Econometrica Supplementary Material

SUPPLEMENT TO "CHEAP TALK WITH TRANSPARENT MOTIVES"

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APPENDIX C

IN THIS APPENDIX, we elaborate on the results mentioned in Section 6 of "Cheap Talk With Transparent Motives" and discuss some additional relevant results.

C.1. *Proof of Proposition 1: Effective Communication*

We now operationalize Chakraborty and Harbaugh's (2010) insight of using fixed-point reasoning to show effective communication is possible, proving Proposition 1. We begin by representing the prior as an average of three posterior beliefs, μ_1, μ_2, and μ_3, such that the three induced estimates of the statistic are noncollinear; one can always find such beliefs because the statistic is itself multivariate. Next, we find a circle of beliefs around the prior within the convex hull of $\{\mu_1, \mu_2, \mu_3\}$. By construction, each belief on said circle yields a different estimate of the statistic. We then document a generalization of the one-dimensional Borsuk–Ulam theorem, which yields an antipodal pair of beliefs μ and μ' on the circle such that $V(\mu) \cap V(\mu')$ is nonempty. Therefore, we can split the prior across μ and μ' to obtain an equilibrium information policy.

In what follows, define the circle $\mathbb{S} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $T\mu$ denote the estimate $\int T \, \mathrm{d}\mu$ of statistic T for any belief $\mu \in \Delta\Theta$.

LEMMA 8: *Let T be a multivariate statistic. Then, a continuous $\varphi : \mathbb{S} \to \Delta\Theta$ exists such that every $z \in \mathbb{S}$ has*:

1. $\frac{1}{2}\varphi(z) + \frac{1}{2}\varphi(-z) = \mu_0$;
2. $T(\varphi(z)) \neq T(\varphi(\hat{z}))$ *for every* $\hat{z} \in \mathbb{S} \setminus \{z\}$;
3. $2\varphi(z) - \mu_0 \in \Delta\Theta$.

PROOF: By assumption, $T(\Theta)$ is noncollinear, and so $T\mu_0 \notin \mathrm{co}\{T\theta_1, T\theta_2\}$ for some distinct $\theta_1, \theta_2 \in \Theta$. Because μ_0 has full support, both $\mu_0(N_1) > 0$ and $\mu_0(N_2) > 0$ for any open neighborhoods N_1 of θ_1 and N_2 of θ_2. We can then define the conditional distribution $\mu_i(\cdot) := \frac{\mu_0(N_i \cap (\cdot))}{\mu_0(N_i)}$ for $i \in \{1, 2\}$. Letting N_1, N_2 be sufficiently small neighborhoods, we may assume $N_1 \cap N_2 = \emptyset$, $T\mu_0 \notin \mathrm{co}\{T\mu_1, T\mu_2\}$, and $\mu(N_1 \cup N_2) < 1$. Therefore, letting $\mu_3(\cdot) := \frac{\mu_0((\cdot) \setminus (N_1 \cup N_2))}{1 - \mu_0(N_1 \cup N_2)}$, we know that $\mu_0 \in \mathrm{co}\{\mu_1, \mu_2, \mu_3\}$, that μ_0 is not in the convex hull any two of $\{\mu_1, \mu_2, \mu_3\}$, and that the three points $\{T\mu_1, T\mu_2, T\mu_3\}$ are affinely independent. So $\mu_0 = \sum_{i=1}^{3} \lambda_i \mu_i$ for some $\mu_1, \mu_2, \mu_3 \in \Delta\Theta$ and $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$. Therefore,

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letting $\epsilon := \frac{1}{2}\min\{\lambda_1, \lambda_2, \lambda_3\}$, define the map

$$\varphi : \mathbb{S} \to \Delta\Theta,$$
$$(x, y) \mapsto (\lambda_1 + \epsilon x)\mu_1 + (\lambda_2 + \epsilon y)\mu_2 + \big[\lambda_3 - \epsilon(x + y)\big]\mu_3.$$

Affine independence of $T\mu_1, T\mu_2, T\mu_3$ ensures $T \circ \varphi$ is injective, and the other desiderata for φ are obviously satisfied. *Q.E.D.*

Next, we document a generalization of the one-dimensional Borsuk–Ulam theorem.

LEMMA 9: *Suppose $f : \mathbb{S} \to \mathbb{R}$ is upper semicontinuous, and every $z \in \mathbb{S}$ has $\max\{f(z), f(-z)\} \geq 0$. Then, some $z \in \mathbb{S}$ exists such that $\min\{f(z), f(-z)\} \geq 0$.*

PROOF: Define $\tilde{f} : \mathbb{S} \to \mathbb{R}$ by letting $\tilde{f}(z) := f(-z)$. By hypothesis, both f and $\tilde{f}$ are upper semicontinuous and $\{\tilde{f} < 0\} \subseteq \{f \geq 0\}$. Assume for a contradiction that the lemma fails, so that $\{\tilde{f} \geq 0\} \subseteq \{f < 0\}$. Because $\{\tilde{f} < 0\} \cup \{\tilde{f} \geq 0\} = \mathbb{S}$ and $\{f \geq 0\} \cap \{f < 0\} = \emptyset$, these containments in fact imply $\{\tilde{f} < 0\} = \{f \geq 0\}$ and $\{\tilde{f} \geq 0\} = \{f < 0\}$. But (given the definition of $\tilde{f}$) the two sets would both be empty if either were, and so would fail to cover $\mathbb{S}$. Therefore, the set $\{f \geq 0\}$ is a nonempty clopen proper subset of the connected space $\mathbb{S}$, a contradiction. *Q.E.D.*

We now complete the proof of the generalization of Chakraborty and Harbaugh's (2010) Theorem 1.

PROOF OF PROPOSITION 1: First, let $\varphi : \mathbb{S} \to \mathbb{R}$ be as delivered by Lemma 8. Next, define the function

$$f : \mathbb{S} \to \mathbb{R},$$
$$z \mapsto \max V\big(\varphi(z)\big) - \min V\big(\varphi(-z)\big).$$

Two properties of f are immediate. First, f is upper semicontinuous because V is upper hemicontinuous. Second, any $z \in \mathbb{S}$ satisfies $f(z) + f(-z) \geq 0$ because $\max V \geq \min V$. Therefore, Lemma 9 delivers $z \in \mathbb{S}$ with $f(z), f(-z) \geq 0$. That is, $\max V(\varphi(z)) \geq \min V(\varphi(-z))$ and $\max V(\varphi(-z)) \geq \min V(\varphi(z))$. Said differently (recall V is convex-valued), $V(\varphi(z)) \cap V(\varphi(-z)) \neq \emptyset$. Lemma 1 then guarantees the existence of an equilibrium that generates information policy $p = \frac{1}{2}\delta_{\varphi(z)} + \frac{1}{2}\delta_{\varphi(-z)}$. In particular, $T\mu$ is not $p(\mu)$-a.s. constant. *Q.E.D.*

Just as Proposition 1 generalizes Chakraborty and Harbaugh's (2010) Theorem 1, the following result generalizes their Theorem 2.

COROLLARY 6: *Let T be any statistic, and suppose $\tilde{u} : \overline{\text{co}}T(\Theta) \to \mathbb{R}$ is a strictly quasiconvex function such that $v(\mu) = \tilde{u}(T\mu)$ for every $\mu \in \Delta\Theta$. If T is multivariate, an S-beneficial equilibrium exists.*

Before proving this result, we note the result follows immediately from Proposition 1 under the additional hypothesis that R has a unique best response to every belief—as assumed in Chakraborty and Harbaugh (2010). Indeed, following Chakraborty and Harbaugh's (2010) argument, strict quasiconvexity of $\tilde{u}$ would imply the binary-message equilibrium constructed above is S-beneficial. The below proof for the general case is similar in spirit, although one additional step is needed.

PROOF OF COROLLARY 6: Again, let $\varphi : \mathbb{S} \to \mathbb{R}$ be as delivered by Lemma 8. Now, define $f := v \circ \varphi - v(\mu_0) : \mathbb{S} \to \mathbb{R}$, which is upper semicontinuous because v is. Moreover, for any $z \in \mathbb{S}$, the distinct estimates $T\varphi(z)$ and $T\varphi(-z)$ have $T\mu_0$ as their midpoint, and so $\max\{f(z), f(-z)\} \geq 0$ by quasiconvexity of $\tilde{u}$. Applying Lemma 9 to f then delivers a $z \in \mathbb{S}$ such that $v \circ \varphi(z), v \circ \varphi(-z) \geq v(\mu_0)$.

By Lemma 8 Part 3, both $\mu := 2\varphi(z) - \mu_0$ and $\mu' := 2\varphi(-z) - \mu_0$ are in $\Delta\Theta$. Because $T\varphi(z) = \frac{1}{2}T\mu + \frac{1}{2}T\mu_0$, strict quasiconvexity of $\tilde{u}$ delivers the following inequality chain:

$$v(\mu_0) \leq v \circ \varphi(z) = \tilde{u}\big(T\varphi(z)\big) < \max\big\{\tilde{u}(T\mu), \tilde{u}(T\mu_0)\big\} = \max\big\{v(\mu), v(\mu_0)\big\}.$$

It follows $v(\mu) > v(\mu_0)$. By the same argument, $v(\mu') > v(\mu_0)$. Thus, the information policy $p = \frac{1}{2}\delta_\mu + \frac{1}{2}\delta_{\mu'}$ secures $\min\{v(\mu), v(\mu')\} > v(\mu_0)$. The result then follows from Theorem 1. *Q.E.D.*

C.2. *The Equilibrium Payoff Set*

In this subsection, we briefly comment on how our tools, and the belief-based approach more broadly, can generate a more complete picture of the world of cheap talk with state-independent S preferences. As will be clear, the results outlined herein are all straightforward to derive given earlier results in the paper.

C.2.1. *Other Sender Payoffs*

Following the recent literature on communication with S commitment, our focus has largely been on high equilibrium S values, that is, those providing payoffs at least as high as those attainable under uninformative communication. However, the tools developed in our paper work equally well to characterize bad sender payoffs. Indeed, the proof of Lemma 1 used no special features of V other than it being a Kakutani correspondence, which $-V$ is as well. Therefore, our game has the same equilibrium distributions over $A \times \Theta$ as the game with S objective $-u_S$. To deliver the mirror-image versions of our main results, define the value function from S-adversarial tiebreaking, $w := \min V : \Delta\Theta \to \mathbb{R}$.

Theorem 1 implies a sender payoff $s \leq w(\mu_0)$ is an equilibrium payoff if and only if some $p \in \mathcal{I}(\mu_0)$ exists such that $p\{w \leq s\} = 1$. Combining this observation with the original statement of the securability theorem tells us $s \in \mathbb{R}$ is an equilibrium S payoff if and only if $p_+, p_- \in \mathcal{I}(\mu_0)$ exist such that $p_+\{v \geq s\} = p_-\{w \leq s\} = 1$. An easy consequence is that the equilibrium S payoff set is convex, which we document in Corollary 3. Corollary 1 has a mirror image as well, telling us the set of S equilibrium payoffs is exactly

$$\Big[\min_{p \in \mathcal{I}(\mu_0)} \sup w(\operatorname{supp} p),\ \max_{p \in \mathcal{I}(\mu_0)} \inf v(\operatorname{supp} p)\Big].$$

Note convexity of the set of attainable S payoffs is special to the case in which S's payoffs are state independent; indeed, the leading example of Crawford and Sobel (1982) does not share this feature.

The mirrored counterpart of our geometric Theorem 2 is that the lowest S payoff attainable in equilibrium is $\underline{w}(\mu_0)$, where $\underline{w}$ is the quasiconvex envelope of w, that is, the pointwise highest quasiconvex and lower semicontinuous function that minorizes w. Therefore, we can geometrically characterize S's equilibrium payoff set as $[\underline{w}(\mu_0), \bar{v}(\mu_0)]$.

C.2.2. *Receiver Payoffs*

Our most powerful tools (the securability theorem and its descendants) pertain to S payoffs. However, the belief-based approach (i.e., Lemma 1) can be used to describe R payoffs as well. Indeed, let $v_R : \Delta\Theta \to \mathbb{R}$ be R's value function, given by $v_R(\mu) := \max_{a\in A} \int_\Theta u_R(a, \cdot)\,\mathrm{d}\mu$. It follows from R's interim rationality that any equilibrium that generates outcome (p, s) will deliver a payoff of $r = \int_{\Delta\Theta} v_R\,\mathrm{d}p$ to R.

Given equilibrium S payoff s, we can then more explicitly derive the set of equilibrium R payoffs compatible with an equilibrium in which S gets payoff s. Let

$$B_s := \{w \le s \le v\} = \left\{\mu \in \Delta\Theta : \exists a_+, a_- \in \arg\max_{a\in A} \int_\Theta u_R(a, \cdot)\,\mathrm{d}\mu \text{ s.t. } u_S(a_-) \le s \le u_S(a_+)\right\}.$$

Then, (s, r) is an equilibrium payoff profile if and only if $r = \int_{\Delta\Theta} v_R\,\mathrm{d}p$ for some $p \in \mathcal{I}(\mu_0) \cap \Delta(B_s)$. The best such R payoff (given s) is given by $\widehat{v_R^s}(\mu_0)$, where $v_R^s : B_s \to \mathbb{R}$ is the restriction of v_R and $\widehat{v_R^s} : \overline{\text{co}}B_s \to \mathbb{R}$ is the concave envelope of v_R^s.

C.2.3. *Implementing Equilibrium Payoffs*

In addition to their role in proving Theorem 1, barely securing policies generate a straightforward way of implementing any equilibrium S payoff.[41] If S could commit, we could apply the revelation principle[42] to implement any S commitment payoff with a commitment protocol in which S makes a pure action recommendation to R, and R always complies. Using barely securing policies, we can show a similar result holds with cheap talk, with one important caveat: R must be allowed to mix. To state this result, for any S strategy σ, define $\mathcal{M}_\sigma$ as the set of messages in σ's support.[43]

PROPOSITION 2: *Fix some S payoff s. Then, the following are equivalent*:
1. *s is generated by an equilibrium.*
2. *s is generated by an equilibrium with $\mathcal{M}_\sigma \subseteq \Delta A$ and $\rho(\alpha) = \alpha\ \forall \alpha \in \mathcal{M}_\sigma$.*
3. *s is generated by an equilibrium with $\mathcal{M}_\sigma \subseteq A$ and $\rho(a|a) > 0\ \forall a \in \mathcal{M}_\sigma$.*

The proposition suggests two ways in which one can implement a payoff of s via incentive-compatible recommendations. The first way has S giving R a *mixed* action recommendation that R *always* follows. The second way has S giving R a *pure* action recommendation that R *sometimes* follows. Both ways can result in R mixing.

That 1 implies 2 follows from standard revelation principle logic. To prove 1 implies 3,[44] we start with a minimally informative information policy that secures s. Because p is minimally informative, it must barely secure s, meaning (p, s) is an equilibrium. Let $\mathcal{E}$

[41]For S payoffs $s \le \min V(\mu_0)$, we use the mirror image of barely securing policies, that is, information policies p such that $\{\min V(\cdot) \le s\} \cap \text{co}\{\mu, \mu_0\} = \{\mu\}$ holds for p-a.e. μ.

[42]See, for example, Myerson (1986), Kamenica and Gentzkow (2011), and Bergemann and Morris (2016).

[43]That is, let $\mathcal{M}_\sigma = \bigcup_{\theta\in\Theta} \text{supp}\,\sigma(\cdot|\theta)$.

[44]The equivalence between 1 and 3 echoes an important result of Bester and Strausz (2001), who studied a mechanism-design setting with one agent, finitely many types, and partial commitment by the principal.

be Part 2's implementation of (p, s), and take $\mathbf{a}(\mu)$ to be some S-preferred action among all those that R plays in $\mathcal{E}$ at belief μ. By minimality of p, $\mathbf{a}(\cdot)$ must be p-essentially one-to-one, because pooling any posteriors that induce the same $\mathbf{a}(\cdot)$ value would yield an even less informative policy that secures s. Thus, $\mathbf{a}(\cdot)$ takes distinct beliefs to distinct (on-path) actions: R can infer μ from $\mathbf{a}(\mu)$. One can then conclude the proof by having S recommend $\mathbf{a}(\mu)$ and R respond to $\mathbf{a}(\mu)$ as he would have responded to μ under $\mathcal{E}$.

The formal proof is below.

PROOF OF PROPOSITION 2: Because 2 and 3 each immediately imply 1, we show the converses.

Suppose s is an equilibrium S payoff. Now take some $p \in \mathcal{I}(\mu_0)$ Blackwell-minimal among all policies securing payoff s, and let $D := \text{supp}(p) \subseteq \Delta\Theta$.[45] Lemma 4 guarantees (p, s) is an equilibrium outcome, say, witnessed by equilibrium $\mathcal{E}_1 = (\sigma_1, \rho_1, \beta_1)$. Letting $\alpha = \alpha_s : D \to \Delta A$ be as delivered by Lemma 2, we may assume $\rho_1(\cdot|m) = \alpha(\cdot|\beta(m))$. In particular, ρ_1 specifies finite-support play for every message.

Let $\mathbb{M} := \text{marg}_M \mathbb{P}_{\mathcal{E}_1}$ and $X := \text{supp}[\mathbb{M} \circ \hat{\rho}^{-1}] \subseteq \Delta A$, and fix arbitrary $(\hat{\alpha}, \hat{\mu}) \in \text{supp}[\mathbb{M} \circ (\rho_1, \beta_1)^{-1}]$; in particular, $\hat{\alpha} \in X$. By continuity of u_R and receiver incentive compatibility, $\hat{\alpha} \in \arg\max_{\alpha \in \Delta A} u_R(\alpha \otimes \hat{\mu})$. Defining $\rho' : M \to \Delta A$ (resp. $\beta' : M \to \Delta\Theta$) to agree with ρ_1 (β_1) on path and take value $\hat{\alpha}$ ($\hat{\mu}$) off path, an equilibrium $\mathcal{E}' = (\sigma_1, \rho', \beta')$ exists such that $\mathbb{P}_{\mathcal{E}'} = \mathbb{P}_{\mathcal{E}_1}$ and $\rho'(\cdot|m) \in X$ for every $m \in M$.

Now define

$$
\begin{aligned}
\sigma_2 : \Theta &\to \Delta X \subseteq \Delta M, \\
\theta &\mapsto \sigma_1(\cdot|\theta) \circ \rho'^{-1}, \\
\rho_2 : M &\to X \subseteq \Delta A, \\
m &\mapsto \begin{cases} m & : m \in X, \\ \hat{\alpha} & : m \notin X, \end{cases} \\
\beta_2 : M &\to \Delta\Theta, \\
m &\mapsto \begin{cases} \mathbb{E}_{m \sim \mathbb{M}}\big[\beta(m)|\rho(m)\big] & : m \in X, \\ \hat{\mu} & : m \notin X. \end{cases}
\end{aligned}
$$

By construction, $(\sigma_2, \rho_2, \beta_2)$ is an equilibrium that generates outcome (p, s),[46] proving 1 implies 2.

Now define the (A- and D-valued, respectively) random variables $\mathbf{a}, \boldsymbol{\mu}$ on $\langle D, p \rangle$ by letting $\mathbf{a}(\mu) := \arg\max_{a \in \text{supp}\, \alpha(\mu)} u_S(a)$ and $\boldsymbol{\mu}(\mu) := \mu$ for $\mu \in D$. Next define the conditional expectation $\mathbf{f} := \mathbb{E}_p[\boldsymbol{\mu}|\mathbf{a}] : D \to D$, which is defined only up to a.e.-p equivalence. By construction, the distribution of $\boldsymbol{\mu}$ is a mean-preserving spread of the distribution of $\mathbf{f}$. That is, p is weakly more informative than $p \circ \mathbf{f}^{-1}$. By hypothesis, $\mathbf{a}(\mu)$ is incentive compatible

Applying a graph-theoretic argument, they showed one can restrict attention to direct mechanisms in which the agent reports truthfully with positive probability. Although the proof techniques are quite different, a common lesson emerges. Agent mixing helps circumvent limited commitment by the principal: in Bester and Strausz's (2001) setting, by limiting the principal's information, and in ours, by limiting her control.

[45] Some policy secures s if s is an equilibrium payoff. The set of such policies is closed (and so compact) because v is upper semicontinuous. Therefore, because the Blackwell order is closed-continuous, a Blackwell-minimal such policy exists.

[46] It generates $(\tilde{p}, s)$ for some garbling $\tilde{p}$ of p. Minimality of p then implies $\tilde{p} = p$.

for R at every $\mu \in D$. But $D = \text{supp}(p \circ \mathbf{f}^{-1})$, which implies $p \circ \mathbf{f}^{-1}$ secures s. But minimality of p implies $p \circ \mathbf{f}^{-1} = p$. So $\mathbf{f} = \mathbb{E}_p[\boldsymbol{\mu}|\mathbf{a}]$ and $\boldsymbol{\mu}$ have the same distribution, which implies $\mathbf{f} = \boldsymbol{\mu}$ a.s.-p. By definition, $\mathbf{f}$ is $\mathbf{a}$-measurable, so that Doob–Dynkin delivers some measurable $\mathbf{b} : A \to D$ such that $\mathbf{f} = \mathbf{b} \circ \mathbf{a}$.

Summing up, we have some measurable $\mathbf{b} : A \to D$ such that $\mathbf{b} \circ \mathbf{a} =_{\text{a.e.}-p} \boldsymbol{\mu}$. Now define

$$\begin{aligned}
\sigma_3 &: \Theta \to \Delta A \subseteq \Delta M, \\
&\quad \theta \mapsto \sigma_2(\cdot|\theta) \circ (\mathbf{a} \circ \beta_2)^{-1}, \\
\rho_3 &: M \to X \subseteq \Delta A, \\
&\quad m \mapsto \begin{cases} \alpha\big(\mathbf{b}(m)\big) & : m \in A, \\ \hat{\alpha} & : m \notin A, \end{cases} \\
\beta_3 &: M \to \Delta\Theta, \\
&\quad m \mapsto \begin{cases} \mathbf{b}(m) & : m \in A, \\ \hat{\mu} & : m \notin A. \end{cases}
\end{aligned}$$

By construction, $(\sigma_3, \rho_3, \beta_3)$ is an equilibrium that generates outcome (p, s), proving 1 implies 3. *Q.E.D.*

Proposition 2 shows some forms of communication are without loss as far as S payoffs are concerned. First, any S equilibrium payoff is attainable in an equilibrium in which S recommends mixed actions that are (on path) followed exactly. This equivalence extends to equilibrium payoff *pairs*, with the same argument: Pooling messages that lead to the same R behavior relaxes incentive constraints and generates the same joint distribution over actions and states, preserving payoffs. Second, any S equilibrium payoff is attainable in an equilibrium in which S recommends pure actions that are followed with positive probability. Whether this result holds in general for payoff pairs is an open question. It is easy to see why, at least, our argument does not go through as stated. The proof begins by considering an information policy that gives no "extraneous" information to R, subject to securing the relevant S value. But taking information away from R in this way can result in a payoff loss.

Still, we can leverage Lemma 1 to show a result of a similar spirit: To implement an equilibrium payoff profile, it is sufficient for R to only use binary mixed actions, the support of which is S's message.

PROPOSITION 3: *Fix some payoff profile* (s, r). *Then, the following are equivalent*:

1. (s, r) *is generated by an equilibrium*.
2. (s, r) *is generated by an equilibrium with* $\mathcal{M}_\sigma \subseteq \Delta A$ *and* $\rho(\alpha) = \alpha \ \forall \alpha \in \mathcal{M}_\sigma$.
3. (s, r) *is generated by an equilibrium with* $\mathcal{M}_\sigma \subseteq \{\frac{1}{2}\delta_a + \frac{1}{2}\delta_{a'} : a, \hat{a} \in A\}$ *and* $\text{supp}[\rho(\alpha)] = \text{supp}(\alpha) \ \forall \alpha \in \mathcal{M}_\sigma$.

We can interpret 3 as describing equilibria in which S tells R, "Play a or $\hat{a}$," for some pair of actions, but does not suggest mixing probabilities.

To see the equivalence between 1 and 3, Lemma 2 from the Appendix can be used to show equilibrium payoff profile (s, r) can be implemented with an equilibrium in which R only ever uses pure actions or binary-support mixtures, with the latter only being used when S is not indifferent between the two supported actions. Without loss, say such equilibrium is as in 2, with S suggesting an incentive-compatible mixture to R. But S rationality implies no two on-path recommendations can have the same support, because then S would have an incentive to deviate to the one putting a higher probability on the preferred action. Therefore, the same behavior could be induced by having every message replaced with a uniform distribution over its (at most binary) support, and the result follows.

With finitely many actions, Proposition 3 yields an a priori upper bound on the number of distinct messages required in equilibrium, similar to Proposition 2. Still, the upper bound of Proposition 2 is significantly smaller: Whereas Proposition 2 says no more than $n := |A|$ messages are required to span the set of equilibrium S values, Proposition 3 guarantees any equilibrium payoff pair can be attained with at most $\frac{n(n-1)}{2}$ messages.

C.3. *Long Cheap Talk*

Let us define the long-cheap-talk game. In addition to the objects in our model section, R has some message space $\tilde{M}$, which we assume is compact metrizable. Let $\mathcal{H}_{<\infty} := \bigsqcup_{t=0}^{\infty}(M \times \tilde{M})^t$, $\mathcal{H}_\infty := (M \times \tilde{M})^{\mathbb{N}}$, and $\Omega := \mathcal{H}_\infty \times A \times \Theta$. In a long-cheap-talk game, S first sees the state $\theta \in \Theta$. Then, at each time $t \in \mathbb{Z}_+$, players send simultaneous messages: S sends $m_t \in M$ and R sends $\tilde{m}_t \in \tilde{M}$. Finally, after seeing the sequence of messages, R chooses an action $a \in A$. Formally, a (behavior) strategy for S is a measurable function $\sigma : \Theta \times \mathcal{H}_{<\infty} \to \Delta M$, and a strategy for R is a pair of measurable functions $(\tilde{\sigma}, \rho)$, where $\tilde{\sigma} : \mathcal{H}_{<\infty} \to \Delta\tilde{M}$ and $\rho : \mathcal{H}_\infty \to \Delta A$. These maps induce (together with the prior μ_0) a unique distribution, $\mathbb{P}_{\sigma,\tilde{\sigma},\rho} \in \Delta\Omega$, which induces payoff $u_S(\text{marg}_A \mathbb{P}_{\sigma,\tilde{\sigma},\rho})$ and $u_R(\text{marg}_{A\times\Theta} \mathbb{P}_{\sigma,\tilde{\sigma},\rho})$ for S and R, respectively.

C.3.1. *Extra Rounds Cannot Help the Sender*

Below, we use our Theorem 1 to show that any S payoff attainable under long cheap talk is also attainable under one-shot communication.[47]

PROPOSITION 4: *Every sender payoff attainable in a Nash equilibrium of the long-cheap-talk game is also attainable in a perfect Bayesian equilibrium of the one-shot cheap-talk game.*

To prove the proposition, fix a payoff s^* that S cannot attain in the one-shot game, and use our securability theorem to construct a continuous biconvex function on $\Delta\Theta \times \mathbb{R}$ that is strictly positive at (μ_0, s^*) and zero on V's graph. Mimicking Appendix A.3 of Aumann and Hart (2003), we then take an arbitrary equilibrium of the long-cheap-talk game, and construct a bimartingale $\{\boldsymbol{\mu}_k, \mathbf{s}_k\}_k$, that is, a martingale over the graph of V such that only one coordinate ever moves at a time.[48] The bimartingale converges to a measure over V's graph and has a time-zero value of $(\boldsymbol{\mu}_0, \mathbf{s}_0) = (\mu_0, s_0)$, where s_0 is S's payoff in said equilibrium. We then follow the easy direction of Aumann and Hart's (1986) characterization of the bi-span of a set, noting the expectations of continuous biconvex functions of a bimartingale grow over time, and so the function constructed at the beginning of the

[47]To ease notational overhead, we employ Nash equilibrium as our solution concept in studying long cheap talk, and so have no need to define a belief map for the receiver. We therefore obtain a stronger result, because any perfect Bayesian equilibrium is also Bayes Nash.

[48]Although the bimartingale we construct is related to the stochastic process of pairs of R beliefs and S payoffs, the two processes are not the same: Each round of communication corresponds to two periods under the bimartingale. Aumann and Hart (2003) used the same construction.

proof assigns (μ_0, s_0) a weakly negative value. It follows that $(\mu_0, s_0) \neq (\mu_0, s^*)$. Because the chosen long-cheap-talk equilibrium was arbitrary, no such equilibrium can yield S a payoff of s^*.

Other than our construction of a biconvex function, the proof follows the logic presented in Aumann and Hart (2003) and Aumann and Hart (1986). Because both papers assume a finite state space, the results of Aumann and Hart (1986) and Aumann and Hart (2003) do not apply directly. We therefore provide a self-contained proof below.

PROOF OF PROPOSITION 4: Take any $s_* \in \mathbb{R}$ that is not an equilibrium payoff for prior μ_0 in the one-shot cheap-talk game. In particular, $s_* \notin V(\mu_0)$. Focus on the case of $s_* > v^*(\mu_0)$, the mirror-image case being analogous. Fix some payoff $s' \in (v^*(\mu_0), s_*)$. Letting B be the closed convex hull of $v^{-1}[s', \infty)$, Theorem 1 tells us $\mu_0 \notin B$. Hahn–Banach then gives an affine continuous $\varphi : \Delta\Theta \to \mathbb{R}$ such that $\varphi(\mu_0) > \max \varphi(B)$. Now define the function[49]

$$\begin{aligned} F &: \Delta\Theta \times \mathbb{R} \to \mathbb{R}_+, \\ &(\mu, s) \mapsto \big[\varphi(\mu) - \max \varphi(B)\big]_+ \big[s - s'\big]_+. \end{aligned}$$

Observe that F is biconvex and continuous. Moreover, $F(\mu, s) = 0$ whenever $s \in V(\mu)$: either $s < s'$ because $\mu \notin B$, or $\mu \in B$ and so $\varphi(\mu) \leq \max \varphi(B)$.

Now consider any Nash equilibrium $(\sigma, (\tilde{\sigma}, \rho))$ of the long-cheap-talk game. Let us define several random variables on the Borel probability space $\langle \Omega, \mathbb{P}_{\sigma,\tilde{\sigma},\rho}\rangle$. For $\omega = ((m_t, \tilde{m}_t)_{t=0}^{\infty}, a, \theta) \in \Omega$, let $\boldsymbol{\theta}(\omega) := \theta$ and $\mathbf{a}(\omega) := a$; and, for $t \in \mathbb{Z}_+$, let $\mathbf{m}_{2t}(\omega) := m_t$ and $\mathbf{m}_{2t+1}(\omega) := \tilde{m}_t$. From these, we define a filtration $(\mathcal{F}_k)_{k\in K}$ with index set $K = \mathbb{Z}_+ \cup \{\infty\}$ by letting each $\mathcal{F}_k$ be the σ-algebra generated by $\{\mathbf{m}_\ell\}_{\ell\in\mathbb{Z}_+, \ell<k}$. Finally, for each $k \in K$, define the ($\Delta\Theta$-valued and $\mathbb{R}$-valued, respectively) random variables $\boldsymbol{\mu}_k := \mathbb{E}[\delta_{\boldsymbol{\theta}}|\mathcal{F}_k]$ and $\mathbf{s}_k := \mathbb{E}[u_S(\mathbf{a})|\mathcal{F}_k]$; and let $P_k \in \Delta(\Delta\Theta \times \mathbb{R})$ denote the distribution of $(\boldsymbol{\mu}_k, \mathbf{s}_k)$. Note that, by construction, P_0 has a distribution $\delta_{(\mu_0,s_0)}$ for some $s_0 \in \mathbb{R}$. Our task is to show $s_0 \neq s_*$.

In what follows, take any statements about the stochastic processes $(\boldsymbol{\mu}_k)_{k\in K}$ and $(\mathbf{s}_k)_{k\in K}$ to hold $\mathbb{P}_{\sigma,\tilde{\sigma},\rho}$-almost surely. By construction, $\boldsymbol{\mu}_{2t+2} = \boldsymbol{\mu}_{2t+1}$ for every $t \in \mathbb{Z}_+$, and both $(\boldsymbol{\mu}_k)_{k\in K}$ and $(\mathbf{s}_k)_{k\in K}$ are martingales. By S rationality, $\mathbf{s}_{2t} = \mathbb{E}[\mathbf{s}_{2t+1}|\mathcal{F}_{2t+1}] = \mathbf{s}_{2t+1}$ for every $t \in \mathbb{Z}_+$. Because F is biconvex and continuous, $\int F \, \mathrm{d}P_0 \leq \int F \, \mathrm{d}P_1 \leq \cdots$. In particular, $\int F \, \mathrm{d}P_k \geq \int F \, \mathrm{d}P_0 = F(\mu_0, s_0)$ for every $k \in \mathbb{Z}_+$. By the martingale convergence theorem, $\mathbf{s}_k$ converges to $\mathbf{s}_\infty$. By the same, every continuous $g : \Theta \to \mathbb{R}$ has $\int_\Theta g \, \mathrm{d}\boldsymbol{\mu}_k$ converging to $\int_\Theta g \, \mathrm{d}\boldsymbol{\mu}_\infty$; so $\boldsymbol{\mu}_k$ converges (weak*) to $\boldsymbol{\mu}_\infty$. But P_k converges (weak*) to P_∞. Therefore, $\int F \, \mathrm{d}P_\infty = \lim_{k\to\infty} \int F \, \mathrm{d}P_k \geq F(\mu_0, s_0)$. By R rationality, $\mathbf{s}_\infty \in V(\boldsymbol{\mu}_\infty)$, implying $F(\boldsymbol{\mu}_\infty, \mathbf{s}_\infty) = 0$, so that $\int F \, \mathrm{d}P_\infty = 0$, too. Therefore, $F(\mu_0, s_0) \leq 0 < F(\mu_0, s_*)$. So $s_0 \neq s_*$, as required. *Q.E.D.*

C.3.2. *Extra Rounds Can Help the Receiver*

Unlike S, R may benefit from long cheap talk when S's preferences are state independent. To see this, consider the following example, which we describe informally. Let $\Theta = \{0, 1\}$; $\mu_0(1) = \frac{1}{8}$; $A = \{\ell, b, t, r\}$; $u_S(b) = 0$, $u_S(\ell) = 1$, $u_S(t) = u_S(r) = 2$; and $u_R(a, \theta) = -(z_a - \theta)^2$, where $z_\ell = 0$, $z_r = 1$, and $z_b = z_t = \frac{1}{2}$. The associated value correspondence V and prior belief μ_0 are depicted in Figure 3.

[49]Recall that $[\cdot]_+ := \max\{\cdot, 0\}$.

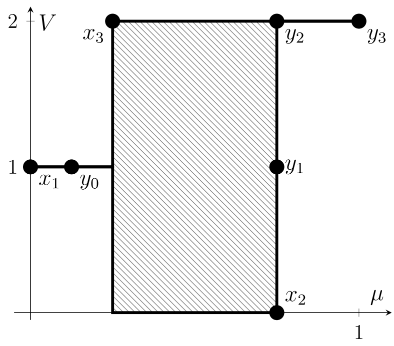

FIGURE 3.—S's value correspondence in an example where R strictly benefits from long cheap talk.

Because every $\mu \in \Delta\Theta$ with $\mu(1) \leq \mu_0(1)$ has $V(\mu) = \{1\}$, Lemma 1 immediately implies every equilibrium outcome (p, s) of the one-shot cheap-talk game has $s = 1$ and $p\{\mu : \mu(1) \leq \frac{3}{4}\} = 1$. In particular, every equilibrium of the long-cheap-talk game generates a "mean outcome" of y_0, as depicted in the figure.

Given the above observations, an equilibrium exists with one round of communication with R beliefs supported on $\{0, \frac{3}{4}\}$, and every other one-shot equilibrium generates less information (in a Blackwell sense) for R; we can depict this equilibrium as generating support $\{x_1, y_1\}$ in the figure. But now, with a jointly controlled lottery, this y_1 can be split in the next round to $\{x_2, y_2\}$.[50] Finally, S can provide additional information in the next round to split y_2 into $\{x_3, y_3\}$. Because action t is optimal for R at belief $\frac{3}{4}$ (i.e., that associated with y_2) but not at belief 1 (i.e., that associated with y_3), this additional information is instrumental to R. Therefore, our equilibrium is strictly better for R than any one-round equilibrium.

Thus, although additional rounds of communication do not change S's equilibrium payoff set, the static and long-cheap-talk models are economically distinct, even under state-independent S preferences.

C.4. *Optimality of Full Revelation*

This section presents formal results discussed in Section 6.4. This section's main result is Proposition 5, which shows two things when v is nowhere quasiconcave: First, full revelation is an S-favorite equilibrium; and second, every S-favorite equilibrium entails full revelation if the state is binary or R's best response is unique for every belief. We also demonstrate, via an example, that nowhere quasiconcavity is insufficient for full revelation to be uniquely S-optimal. The example also illustrates S-unfavorable tie breaking can create a benefit from commitment even when full revelation is both S's favorite equilibrium and S's favorite commitment policy. We conclude the section by discussing conditions under which v is nowhere quasiconcave. In particular, we show a strictly quasiconvex v is nowhere quasiconcave if and only if it is nowhere quasiconcave on each of the simplex's one-dimensional extreme subsets (Corollary 7).

[50]Informally, following Aumann and Hart (2003), each player could toss a fair coin (independent of the state for S) and announce its outcome. Then, the players move to x_2 if the coins come up the same, and y_2 otherwise. Such jointly controlled randomization could be done simultaneously with the information that S initially conveys, so that our three-round example can be converted into a slightly more complicated two-round example.

The next few lemmas serve as preliminary steps toward Proposition 5. Lemma 10 provides a way of constructing a measurable correspondence. Using this lemma, we show every non-full revelation commitment policy can be improved upon when v is nowhere concave, by splitting non-extreme beliefs. Similarly, one can split such beliefs to weakly increase a policy's secured value whenever v is nowhere quasiconcave (Lemma 11). An immediate consequence is that under nowhere quasiconcavity, full revelation secures S's highest equilibrium value (Lemma 12). Nowhere quasiconcavity also implies S can do better than no information at every non-extreme belief (Lemma 13). We then combine these lemmas with the observation that the payoff secured by full revelation depends only on the prior's support to show full revelation barely secures S's highest equilibrium payoff.

We now proceed with proving Lemma 10. This lemma is based on Aliprantis and Border's (2006) discussion concerning measurability of correspondences. All measurability statements are made with respect to the appropriate Borel σ-algebras.

LEMMA 10: *Let X and Y be compact metrizable spaces, $\Xi : X \to \mathbb{R}$ upper semicontinuous, and $\Upsilon : Y \to \mathbb{R}$ measurable. Then,*

$$\Gamma : Y \rightrightarrows X, \quad y \mapsto \Xi^{-1}[\Upsilon(y), \infty),$$

is weakly measurable.

PROOF: Recall that a nonempty-compact-valued correspondence into X is weakly measurable if and only if it is measurable when viewed as a $\mathcal{K}_X$-valued function (Theorem 18.10 from Aliprantis and Border (2006)).[51] We now proceed with proving the lemma. To begin, let $\bar{z} = \max \Xi(X)$, and observe that

$$\Lambda : (-\infty, \bar{z}] \rightrightarrows X, \quad z \mapsto \Xi^{-1}[z, \infty) = \{\Xi \geq z\},$$

is nonempty-compact-valued because Ξ is upper semicontinuous. We claim below that Ξ is weakly measurable. It follows that $y \mapsto \Lambda \circ \Upsilon(y)$ is a measurable function from Y into $\mathcal{K}_X$, and so is weakly measurable when viewed as a correspondence. Noting $\Gamma = \Lambda \circ \Upsilon$ completes the proof.

We now argue Ξ is weakly measurable. To do so, consider any open $G \subseteq X$. The lower inverse image of G under Λ is

$$\begin{aligned} \Lambda^l(G) &= \{z \leq \bar{z} : \Lambda(z) \cap G \neq \emptyset\} \\ &= \{z \leq \bar{z} : \{\Xi \geq z\} \cap G \neq \emptyset\} \\ &= \{z \leq \bar{z} : \Xi(G) \nsubseteq (-\infty, z)\}, \end{aligned}$$

which is an interval. *Q.E.D.*

When v is nowhere (quasi)concave, Lemma 10 gives a splitting of each non-extreme belief that increases v's expected (secured) value. We present this result below.

[51] $\mathcal{K}_X$ denotes all nonempty compact subsets of X, equipped with the Hausdorff metric.

LEMMA 11: *Suppose v is nowhere (quasi)concave. Then, a measurable selector r of $\mathcal{I}:\Delta\Theta \rightrightarrows \Delta\Delta\Theta$ exists such that* $\int v\,\mathrm{d}r(\mu) > v(\mu)$ ($\inf v(\operatorname{supp} r(\mu)) > v(\mu)$) *for all* $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$.

PROOF: Observe that $\hat{v}(\cdot)$ ($\bar{v}(\cdot)$) is upper semicontinuous and therefore measurable. Moreover, $p \mapsto \int v\,\mathrm{d}p$ ($p \mapsto \inf v(\operatorname{supp} p)$) is an upper semicontinuous function from $\Delta\Delta\Theta$ to $\mathbb{R}$. Therefore, Lemma 10 implies $\mu \mapsto \{p \in \Delta\Delta\Theta : \int v\,\mathrm{d}p \geq \hat{v}(\mu)\}$ ($\mu \mapsto \{p \in \Delta\Delta\Theta : \inf v(\operatorname{supp} p) \geq \bar{v}(\mu)\}$) is weakly measurable. Noting $\mathcal{I}$ is also weakly measurable (by upper hemicontinuity) implies

$$\mu \mapsto \mathcal{I}(\mu) \cap \left\{p \in \Delta\Delta\Theta : \int v\,\mathrm{d}p \geq \hat{v}(\mu)\right\}$$

$$\big(\mu \mapsto \mathcal{I}(\mu) \cap \big\{p \in \Delta\Delta\Theta : \inf v(\operatorname{supp} p) \geq \bar{v}(\mu)\big\}\big)$$

is weakly measurable. Because the latter correspondence is nonempty-valued, it admits a measurable selector, r, by the Kuratowski and Ryll–Nardzewski selection theorem (Theorem 18.13 from Aliprantis and Border (2006)). The result follows from noting $\hat{v}(\mu) > v(\mu)$ ($\bar{v}(\mu) > v(\mu)$) holds for all $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$ whenever v is nowhere (quasi)concave (appealing to Corollary 1). *Q.E.D.*

Lemma 11 above immediately implies full revelation is S's uniquely optimal commitment protocol whenever v is nowhere concave. The reason is that any other information policy can be strictly improved upon via the splitting generated by the lemma. Lemma 11 also implies that when v is nowhere quasiconcave, full revelation secures S's maximal equilibrium. We prove the latter result in the lemma below.

LEMMA 12: *If v is nowhere quasiconcave,* $\bar{v}(\mu) = \inf_{\theta\in\operatorname{supp}(\mu)} v(\delta_\theta)$ *for all* $\mu \in \Delta\Theta$; *that is, full information secures S's maximal equilibrium value.*

PROOF: Fix $\mu \in \Delta\Theta$. A unique $p^F \in \mathcal{I}(\mu)$ exists with $p^F\{\delta_\theta\}_{\theta\in\Theta} = 1$; clearly, p^F has support $\{\delta_\theta\}_{\theta\in\operatorname{supp}(\mu)}$. By Corollary 1, we know $\bar{v}(\mu)$ is the highest securable value at prior μ. Thus, letting $\mathcal{P} := \{p \in \mathcal{I}(\mu) : p \text{ secures } \bar{v}(\mu)\}$, our aim is to show $p^F \in \mathcal{P}$. Corollary 1 tells us $\mathcal{P}$ is nonempty, and upper semicontinuity of v implies $\mathcal{P}$ is closed. The mean-preserving spread order being closed-continuous, $\mathcal{P}$ contains some maximal element, p, with respect to this order. Letting r be as delivered by Lemma 11, the policy $\int r\,\mathrm{d}p$ belongs to $\mathcal{P}$ as well.[52] But maximality of p requires that $p = \int r\,\mathrm{d}p$, implying $p = p^F$. *Q.E.D.*

The next lemma establishes that under nowhere quasiconcavity, S can always benefit from cheap talk.

LEMMA 13: *If v is nowhere quasiconcave,* $\bar{v}(\mu) > v(\mu)$ *for all* $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$.

PROOF: Fix any $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$. By hypothesis, $\mu', \mu'' \in \Delta\Theta$ and $\lambda \in (0,1)$ exist such that $\mu = \lambda\mu' + (1-\lambda)\mu''$ and $v(\mu) < v(\mu'), v(\mu'')$. Therefore, $p = \lambda\delta_{\mu'} + (1-\lambda)\delta_{\mu''} \in \mathcal{I}(\mu)$ secures a value strictly above $v(\mu)$, and so $\bar{v}(\mu) > v(\mu)$ by Theorem 1. *Q.E.D.*

We now prove our main result regarding nowhere quasiconcavity.

[52]Here, $\int r\,\mathrm{d}p \in \mathcal{I}(\mu)$ is given by $[\int r\,\mathrm{d}p](D) := \int r(D|\cdot)\,\mathrm{d}p$ for Borel $D \subseteq \Delta\Theta$.

PROPOSITION 5: *Suppose v is nowhere quasiconcave. Then,*

1. *Some S-preferred equilibrium entails full information.*
2. *If Θ is binary, or if R has a unique best response to every belief, every S-preferred equilibrium entails full information.*

PROOF: We begin by showing full revelation barely secures $\bar{v}(\mu_0)$. Fix some $\theta \in \operatorname{supp}\mu_0$. Consider any $\mu \in \operatorname{co}\{\delta_\theta, \mu_0\}\backslash\{\delta_\theta\}$. We argue $\bar{v}(\mu_0) > v(\mu)$, and so $v^{-1}[\bar{v}(\mu_0), \infty) \cap \operatorname{co}\{\delta_\theta, \mu_0\} = \{\delta_\theta\}$, as required. Because the support of μ and μ_0 is the same, full revelation secures the same value for both beliefs. Therefore, Lemma 12 and Lemma 13 yield

$$v(\mu) < \bar{v}(\mu) = \inf\sup v\big(\{\delta_\theta\}_{\theta\in\operatorname{supp}\mu_0}\big) = \bar{v}(\mu_0).$$

In other words, full revelation barely secures $\bar{v}(\mu_0)$. The securability theorem (more precisely, Lemma 4) then delivers the first point.

To show the second part, we claim below $\bar{v}(\mu) \le \bar{v}(\mu_0)$ for each $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$. Lemma 13 then implies $v(\mu) < \bar{v}(\mu) \le \bar{v}(\mu_0)$ for all $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$. As such, $p \in \mathcal{I}(\mu_0)$ secures $\bar{v}(\mu_0)$ only if $\operatorname{supp} p \subseteq \{\delta_\theta\}_{\theta\in\Theta}$, that is, p provides full information. To conclude the proof, we note $(p, \bar{v}(\mu_0))$ is an equilibrium outcome only if p secures $\bar{v}(\mu_0)$, meaning no p other than full revelation can yield S a payoff of $\bar{v}(\mu_0)$.

All that remains is to show $\bar{v}(\mu) \le \bar{v}(\mu_0)$ for all $\mu \in \Delta\Theta\backslash\{\delta_\theta\}_{\theta\in\Theta}$. When $|\Theta| = 2$, this inequality holds with equality by Lemma 12. If R's best response is unique, v is continuous, and so every $\theta \in \Theta$ has

$$v(\delta_\theta) = \lim_{n\to\infty} v\left(\frac{n-1}{n}\delta_\theta + \frac{1}{n}\mu_0\right) \le \lim_{n\to\infty} \bar{v}\left(\frac{n-1}{n}\delta_\theta + \frac{1}{n}\mu_0\right) = \bar{v}(\mu_0),$$

where the last equality follows from Lemma 12. The same lemma then implies $\bar{v}(\mu) = \inf v(\{\delta_\theta\}_{\theta\in\operatorname{supp}\mu}) \le \bar{v}(\mu_0)$, as required. *Q.E.D.*

We now provide an example that witnesses two properties. First, it shows nowhere quasiconcavity alone is insufficient for uniqueness of full revelation as an S-favorite equilibrium. Second, it is possible for S to benefit from commitment despite full revelation being best for S both with and without commitment.

EXAMPLE 4: Let $\Theta := \{-1, 0, 1\}$, $A := \{0, 1\} \times \Delta\Theta$, $\mu^* := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, $\mu_0 := \frac{1}{2}\delta_0 + \frac{1}{2}\mu^*$, and $H : \Delta\Theta \to \mathbb{R}_+$ a continuous and strictly concave function with $H(\delta_\theta) = 0\ \forall\theta \in \Theta$. Let players utilities $u_S : A \to \mathbb{R}$ and $u_R : A \times \Theta \to \mathbb{R}$ be given by

$$u_S\big((x, \hat{\mu})\big) := xH\big(\mu^*\big) - H(\hat{\mu})$$

and

$$u_R\big((x, \hat{\mu}), \theta\big) := -\sum_{\tilde{\theta}\in\Theta}\big[\hat{\mu}(\tilde{\theta}) - \mathbf{1}_{\tilde{\theta}=\theta}\big]^2 - x\big(1 - \theta^2\big).$$

Observe $(x, \hat{\mu})$ is a best response to R belief μ if and only if $\hat{\mu} = \mu$ and $x\mu(0) = 0$. Therefore, the value function is given by $v(\mu) = H(\mu^*)\mathbf{1}_{\mu(0)=0} - H(\mu)$. By construction, this function is strictly quasiconvex because $-H$ is. Appealing to Corollary 7 (see below), the value function is then nowhere quasiconcave, and so full information is an S-preferred equilibrium, yielding S payoff $\min\{H(\mu^*), 0\} = 0$.

Observe that, in an S-preferred equilibrium, R breaks indifferences against S when the state is nonzero. Therefore, S gets a payoff strictly lower than her commitment value of $\frac{1}{2}H(\mu^*)$. Moreover, full information is not the only S-preferred equilibrium information policy, because Lemma 1 implies $(\frac{1}{2}\delta_{\delta_0} + \frac{1}{2}\delta_{\mu^*}, 0)$ is an equilibrium outcome.

We conclude this section with sufficient conditions for v to be nowhere quasiconcave. In particular, we show a strictly quasiconvex v is nowhere quasiconcave if and only if it is nowhere quasiconcave on every one-dimensional extreme subset of $\Delta\Theta$.

COROLLARY 7: *Let v be strictly quasiconvex. The following are equivalent*:
(i) *v is nowhere quasiconcave.*
(ii) *$v|_{\Delta\{\theta,\theta'\}}$ is nowhere quasiconcave for every $\theta, \theta' \in \Theta$.*

PROOF: Clearly, (i) implies (ii). That (ii) implies (i) follows from applying Corollary 6 with $T(\theta) := \delta_\theta$. Indeed, for any prior $\mu \in \Delta\Theta$ with $|\operatorname{supp}\mu| \geq 3$, Corollary 6's proof delivers a pair of beliefs μ', μ'' with μ as their midpoint such that $v(\mu) < v(\mu'), v(\mu'')$. Therefore, the definition of nowhere quasiconcavity need only be verified at binary-support beliefs whenever v is strictly quasiconvex. *Q.E.D.*

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