (57) and (67) also allow us to express the second-order welfare loss from within-sector misallocation as a quadratic function of sectoral inflation rates:

$$\sum_t \rho^t \boldsymbol{\lambda}^T \boldsymbol{\xi}_t \approx -\frac{1}{2}\sum_t \rho^t \pi_t^T (I-\Delta)\Delta^{-1}\operatorname{diag}(\boldsymbol{\lambda})\operatorname{diag}(\boldsymbol{\epsilon})\pi_t.$$

Q.E.D.

PROOF OF LEMMA 4: Denote

$$\mathcal{A} \equiv \begin{pmatrix} \frac{1}{\rho}(I-\mathcal{V})^{-1} & \frac{1}{\rho}(I-\mathcal{V})^{-1}\mathcal{V} & -\frac{1}{\rho}(I-\mathcal{V})^{-1}\boldsymbol{B}(\gamma+\varphi) \\ I & I & \mathbf{0} \\ \frac{\zeta}{\gamma}\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1} & \mathbf{0}^T & 1 \end{pmatrix}.$$

The dynamic system that governs the economy under the given Taylor rule is

$$\begin{pmatrix} \mathbb{E}\boldsymbol{\pi}_{t+1} \\ \mathbf{p}_t \\ \mathbb{E}\tilde{y}_{t+1} \end{pmatrix} = \mathcal{A}\begin{pmatrix} \boldsymbol{\pi}_t \\ \mathbf{p}_{t-1} \\ \tilde{y}_t \end{pmatrix} - \begin{pmatrix} (I-\mathcal{V})^{-1}\mathcal{V} \\ \mathbb{O} \\ \mathbf{0}^T \end{pmatrix} d\log \mathbf{A}_t. \tag{68}$$

The matrix $\mathcal{A}$ has a unit eigenvalue, with eigenvector $\boldsymbol{\pi} = \mathbf{0}$, $\mathbf{p} = \mathbf{1}$, $\tilde{y} = 0$. Lemma 6 below demonstrates that the system (68) has a unique bounded solution for any given past prices $\mathbf{p}_{t-1}$ and productivity shocks $d\log\mathbf{A}_t$, if and only if the matrix $\mathcal{A}$ has $N+1$ additional eigenvalues strictly larger than 1, and $N-1$ strictly smaller than 1 in modulus. For now, let us take this result as given and prove that the condition is satisfied.

Any eigenvector $\begin{pmatrix} \boldsymbol{\pi} \\ \mathbf{p} \\ \tilde{y} \end{pmatrix}$ of the matrix $\mathcal{A}$, with corresponding eigenvalue $\nu \neq 1$, must satisfy

$$\mathbf{p} = \frac{\boldsymbol{\pi}}{\nu - 1},$$

$$\tilde{y} = \frac{\zeta}{\nu-1}\frac{\gamma+\varphi}{\gamma}\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\boldsymbol{\pi},$$

$$\rho\nu(I-\mathcal{V})\boldsymbol{\pi} = \left[I + \frac{\mathcal{V}}{\nu-1} - \frac{\zeta}{\nu-1}\frac{\gamma+\varphi}{\gamma}\boldsymbol{B}\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\right]\boldsymbol{\pi}. \tag{69}$$

The last equation can be rearranged to obtain the condition

$$\frac{1-\rho\nu+\rho\nu^2}{\nu}(I-\mathcal{V})\boldsymbol{\pi} = \left[I - \frac{\zeta}{\nu}\frac{\gamma+\varphi}{\gamma}\boldsymbol{B}\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\right]\boldsymbol{\pi}. \tag{70}$$

Denote the eigenvectors of the matrix $I-\mathcal{V}$ by $\mathbf{v}_i$, $i \in \{1,\dots,N\}$, ordering them so that $\mathbf{v}_N = \mathbf{1}$, and denote by $\{\xi_i\}_{i=1}^N$ the corresponding eigenvalues, recalling that $|\xi_i| \in (0,1)$

for $i = 1, \dots, N-1$.[4] It is easy to verify that $\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\mathbf{v}_i = 0\ \forall i \in \{1, \dots, N-1\}$. Therefore, the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_{N-1}$ satisfy condition (70), setting

$$\frac{1-\rho\nu+\rho\nu^2}{\nu} = \xi_i. \tag{71}$$

For $|\xi_i| \in (0, 1)$, equation (71) has two roots, ν_i^+ and ν_i^-, such that $|\nu_i^+| > 1$ and $|\nu_i^-| < 1$. Hence, $\{\nu_i^+, \nu_i^-\}_{i=1}^{N-1}$ are $N-1$ pairs of eigenvalues of $\mathcal{A}$, one with modulus strictly greater than 1, and the other with modulus strictly smaller than 1. To construct the remaining pair, note that premultiplying the first equation in (68) times the vector $\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}$ implies the divine coincidence result:[5]

$$(\gamma+\varphi)\tilde{y}_t = \boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}(\boldsymbol{\pi}_t - \rho\mathbb{E}\boldsymbol{\pi}_{t+1}).$$

This condition must hold for the eigenvectors of $\mathcal{A}$ as well, implying

$$(\gamma+\varphi)\tilde{y}_t = \frac{\zeta}{\nu-1}\frac{\gamma+\varphi}{\gamma}\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\boldsymbol{\pi} = (1-\rho\nu)\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\boldsymbol{\pi}_t.$$

Rearranging the equality yields

$$\left[(1-\rho\nu)(1-\nu) + \zeta\frac{\gamma+\varphi}{\gamma}\right]\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\boldsymbol{\pi} = 0.$$

We know that this condition is satisfied when setting $\boldsymbol{\pi} = \mathbf{v}_i$, $i = \{1, \dots, N-1\}$. Moreover, it is also satisfied when ν is such that

$$(1-\rho\nu)(1-\nu) + \zeta\frac{\gamma+\varphi}{\gamma} = 0. \tag{72}$$

For $\zeta = 0$, the solutions to this equation are $\nu_N^- = 1$ and $\nu_N^+ = \frac{1}{\rho}$. For $\zeta > 0$, therefore, both solutions must have modulus larger than 1. From equation (70), the inflation component of the corresponding eigenvectors is given by

$$\boldsymbol{\pi} = \left[I - \frac{1-\rho\nu+\rho\nu^2}{\nu}(I-\Delta\Omega)^{-1}(I-\Delta)\right]^{-1}\mathcal{B}$$

for $\nu = \nu_N^-$ or $\nu = \nu_N^+$, respectively.[6] The components corresponding to lagged prices and the output gap can then be computed using (69). *Q.E.D.*

---

[4] This follows from the Perron–Frobenius theorem, and from the fact that $I - \mathcal{V}$ is a probability matrix.

[5] To obtain the result, we used the relations

$$\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\mathcal{M} = \boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1},$$
$$\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\mathcal{V} = \mathbf{0}^T,$$
$$\boldsymbol{\lambda}^T(I-\Delta)\Delta^{-1}\boldsymbol{B} = 1.$$

[6] The matrix $I - \frac{1-\rho\nu+\rho\nu^2}{\nu}(I-\Delta\Omega)^{-1}(I-\Delta)$ is always invertible, because equation (72) implies $|\frac{1-\rho\nu+\rho\nu^2}{\nu}| < 1$ for $\nu = \nu_N^+$ or $\nu = \nu_N^-$, while all eigenvalues of the matrix $(I-\Delta\Omega)^{-1}(I-\Delta)$ have modulus greater than or equal to 1. To see this, it is easier to show that all the eigenvalues of the inverse matrix $(I-\Delta)^{-1}(I-\Delta\Omega)$ are weakly smaller than 1 in modulus. This follows from the Perron–Frobenius theorem, because this matrix has

LEMMA 6: *The system* (68) *has a unique bounded solution if and only if the matrix* $\mathcal{A}$ *has* N *eigenvalues strictly smaller than* 1 *in modulus*, *and* N *eigenvalues strictly larger than* 1, *in addition to the unit eigenvalue*.

PROOF: Given our assumptions about the productivity process, we have that

$$\mathbb{E}\lim_{t\to\infty}\begin{pmatrix}\boldsymbol{\pi}_t\\ \mathbf{p}_{t-1}\\ \tilde{y}_t\end{pmatrix}=\lim_{t\to\infty}\mathcal{A}^t\begin{pmatrix}\boldsymbol{\pi}_0\\ \mathbf{p}_{-1}\\ \tilde{y}_0\end{pmatrix}$$
$$-\lim_{t\to\infty}\left(\sum_{s\leq t}\eta^s\mathcal{A}^{t-s}\right)\begin{pmatrix}\frac{1}{\rho}(I-\mathcal{V})^{-1}\mathcal{V}\\ \mathbb{O}\\ \mathbf{0}^T\end{pmatrix}d\log\mathbf{A}_0.$$

We can decompose the productivity term as a linear combination of the eigenvectors of $\mathcal{A}$, $\{w_1,\dots,w_{2N+1}\}$:

$$\begin{pmatrix}\frac{1}{\rho}(I-\mathcal{V})^{-1}\mathcal{V}\\ \mathbb{O}\\ \mathbf{0}^T\end{pmatrix}d\log\mathbf{A}_0=b_1w_1+\cdots+b_{2N}w_{2N+1}.$$

Let us also write $\begin{pmatrix}\boldsymbol{\pi}_0\\ \mathbf{p}_{-1}\\ \tilde{y}_0\end{pmatrix}$ in components with respect to $\{w_1,\dots,w_{2N+1}\}$:

$$\begin{pmatrix}\boldsymbol{\pi}_0\\ \mathbf{p}_{-1}\\ \tilde{y}_0\end{pmatrix}=\sum_{i=1}^{2N+1}x_iw_i.$$

Denote by $\{\nu_1,\dots,\nu_{2N+1}\}$ the eigenvalues corresponding to $\{w_1,\dots,w_{2N+1}\}$. We have

$$\lim_{t\to\infty}\mathcal{A}^t\begin{pmatrix}\boldsymbol{\pi}_0\\ \mathbf{p}_{-1}\\ \tilde{y}_0\end{pmatrix}+\left(\sum_{s\leq t}\eta^s\mathcal{A}^{t-s}\right)\begin{pmatrix}\frac{1}{\rho}\mathcal{M}^{-1}\mathcal{V}(I-\Omega)\\ \mathbb{O}\\ \mathbf{0}^T\end{pmatrix}d\log\mathbf{A}_0$$
$$=\lim_{t\to\infty}\sum_{i/|\nu_i|\geq 1}\left[x_i+\frac{\nu_i}{\nu_i-\eta}b_i\right]\nu_i^tw_i.$$

To have a bounded solution, we need

$$\begin{pmatrix}\boldsymbol{\pi}_0\\ \mathbf{p}_{-1}\\ \tilde{y}_0\end{pmatrix}=-\sum_{i/|\nu_i|>1}\frac{\nu_i}{\nu_i-\eta}b_iw_i+\sum_{i/|\nu_i|\leq 1}x_iw_i. \tag{73}$$

---

all positive elements, and the sum of each row is less than or equal to

$$\frac{1-\min_k\{\hat{\delta}_k\}\sum_j\Omega_{ij}}{1-\min_k\{\hat{\delta}_k\}}=1.$$

This is a system of $2N+1$ equations, with unknowns $\boldsymbol{\pi}_0$, $\tilde{y}_0$, and $\{x_i\}_{i/|\nu_i|\leq 1}$. The system has a unique solution if and only if the matrix $\mathcal{A}$ has exactly $N-1$ eigenvalues with modulus strictly smaller than 1. *Q.E.D.*

Example 7

In this example, we complete the derivation of the horizontal component of the optimal output gap. This component is similar to equation (35), with an extra term coming from wage rigidity ($\delta_L < 1$):

$$y^*_{\text{hor}} = (\epsilon - \sigma)(\gamma + \varphi)\frac{\delta_L\big(1 - \mathbb{E}_\beta(\delta)\big)}{1 - \delta_L \mathbb{E}_\beta(\delta)}\mathbb{E}_{\beta(1-\delta)}\boldsymbol{\pi},$$

where $\mathbb{E}_{\beta(1-\delta)}$ is defined as in Example 6. Note that $y^*_{\text{hor}} = 0$ when wages are fully rigid, because in this case monetary policy has no effect on wages and prices. Solving for inflation as a function of the oil shock, we find

$$\mathbb{E}_{\beta(1-\delta)}\boldsymbol{\pi} = -\bigg[\mathbb{E}_{\beta(1-\delta)}(\delta\omega_{\text{oil}}) - \frac{\delta_L\big(1 - \mathbb{E}_\beta(\delta)\big)}{1 - \delta_L \mathbb{E}_\beta(\delta)}\mathbb{E}_{\beta(1-\delta)}(\delta)\mathbb{E}_{\beta(1-\delta)}(\omega_{i,\text{oil}})\bigg]\log A_{\text{oil}}. \tag{74}$$

The expression in square brackets is similar to the covariance $\text{Cov}_{\beta(1-\delta)}(\delta, \omega_{\text{oil}})$, slightly modified to account for wage rigidity. The expectation $\mathbb{E}_{\beta(1-\delta)}(\delta\omega_{\text{oil}})$ is the direct effect of productivity on marginal costs, while the product $\mathbb{E}_{\beta(1-\delta)}(\delta)\mathbb{E}_{\beta(1-\delta)}(\omega_{\text{oil}})$ is the indirect effect through lower wages. This second effect is muted with wage rigidities ($\delta_L < 1$), so that (74) is always larger than the covariance $\text{Cov}_{\beta(1-\delta)}(\delta, \omega_{\text{oil}})$. In practice, when calibrating sectoral oil shares and price adjustment frequencies to the U.S. data, equation (74) is positive for $\log A_{\text{oil}} < 0$, but small.

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