Econometrica Supplementary Material

SUPPLEMENT TO "CAUTION AND REFERENCE EFFECTS"

(*Econometrica*, Vol. 92, No. 6, November 2024, 2069–2103)

SIMONE CERREIA-VIOGLIO
Department of Decision Sciences and IGIER, Bocconi University

DAVID DILLENBERGER
Department of Economics, University of Pennsylvania

PIETRO ORTOLEVA
Department of Economics and SPIA, Princeton University

THIS APPENDIX includes all the missing proofs and the ancillary facts used in the main body of the paper. We begin with a section on facts instrumental for Theorem 1 and Proposition 5.

Foundation

Recall the definition of $\succcurlyeq'$ in Section 5, that is,

$$p \succcurlyeq' q \quad \overset{\text{def}}{\Longleftrightarrow} \quad \lambda p + (1-\lambda)r \succcurlyeq \lambda q + (1-\lambda)r \quad \forall \lambda \in (0,1], \forall r \in \Delta.$$

The goal of this section is to provide a Multi-Expected Utility representation for $\succcurlyeq'$.

LEMMA 1: *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order. The following statements are true*:

1. *The relation $\succcurlyeq$ satisfies M-NCI if and only if, for each $p \in \Delta$ and for each $m \in \mathbb{R}$,*

$$p \succcurlyeq \delta_{me_1} \quad \Longrightarrow \quad p \succcurlyeq' \delta_{me_1}. \quad (\textit{Equivalently, } p \not\succcurlyeq' \delta_{me_1} \Longrightarrow \delta_{me_1} \succ p.)$$

2. *If $\succcurlyeq$ satisfies Monotonicity, then for each $x, y \in \mathbb{R}^k$,*

$$x > y \quad \Longrightarrow \quad \delta_x \succ' \delta_y. \tag{12}$$

3. *If $\succcurlyeq$ satisfies Monetary equivalent, then, for each $x, y \in \mathbb{R}^k$, there exists $m \in \mathbb{R}_+$ such that*

$$\delta_{y+me_1} \succcurlyeq' \delta_x \succcurlyeq' \delta_{y-me_1}. \tag{13}$$

PROOF: All three points follow from the definition of $\succcurlyeq'$ and M-NCI, Monotonicity, and Monetary equivalent, respectively. *Q.E.D.*

Simone Cerreia-Vioglio: simone.cerreia@unibocconi.it
David Dillenberger: ddill@econ.upenn.edu
Pietro Ortoleva: pietro.ortoleva@princeton.edu

Aumann Utilities and Multi-Expected Utility Representations

In this section, in our formal results, we consider a binary relation $\succcurlyeq^*$ over Δ such that

$$p \succcurlyeq^* q \quad \Longleftrightarrow \quad \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}, \tag{14}$$

where $\mathcal{W} \subseteq C(\mathbb{R}^k)$. Recall that a function $v \in C(\mathbb{R}^k)$ is an Aumann utility if and only if

$$p \succ^* q \quad \Longrightarrow \quad \mathbb{E}_p(v) > \mathbb{E}_q(v) \quad \text{and} \quad p \sim^* q \quad \Longrightarrow \quad \mathbb{E}_p(v) = \mathbb{E}_q(v).$$

We denote by e the vector whose components are all 1's. We endow $C(\mathbb{R}^k)$ with the distance $d : C(\mathbb{R}^k) \times C(\mathbb{R}^k) \to [0, \infty)$ defined by

$$d(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \min\left\{ \max_{x \in [-ne, ne]} \left| f(x) - g(x) \right|, 1 \right\} \quad \forall f, g \in C\left(\mathbb{R}^k\right).$$

It is routine to show that $(C(\mathbb{R}^k), d)$ is separable.[1] Moreover, if $\{f_m\}_{m \in \mathbb{N}} \subseteq C(\mathbb{R}^k)$ is such that $f_m \stackrel{d}{\to} f$, then $\{f_m\}_{m \in \mathbb{N}}$ converges uniformly to f on each compact subset of $\mathbb{R}^k$.

PROPOSITION 7: *If $\succcurlyeq^*$ is as in (14) and such that*

$$x > y \quad \Longrightarrow \quad \delta_x \succ^* \delta_y, \tag{15}$$

then $\succcurlyeq^*$ admits a strictly increasing Aumann utility.

PROOF: By (14), observe that $x > y$ implies $v(x) \geq v(y)$ for all $v \in \mathcal{W}$. This implies that each $v \in \mathcal{W}$ is increasing. By Aliprantis and Border (2006, Corollary 3.5), there exists a countable d-dense subset D of $\mathcal{W}$. Clearly, we have that

$$p \succcurlyeq^* q \quad \Longrightarrow \quad \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{16}$$

Vice versa, consider $p, q \in \Delta$ such that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in D$. Since p and q have compact support, there exists $\bar{n} \in \mathbb{N}$ such that $[-\bar{n}e, \bar{n}e]$ contains both supports. Consider $v \in \mathcal{W}$. Since D is d-dense in $\mathcal{W}$, there exists a sequence $\{v_l\}_{l \in \mathbb{N}} \subseteq D$ such that $v_l \stackrel{d}{\to} v$. It follows that v_l converges uniformly on $[-\bar{n}e, \bar{n}e]$. This implies that

$$\begin{aligned} \mathbb{E}_p(v) &= \int_{[-\bar{n}e, \bar{n}e]} v \, \mathrm{d}p = \lim_l \int_{[-\bar{n}e, \bar{n}e]} v_l \, \mathrm{d}p = \lim_l \mathbb{E}_p(v_l) \\ &\geq \lim_l \mathbb{E}_q(v_l) = \lim_l \int_{[-\bar{n}e, \bar{n}e]} v_l \, \mathrm{d}q = \int_{[-\bar{n}e, \bar{n}e]} v \, \mathrm{d}q = \mathbb{E}_q(v). \end{aligned}$$

By (14) and (16) and since v was arbitrarily chosen, we can conclude that

$$p \succcurlyeq^* q \Longleftrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{17}$$

Since D is countable, we can list its elements: $D = \{v_m\}_{m \in \mathbb{N}}$. Set $b_l = l + \max\{|v_l(-le)|, |v_l(le)|\}$ for all $l \in \mathbb{N}$ and $a_m = \Pi_{l=1}^m b_l \geq b_m$ for all $m \in \mathbb{N}$. Finally, define $v : \mathbb{R}^k \to \mathbb{R}$ by

$$v(x) = \sum_{m=1}^{\infty} \frac{v_m(x)}{a_m} \quad \forall x \in \mathbb{R}^k. \tag{18}$$

[1] A proof is available upon request.

We first prove that v is a well-defined continuous function. Fix $x \in \mathbb{R}^k$. It follows that there exists $\bar{m} \in \mathbb{N}$ such that $x \in [-me, me]$ for all $m \geq \bar{m}$. Since each v_m is increasing, we have that $|v_m(x)| \leq \max\{|v_m(-me)|, |v_m(me)|\} \leq b_m \leq a_m$ for all $m \geq \bar{m}$. Since $a_m \geq m!$ for all $m \in \mathbb{N}$, it follows that

$$\frac{|v_m(x)|}{a_m} = \frac{|v_m(x)|}{b_m a_{m-1}} \leq \frac{1}{a_{m-1}} \leq \frac{1}{(m-1)!} \quad \forall m \geq \bar{m}+1.$$

This implies that the right-hand side of (18) converges. Since x was arbitrarily chosen, v is well-defined. Next, consider $n \in \mathbb{N}$. From the same argument above, we have that

$$\frac{|v_m(x)|}{a_m} \leq \frac{1}{(m-1)!} \quad \forall x \in [-ne, ne], \forall m \geq n+1.$$

By Weierstrass's M-test and since $\{v_m/a_m\}_{m\in\mathbb{N}}$ is a sequence of continuous functions, we can conclude that $v = \sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on $[-ne, ne]$, yielding that v is continuous on $[-ne, ne]$. Since n was arbitrarily chosen, it follows that v is continuous.

Finally, assume that $p \succ^* q$ (resp. $p \sim^* q$). By (17), we have that $\mathbb{E}_p(v_m) \geq \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m) = \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$). In particular, we have that $\mathbb{E}_p(v_m/a_m) \geq \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}/a_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}}/a_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m/a_m) = \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$). Since $\sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on compacta and the supports of p and q are compact, we can conclude that

$$\begin{aligned}
\mathbb{E}_p(v) - \mathbb{E}_q(v) &= \mathbb{E}_p\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) = \lim_l \sum_{m=1}^{l} \mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \lim_l \sum_{m=1}^{l} \mathbb{E}_q\left(\frac{v_m}{a_m}\right) \\
&= \lim_l \left[\sum_{m=1}^{l}\left(\mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\frac{v_m}{a_m}\right)\right)\right].
\end{aligned}$$

This implies that if $p \succ^* q$ (resp. $p \sim^* q$), then $\mathbb{E}_p(v) > \mathbb{E}_q(v)$ (resp. $\mathbb{E}_p(v) = \mathbb{E}_q(v)$), proving that v is an Aumann utility. In particular, by (15), v is strictly increasing. *Q.E.D.*

Consider a binary relation $\succcurlyeq^*$ on Δ. Define $\mathcal{W}_{\max}(\succcurlyeq^*)$ as the set of all strictly increasing functions $v \in C(\mathbb{R}^k)$ such that $v(0) = 0$ and $p \succcurlyeq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. We say that a set $\mathcal{W}$ in $C(\mathbb{R}^k)$ has full image if and only if

$$\forall x, y \in \mathbb{R}^k, \exists m \in \mathbb{R}_+ \text{ s.t. } v(y + me_1) \geq v(x) \geq v(y - me_1) \quad \forall v \in \mathcal{W}.$$

PROPOSITION 8: *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in* (14). *If $\succcurlyeq^*$ satisfies* (12) *and* (13), *then $\mathcal{W}_{\max}(\succcurlyeq^*)$ is a nonempty convex set with full image that satisfies* (14).

PROOF: Consider $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and $\lambda \in (0, 1)$. Since both functions are strictly increasing and continuous and such that $v_1(0) = 0 = v_2(0)$, it follows that $\lambda v_1 + (1-\lambda)v_2$ is strictly increasing, continuous, and takes value 0 in 0. Since $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$ and $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$. This implies that

$$\begin{aligned}
\mathbb{E}_p\big(\lambda v_1 + (1-\lambda)v_2\big) &= \lambda \mathbb{E}_p(v_1) + (1-\lambda)\mathbb{E}_p(v_2) \\
&\geq \lambda \mathbb{E}_q(v_1) + (1-\lambda)\mathbb{E}_q(v_2) = \mathbb{E}_q\big(\lambda v_1 + (1-\lambda)v_2\big),
\end{aligned}$$

proving that $\lambda v_1 + (1-\lambda)v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, $\mathcal{W}_{\max}(\succcurlyeq^*)$ is convex. By Proposition 7, there exists a strictly increasing $\hat{v} \in C(\mathbb{R}^k)$ such that

$$p \succ^* q \quad \Longrightarrow \quad \mathbb{E}_p(\hat{v}) > \mathbb{E}_q(\hat{v}) \quad \text{and} \quad p \sim^* q \quad \Longrightarrow \quad \mathbb{E}_p(\hat{v}) = \mathbb{E}_q(\hat{v}).$$

Without loss of generality, we can assume that $\hat{v}(0) = 0$ (given $\hat{v}$, set $v = \hat{v} - \hat{v}(0)$) and, in particular, we have that $\hat{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is nonempty. Since $\succcurlyeq^*$ satisfies (13), it follows that $\mathcal{W}_{\max}(\succcurlyeq^*)$ has full image. Since $\succcurlyeq^*$ satisfies (12), v is increasing for all $v \in \mathcal{W}$. This implies that for each $v \in \mathcal{W}$ and for each $n \in \mathbb{N}$, the function $v_n = (1 - \frac{1}{n})v + \frac{1}{n}\hat{v} - [(1-\frac{1}{n})v(0) + \frac{1}{n}\hat{v}(0)] \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. Vice versa, we have that

$$\begin{aligned} &\mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*) \\ &\quad\Longrightarrow \quad \mathbb{E}_p\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \geq \mathbb{E}_q\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \quad \forall v \in \mathcal{W}, \forall n \in \mathbb{N} \\ &\quad\Longrightarrow \quad \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \quad \Longrightarrow \quad p \succcurlyeq^* q, \end{aligned}$$

proving that (14) holds with $\mathcal{W}_{\max}(\succcurlyeq^*)$ in place of $\mathcal{W}$. *Q.E.D.*

We conclude by discussing Multi-Expected Utility representations which feature odd sets. To do this, we make two simple observations. First, recall the map $\sigma : \Delta \to \Delta$, which swaps gains with losses, defined by

$$\sigma(p)(B) = p(-B) \quad \text{for all Borel subsets } B \text{ of } \mathbb{R}^k \text{ and for all } p \in \Delta.$$

It is immediate to see that σ is affine and $\sigma(\sigma(p)) = p$ for all $p \in \Delta$. Second, by the Change of Variable Theorem (see, e.g., Aliprantis and Border 2006, Theorem 13.46), we have that

$$\mathbb{E}_{\sigma(r)}(v) = \int_{\mathbb{R}^k} v \, \mathrm{d}\sigma(r) = -\int_{\mathbb{R}^k} \bar{v} \, \mathrm{d}r = -\mathbb{E}_r(\bar{v}) \quad \forall r \in \Delta, \forall v \in C(\mathbb{R}^k), \tag{19}$$

where $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$ and for all $v \in C(\mathbb{R}^k)$.

PROPOSITION 9: *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in* (14) *which satisfies* (12) *and* (13). *The following statements are equivalent*:

(i) *For each $p, q \in \Delta$,*

$$p \succcurlyeq^* q \quad \Longleftrightarrow \quad \sigma(q) \succcurlyeq^* \sigma(p).$$

(ii) *For each $p, q \in \Delta$,*

$$p \succcurlyeq^* q \quad \Longrightarrow \quad \sigma(q) \succcurlyeq^* \sigma(p).$$

(iii) $\mathcal{W}_{\max}(\succcurlyeq^*)$ *is odd*.

Moreover, if $\mathcal{W}$ in (14) *is odd, then* (i) *and* (ii) *hold*.

For the last part of the statement, that is, proving that if $\mathcal{W}$ is odd, then (i) and (ii) hold, we can dispense with the assumption that $\succcurlyeq^*$ satisfies (12) and (13). The proof will clarify.

PROOF: By Proposition 8, we have that

$$p \succcurlyeq^* q \quad \iff \quad \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*).$$

In other words, for the first part of the statement, we can replace $\mathcal{W}$ in (14) with $\mathcal{W}_{\max}(\succcurlyeq^*)$.
(i) implies (ii). It is obvious.
(ii) implies (iii). Fix $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition of $\bar{v}$ and since each v in $\mathcal{W}_{\max}(\succcurlyeq^*)$ is strictly increasing, continuous, and such that $v(0) = 0$, we have that $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. By assumption and (19), we have that

$$\begin{aligned} p \succcurlyeq^* q \quad &\implies \quad \sigma(q) \succcurlyeq^* \sigma(p) \quad \implies \quad \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \\ &\implies \quad -\mathbb{E}_q(\bar{v}) \geq -\mathbb{E}_p(\bar{v}) \quad \implies \quad \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}). \end{aligned}$$

By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, we can conclude that $\bar{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is odd.
(iii) implies (i). By (19) and since $\mathcal{W}$ is odd and represents $\succcurlyeq^*$, we have that

$$\begin{aligned} p \succcurlyeq^* q \quad &\iff \quad \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \quad \iff \quad \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}) \quad \forall v \in \mathcal{W} \\ &\iff \quad \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \quad \forall v \in \mathcal{W} \quad \iff \quad \sigma(q) \succcurlyeq^* \sigma(p), \end{aligned}$$

proving the implication (since $\mathcal{W}_{\max}(\succcurlyeq^*)$ represents $\succcurlyeq^*$) and also the second part of the statement. *Q.E.D.*

Representing $\succcurlyeq'$

We can finally provide a Multi-Expected Utility representation for $\succcurlyeq'$.

PROPOSITION 10: *If $\succcurlyeq$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent, then*

$$p \succcurlyeq' q \quad \iff \quad \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq').$$

Moreover, $\mathcal{W}_{\max}(\succcurlyeq')$ is a nonempty convex set with full image.

PROOF: By the same techniques of Cerreia-Vioglio (2009, Proposition 22) (see also Cerreia-Vioglio, Maccheroni, and Marinacci 2017, Lemma 1 and Footnote 10), $\succcurlyeq'$ is a preorder that satisfies Sequential Continuity and Independence.[2] By Evren (2008, Theorem 2), there exists a set $\mathcal{W} \subseteq C(\mathbb{R}^k)$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Lemma 1 and since $\succcurlyeq$ is a Weak Order which satisfies Monotonicity and Monetary equivalent, we have that $\succcurlyeq'$ satisfies (12) and (13). By Proposition 8 and considering $\succcurlyeq'$ in place of $\succcurlyeq^*$, $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max}(\succcurlyeq')$, proving the statement. *Q.E.D.*

[2]That is, for each two generalized sequences $\{p_\alpha\}_{\alpha \in A}$ and $\{q_\alpha\}_{\alpha \in A}$ in Δ,

$$p_\alpha \succcurlyeq' q_\alpha \quad \forall \alpha \in A, \quad p_\alpha \to p, \quad \text{and} \quad q_\alpha \to q \quad \implies \quad p \succcurlyeq' q.$$

Other Results

In this last section, we prove an ancillary lemma which is instrumental in proving Proposition 4. We show that if $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical.

LEMMA 2: *If $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical.*

PROOF: Define $\succcurlyeq^*$ to be such that $p \succcurlyeq^* q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$, where $\mathcal{W}$ is a finite essential Cautious Utility representation of $\succcurlyeq$. Since $\mathcal{W}$ is finite, we have that the smallest convex cone containing $\mathcal{W}$, denoted by $\operatorname{cone}(\mathcal{W})$, is closed with respect to the $\sigma(C(\mathbb{R}^k), \Delta)$-topology and so is the set $\operatorname{cone}(\mathcal{W}) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}$. By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, it follows that $\operatorname{cone}(\mathcal{W}) \backslash \{0\} \subseteq \mathcal{W}_{\max}(\succcurlyeq^*)$. By Proposition 8, Remark 3, and Evren (2008, Theorem 5) and since $\mathcal{W}$ is a Cautious Utility representation, we have that (where the closure is in the $\sigma(C(\mathbb{R}^k), \Delta)$-topology)

$$\begin{aligned}\operatorname{cone}(\mathcal{W}) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}} &= \operatorname{cl}\big(\operatorname{cone}\big(\mathcal{W}_{\max}\big(\succcurlyeq^*\big)\big) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}\big) \\ &\supseteq \operatorname{cl}\big(\mathcal{W}_{\max}\big(\succcurlyeq^*\big) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}}\big) \\ &\supseteq \mathcal{W}_{\max}\big(\succcurlyeq^*\big) + \{\theta 1_{\mathbb{R}^k}\}_{\theta \in \mathbb{R}},\end{aligned}$$

yielding that $\operatorname{cone}(\mathcal{W}) \backslash \{0\} \supseteq \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, $\operatorname{cone}(\mathcal{W}) \backslash \{0\} = \mathcal{W}_{\max}(\succcurlyeq^*)$. Since the functional $v \mapsto c(p, v)$ is quasiconcave over $\operatorname{cone}(\mathcal{W}) \backslash \{0\}$ for all $p \in \Delta$, it is immediate to see that

$$V(p) = \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \operatorname{cone}(\mathcal{W}) \backslash \{0\}} c(p, v) \quad \forall p \in \Delta.$$

By Remark 3 and since $\mathcal{W} = \{v_i\}_{i=1}^n$ is a finite Cautious Utility representation, we have that $\succcurlyeq$ satisfies Axioms 1–5. By Theorem 1 and its proof, $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Cautious Utility representation for $\succcurlyeq$. In particular, we have that

$$V(p) = \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \operatorname{cone}(\mathcal{W}) \backslash \{0\}} c(p, v) = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p, v) \quad \forall p \in \Delta.$$

Since $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies the Independence axiom and $p \succcurlyeq^* q$ implies $p \succcurlyeq q$, we have that $\succcurlyeq^*$ is a subrelation of $\succcurlyeq'$ and $\mathcal{W}_{\max}(\succcurlyeq') \subseteq \mathcal{W}_{\max}(\succcurlyeq^*) = \operatorname{cone}(\mathcal{W}) \backslash \{0\}$. By contradiction, assume that $\mathcal{W}_{\max}(\succcurlyeq') \neq \operatorname{cone}(\mathcal{W}) \backslash \{0\}$. Since $\mathcal{W}_{\max}(\succcurlyeq')$ is a convex set closed with respect to strictly positive scalar multiplications, this implies that $\mathcal{W} \nsubseteq \mathcal{W}_{\max}(\succcurlyeq')$. If $\mathcal{W}$ is a singleton, then $\succcurlyeq$ is Expected Utility and, in particular, $\succcurlyeq'$ is complete and coincides with $\succcurlyeq$. This implies that $\mathcal{W} = \{v_1\}$ and $\mathcal{W}_{\max}(\succcurlyeq') = \{\lambda v_1\}_{\lambda > 0} = \operatorname{cone}(\mathcal{W}) \backslash \{0\}$, a contradiction. Assume $\mathcal{W}$ is not a singleton. Consider $\breve{v} \in \mathcal{W} \backslash \mathcal{W}_{\max}(\succcurlyeq')$. Since $\mathcal{W}$ is essential, there exists $\bar{p} \in \Delta$ such that $\min_{v \in \mathcal{W}} c(\bar{p}, v) < \min_{v \in \mathcal{W} \backslash \{\breve{v}\}} c(\bar{p}, v)$. Since $\mathcal{W} = \{v_i\}_{i=1}^n$ and $n \geq 2$, without loss of generality, we can set $\breve{v} = v_n \notin \mathcal{W}_{\max}(\succcurlyeq')$. In particular, we have that

$$\inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = \min_{v \in \mathcal{W}} c(\bar{p}, v) = c(\bar{p}, v_n) < c(\bar{p}, v_i) \quad \forall i \in \{1, \ldots, n-1\}. \tag{20}$$

Consider a sequence $\{\hat{v}_m\}_{m \in \mathbb{N}} \subseteq \mathcal{W}_{\max}(\succcurlyeq')$ such that $c(\bar{p}, \hat{v}_m) \downarrow \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. By construction and since $\mathcal{W}_{\max}(\succcurlyeq') \subseteq \operatorname{cone}(\mathcal{W}) \backslash \{0\}$, there exists a collection of scalars

$\{\lambda_{m,i}\}_{m\in\mathbb{N},i\in\{1,\dots,n\}} \subseteq [0,\infty)$ such that $\hat{v}_m = \sum_{i=1}^n \lambda_{m,i} v_i$ for all $m \in \mathbb{N}$. Since $\hat{v}_m$ is strictly increasing, we have that, for each $m \in \mathbb{N}$, there exists $i \in \{1,\dots,n\}$ such that $\lambda_{m,i} > 0$. Define $\lambda_{m,\sigma} = \sum_{i=1}^n \lambda_{m,i} > 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and for each $i \in \{1,\dots,n\}$, define also $\bar{\lambda}_{m,i} = \lambda_{m,i}/\lambda_{m,\sigma}$ as well as $\tilde{v}_m = \sum_{i=1}^n \bar{\lambda}_{m,i} v_i = \hat{v}_m/\lambda_{m,\sigma}$. Since $\lambda_{m,\sigma} > 0$ for all $m \in \mathbb{N}$, it is immediate to see that $c(\bar{p}, \tilde{v}_m) = c(\bar{p}, \hat{v}_m)$ for all $m \in \mathbb{N}$ and, in particular, $c(\bar{p}, \tilde{v}_m) \downarrow \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. For each $m \in \mathbb{N}$, denote by $\bar{\lambda}_m$ the $\mathbb{R}^n$ vector whose ith component is $\bar{\lambda}_{m,i}$. Since $\{\bar{\lambda}_m\}_{m\in\mathbb{R}}$ is a sequence in the $\mathbb{R}^n$ simplex, there exists a subsequence $\{\bar{\lambda}_{m_l}\}_{l\in\mathbb{N}}$ such that $\bar{\lambda}_{m_l,i} \to \bar{\lambda}_i \in [0,1]$ for all $i \in \{1,\dots,n\}$ and $\sum_{i=1}^n \bar{\lambda}_i = 1$. It is immediate to see that $\tilde{v}_{m_l} = \sum_{i=1}^n \bar{\lambda}_{m_l,i} v_i \overset{\sigma(C(\mathbb{R}^k),\Delta)}{\to} \sum_{i=1}^n \bar{\lambda}_i v_i = \tilde{v}$, where $\tilde{v}$ is continuous, strictly increasing, and such that $\tilde{v}(0) = 0$. Moreover, for each $p, q \in \Delta$, we have that $p \succcurlyeq' q$ implies $\mathbb{E}_p(\tilde{v}) \geq \mathbb{E}_q(\tilde{v})$, proving that $\tilde{v} \in \mathcal{W}_{\max}(\succcurlyeq')$. Note that $\bar{\lambda}_n < 1$; otherwise, we would have that $v_n = \tilde{v} \in \mathcal{W}_{\max}(\succcurlyeq')$, a contradiction. By (20) and since $\bar{\lambda}_n < 1$ and the functional $v \mapsto c(p, v)$ is explicitly quasiconcave over $\mathrm{co}(\mathcal{W})$ for all $p \in \Delta$,[3] we have that

$$c(\bar{p}, v_n) < c(\bar{p}, \tilde{v}) = \lim_l c(\bar{p}, \tilde{v}_{m_l}) = \lim_m c(\bar{p}, \tilde{v}_m) = \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = c(\bar{p}, v_n),$$

a contradiction. It follows that $\mathcal{W}_{\max}(\succcurlyeq') = \mathrm{cone}(\mathcal{W})\backslash\{0\}$ and, in particular, $\mathcal{W}$ represents also $\succcurlyeq'$. This implies that $\mathcal{W}$ is canonical. *Q.E.D.*

REFERENCES

ALIPRANTIS, CHARALAMBOS D., AND KIM C. BORDER (2006): *Infinite Dimensional Analysis*. Berlin Heidelberg New York: Springer. [0002,0004,0007]

CERREIA-VIOGLIO, SIMONE (2009): "Maxmin Expected Utility on a Subjective State Space: Convex Preferences Under Risk," Mimeo, Bocconi University. [0005]

CERREIA-VIOGLIO, SIMONE, FABIO MACCHERONI, AND MASSIMO MARINACCI (2017): "Stochastic Dominance Analysis Without the Independence Axiom," *Management Science*, 63 (4), 1097–1109. [0005]

EVREN, ÖZGÜR (2008): "On the Existence of Expected Multi-Utility Representations," *Economic Theory*, 35 (3), 575–592. [0005,0006]

Co-editor Barton L. Lipman handled this manuscript.

Manuscript received 4 April, 2023; final version accepted 30 May, 2024; available online 30 May, 2024.

[3]Formally (see, e.g., Aliprantis and Border (2006, p. 300)), given $p \in \Delta$, for each $h \in \mathbb{N}\backslash\{1\}$, for each $\{v_l\}_{l=1}^h \subseteq \mathrm{co}(\mathcal{W})$, and for each $\{\lambda_l\}_{l=1}^h \subseteq [0,1]$ such that $\sum_{l=1}^h \lambda_l = 1$ and $\lambda_h < 1$,

$$c(p, v_i) > c(p, v_h) \quad \forall i \in \{1,\dots,h-1\} \implies c\left(p, \sum_{i=1}^h \lambda_i v_i\right) > c(p, v_h).$$