Econometrica Supplementary Material

SUPPLEMENT TO "REVISION GAMES"

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APPENDIX B: APPENDIX FOR SECTION 4

B.1. *Proofs and a Calculation for Section 4.1*

B.1.1. *The Proofs for the Reduction Argument*

WE BEGIN with the formal definitions of trigger strategy equilibria for the supply-schedule game.

Formulation of the objective.
As in Section 3, we define

$$X := \{x_i : [0, T] \to S | \pi_i(x_i(\cdot), x_i(\cdot)) \text{ is measurable}\}.$$

Given a pair $(x_1, x_2) \in X^2$ and a Nash equilibrium (s_1, s_2), we define seller i's *trigger strategy with plan* (x_1, x_2) *and a Nash equilibrium* (s_1, s_2), denoted by $\sigma_i((x_1, x_2), s_i)$, to be a strategy in which the following hold for each time $-t \in [-T, 0]$ such that there is an opportunity:

1. If each seller submits an order $x_i(\tau)$ for every $-\tau \in [-T, -t)$ at which there is an opportunity, then seller i submits the order $x_i(t)$.
2. Otherwise, each seller i submits the order s_i.

Let $\Sigma := \{(\sigma_i((x_1, x_2), s_i))_{i=1,2} | x_i \in X, s_i \in S \text{ for each } i = 1, 2\}$. We say that $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma$ is *symmetric* if, for every t, $q_1(x_1(t), x_2(t)) = q_2(x_1(t), x_2(t))$. That is, the realized supplies from the two sellers (after rationing) are the same. Let $\bar{\Sigma} \subseteq \Sigma$ be the set of symmetric $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$. As in Section 3, we can formulate the incentive compatibility condition, which defines subgame-perfect equilibria in trigger strategies. Let $\Sigma^* \subseteq \bar{\Sigma}$ be the set of subgame-perfect equilibria in $\bar{\Sigma}$. *Our objective is to find a strategy profile in* Σ^* *that generates the highest ex ante payoff to each seller.*[1] As in Section 3, there may exist multiple maximizers of the ex ante payoff to each seller. We will show that there is an *essentially unique* optimal plan of quantities, $q_i(x_1(t), x_2(t))$, $t \in [0, T]$ for each $i = 1, 2$, by which we mean that if both $(\sigma_i((x_1, x_2), s_i))_{i=1,2}, (\sigma_i((x_1', x_2'), s_i'))_{i=1,2} \in \Sigma^*$ maximize the expected payoff, then $q_i(x_1(t), x_2(t)) = q_i(x_1'(t), x_2'(t))$ holds for each $i = 1, 2$ for almost all $t \in [0, T)$ and at $t = T$.[2]

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[1]Note that, at this point, the existence of such a strategy profile is not obvious. The existence follows from the reduction argument that follows.

[2]Note that we do not require that the equality holds at $t = 0$ because there are multiple Nash equilibria in the component game.

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Let (s_1^N, s_2^N) be a Nash equilibrium of the supply-schedule game such that $q_i(s_1^N, s_2^N) = \frac{a-c}{2b}$ for each $i = 1, 2$ and hence $\pi_i(s_1^N, s_2^N) = 0$ for each $i = 1, 2$. Such (s_1^N, s_2^N) exists by Lemma 1.

To prove that the reduction is possible, we present a series of lemmas. We first prove the following three lemmas (Lemmas 3, 4, and 5).

LEMMA 3—Severest Punishment: *Fix $(x_1, x_2) \in X^2$ and a Nash equilibrium $(s_1, s_2) \in S^2$, and suppose that $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Then, $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2} \in \Sigma^*$.*

PROOF: Note that for any Nash equilibrium (s_1, s_2), $\pi_i(s_1, s_2) \geq 0 = \pi_i(s_1^N, s_2^N)$. This is because the order s_1' such that $s_1'(p) = 0$ for all p guarantees a payoff of zero against any supply scheme of the opponent, that is, $\pi_1(s_1', s_2) = 0$ for every s_2, so the payoff under any Nash equilibrium must be no less than 0 (the same argument holds for seller 2). Seller 1's incentive compatibility condition at time $-t$ under $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ is

$$\begin{aligned} & e^{-\lambda t} \sup_{\tilde{s}_1 \in S} \big[\pi_1\big(\tilde{s}_1, x_2(t)\big)\big] + \int_0^t \pi_i(s_1, s_2) \lambda e^{-\lambda \tau} \, d\tau \\ & \quad \leq e^{-\lambda t} \pi_1\big(x_1(t), x_2(t)\big) + \int_0^t \pi_i\big(x_1(\tau), x_2(\tau)\big) \lambda e^{-\lambda \tau} \, d\tau. \end{aligned}$$

This and $\pi_i(s_1, s_2) \geq \pi_i(s_1^N, s_2^N)$ imply

$$\begin{aligned} & e^{-\lambda t} \sup_{\tilde{s}_1 \in S} \big[\pi_1\big(\tilde{s}_1, x_2(t)\big)\big] + \int_0^t \pi_i\big(s_1^N, s_2^N\big) \lambda e^{-\lambda \tau} \, d\tau \\ & \quad \leq e^{-\lambda t} \pi_1\big(x_1(t), x_2(t)\big) + \int_0^t \pi_i\big(x_1(\tau), x_2(\tau)\big) \lambda e^{-\lambda \tau} \, d\tau, \end{aligned}$$

which is the incentive compatibility condition at time $-t$ under $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2}$. Hence, $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2}$ is also a SPE. *Q.E.D.*

In what follows, the vertical orders are going to be the key to the reduction. Let $\hat{s}[q] \in S$ for $q \geq 0$ be the order such that $\hat{s}[q](p) = q$ for every price $p \geq 0$.

LEMMA 4—Less Than the Nash Quantity: *Fix $(x_1, x_2) \in X^2$ and a Nash equilibrium $(s_1, s_2) \in S^2$, and suppose that $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Let $\tilde{\mathcal{T}} \subseteq [0, T]$ be the set of times t such that $q_1(x_1(t), x_2(t)) = q_2(x_1(t), x_2(t)) > q^N$ for each $t \in \tilde{\mathcal{T}}$. If $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, then there exists $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$ that gives each seller a strictly greater ex ante payoff than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ such that $q_i(x_1'(t), x_2'(t)) \leq q^N$ for each $t \in [0, T]$.*

PROOF: Take $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Consider a plan $(x_1', x_2') \in X^2$ defined by

$$x_i'(t) = \begin{cases} x_i(t) & \text{if } t \notin \tilde{\mathcal{T}} \\ \hat{s}\big[q^N\big] & \text{if } t \in \tilde{\mathcal{T}} \end{cases}.$$

By definition, $q_i(x_1'(t), x_2'(t)) \leq q^N$ for each $t \in [0, T]$. We first show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$, and then show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ generates a strictly higher payoff to each seller than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$. We focus on seller 1 below. A symmetric argument holds for seller 2.

To show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$, first note that the incentive compatibility condition at time $-t$ for the subgame-perfect equilibrium $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$ can be written as

$$\begin{aligned} & e^{-\lambda t} \sup_{\tilde{s}_1 \in S} \pi_1\big(\tilde{s}_1, x_2(t)\big) + \int_0^t \pi_1\big(s_1^N, s_2^N\big)\lambda e^{-\lambda\tau}\, d\tau \\ & \quad \le e^{-\lambda t}\pi_1\big(x_1(t), x_2(t)\big) + \int_0^t \pi_1\big(x_1(\tau), x_2(\tau)\big)\lambda e^{-\lambda\tau}\, d\tau. \end{aligned} \tag{25}$$

Note that, for any $\tau \in [0, T]$,

$$\begin{aligned} \pi_1\big(x_1'(\tau), x_2'(\tau)\big) &= \bar{\pi}_i\big(q^N, q^N\big) > \bar{\pi}_i\big(q_1\big(x_1(\tau), x_2(\tau)\big), q_2\big(x_1(\tau), x_2(\tau)\big)\big) \\ &= \pi_1\big(x_1(\tau), x_2(\tau)\big). \end{aligned} \tag{26}$$

Equations (25) and (26) then imply that

$$\begin{aligned} & e^{-\lambda t} \sup_{\tilde{s}_1 \in S} \pi_1\big(\tilde{s}_1, x_2(t)\big) + \int_0^t \pi_1\big(s_1^N, s_2^N\big)\lambda e^{-\lambda\tau}\, d\tau \\ & \quad \le e^{-\lambda t}\pi_1\big(x_1'(t), x_2'(t)\big) + \int_0^t \pi_1\big(x_1'(\tau), x_2'(\tau)\big)\lambda e^{-\lambda\tau}\, d\tau. \end{aligned} \tag{27}$$

If $t \notin \tilde{\mathcal{T}}$, then $x_2(t)$ in the left-hand side of (27) can be replaced by $x_2'(t)$, which yields the incentive compatibility condition at time $-t$ for $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$.

To show that the incentive compatibility condition at time $-t$ with $t \in \tilde{\mathcal{T}}$ holds, (27) implies that it suffices to show that $(\hat{s}[q^N], \hat{s}[q^N])$ is a Nash equilibrium of the supply-schedule game. To show this, take any deviation by seller 1, $s_1 \in S$, and let $\hat{p} = p(s_1, \hat{s}[q^N])$:

1. Suppose first that $\hat{p} < c$. Then, the maximized payoff is zero, which is no more than $\pi_1(\hat{s}[q^N], \hat{s}[q^N])$.
2. Suppose that $\hat{p} > c$. In this case, by the "price first" rule, $q_2(s_1, \hat{s}[q^N]) \ge q^N$. This implies that $q_1(s_1, \hat{s}[q^N]) \le D(\hat{p}) - q^N$, so 1's payoff is at most

$$\max\big\{0, (\hat{p} - c)\big(D(\hat{p}) - q^N\big)\big\}.$$

Then, since (q^N, q^N) is a Nash equilibrium of the Cournot competition and $(\hat{p} - c)(D(\hat{p}) - q^N)$ is a payoff from a deviation in the Cournot competition, this upper bound is no more than $\pi_1(\hat{s}[q^N], \hat{s}[q^N])$.

To show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ generates a strictly higher payoff to each seller than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$, recall (26). Thus, the difference of the ex ante payoffs can be calculated as

$$\begin{aligned} & \left[e^{-\lambda T}\pi_1\big(x_1'(T), x_2'(T)\big) + \int_0^T \pi_1\big(x_1'(t), x_2'(t)\big)\lambda e^{-\lambda t}\, dt\right] \\ & \quad - \left[e^{-\lambda T}\pi_1\big(x_1(T), x_2(T)\big) + \int_0^T \pi_1\big(x_1(t), x_2(t)\big)\lambda e^{-\lambda t}\, dt\right] \\ & \ge e^{-\lambda T}\big(\pi_1\big(x_1'(T), x_2'(T)\big) - \pi_1\big(x_1(T), x_2(T)\big)\big) \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\int_{t \in \tilde{\mathcal{T}}} \big(\pi_1\big(x_1'(t), x_2'(t)\big) - \pi_1\big(x_1(t), x_2(t)\big)\big) \lambda e^{-\lambda t}\, dt \right) \\
&> 0,
\end{aligned}
$$

where the last inequality follows because $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, and (26) holds. This completes the proof. *Q.E.D.*

LEMMA 5—No Demand Rationing: *Fix $(x_1, x_2) \in X^2$ and a Nash equilibrium (s_1, s_2), and suppose $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$. Let $\tilde{\mathcal{T}} \subseteq [0, T]$ be the set of times t such that $q_1(x_1(t), x_2(t)) + q_2(x_1(t), x_2(t)) < D(p(x_1(t), x_2(t)))$ and $p(x_1(t), x_2(t)) > c$ for each $t \in \tilde{\mathcal{T}}$. If $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, then there exists $(x_1', x_2') \in X$ such that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2} \in \Sigma^*$ has a strictly greater ex ante payoff to each seller than under $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ and $q_1(x_1'(t), x_2'(t)) + q_2(x_1'(t), x_2'(t)) = D(p(x_1'(t), x_2'(t)))$ for each $t \in [0, T]$.*

PROOF: Consider a plan $(x_1', x_2') \in X^2$ defined by

$$
x_i'(t) = \begin{cases} x_i(t) & \text{if } t \notin \tilde{\mathcal{T}}, \\ \tilde{x}_i(t) & \text{if } t \in \tilde{\mathcal{T}}, \end{cases}
$$

where

$$
\tilde{x}_i(t)(p) = \begin{cases} x_i(t)(p) & \text{if } p < p\big(x_1(t), x_2(t)\big), \\ \dfrac{D\big(p\big(x_1(t), x_2(t)\big)\big)}{2} & \text{if } p \geq p\big(x_1(t), x_2(t)\big). \end{cases}
$$

Then, by the definition of $q_i(\cdot, \cdot)$, $q_1(x_1'(t), x_2'(t)) + q_2(x_1'(t), x_2'(t)) = D(p(x_1'(t), x_2'(t)))$ holds for each $t \in [0, T]$.

To show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ generates a strictly higher payoff to each seller than $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$, first note that, for every $t \in \tilde{\mathcal{T}}$,

$$
\begin{aligned}
\pi_1\big(x_1'(t), x_2'(t)\big) &= \big(p\big(x_1(t), x_2(t)\big) - c\big) \frac{D\big(p\big(x_1(t), x_2(t)\big)\big)}{2} \\
&> \big(p\big(x_1(t), x_2(t)\big) - c\big) q_i\big(x_1(t), x_2(t)\big) = \pi_1\big(x_1(t), x_2(t)\big),
\end{aligned} \tag{28}
$$

where the inequality comes from the assumption that $p(x_1(t), x_2(t)) > c$ and $q_1(x_1(t), x_2(t)) + q_2(x_1(t), x_2(t)) < D(p(x_1(t), x_2(t)))$. Thus, the difference of the ex ante payoffs can be calculated as

$$
\begin{aligned}
&\left[e^{-\lambda T} \pi_1\big(x_1'(T), x_2'(T)\big) + \int_0^T \pi_1\big(x_1'(t), x_2'(t)\big) \lambda e^{-\lambda t}\, dt \right] \\
&\quad - \left[e^{-\lambda T} \pi_1\big(x_1(T), x_2(T)\big) + \int_0^T \pi_1\big(x_1(t), x_2(t)\big) \lambda e^{-\lambda t}\, dt \right] \\
&\geq e^{-\lambda T} \big(\pi_1\big(x_1'(T), x_2'(T)\big) - \pi_1\big(x_1(T), x_2(T)\big)\big) \\
&\quad + \left(\int_{t \in \tilde{\mathcal{T}}} \big(\pi_1\big(x_1'(t), x_2'(t)\big) - \pi_1\big(x_1(t), x_2(t)\big)\big) \lambda e^{-\lambda t}\, dt \right) \\
&> 0,
\end{aligned}
$$

where the last inequality follows because $\tilde{\mathcal{T}}$ has a positive measure or $T \in \tilde{\mathcal{T}}$, and (28) holds.

An analogous proof to the one for Lemma 4 can show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ is a SPE. This completes the proof. *Q.E.D.*

Let

$$\Sigma^{**} = \left\{ \left(\sigma_i\big((x_1, x_2), s_i^N\big)\right)_{i=1,2} \in \Sigma^* \middle| q_i\big(x_1(t), x_2(t)\big) \le q^N \text{ and } \right.$$
$$\left. q_i\big(x_1(t), x_2(t)\big) = \frac{D\big(p\big(x_1(t), x_2(t)\big)\big)}{2}, i = 1, 2 \right\}.$$

Note that $q_1(x_1(t), x_2(t)) = q_2(x_1(t), x_2(t)) \le q^N$ (cf. Lemma 4) implies that $p(x_1(t), x_2(t)) > c$ (cf. Lemma 5). Hence, if there is $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^{**}$ maximizing the ex ante payoff to each seller in Σ^{**}, then it maximizes the ex ante payoff in Σ^* as well. Moreover, for any maximizer of the ex ante payoff to each seller $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ in Σ^*, there exists a maximizer $(\sigma_i((x_1', x_2'), s_i'))_{i=1,2}$ in Σ^* such that $(\sigma_i((x_1', x_2'), s_i'))_{i=1,2} \in \Sigma^{**}$ and $q_i(x_1(t), x_2(t)) = q_i(x_1'(t), x_2'(t))$ for each i for almost all $t \in [0, T)$ and at $t = T$. The next lemma proves this point.

LEMMA 6—Restricting Attention to Σ^{**}: *For any maximizer of the ex ante payoff to each seller $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ in Σ^*, there exists a maximizer $(\sigma_i((x_1', x_2'), s_i'))_{i=1,2}$ in Σ^* such that $(\sigma_i((x_1', x_2'), s_i'))_{i=1,2} \in \Sigma^{**}$ and $q_i(x_1(t), x_2(t)) = q_i(x_1'(t), x_2'(t))$ for each i for almost all $t \in [0, T)$ and at $t = T$.*

PROOF: Fix $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^*$ maximizing the ex ante payoff to each seller in Σ^{**}. Let $\tilde{\mathcal{T}}$ be the set of times t such that $q_i(x_1(t), x_2(t)) > q^N$ for each $i = 1, 2$ or $q_1(x_1(t), x_2(t)) + q_2(x_1(t), x_2(t)) < D(p(x_1(t), x_2(t)))$. Lemmas 4 and 5 imply that $\tilde{\mathcal{T}}$ has measure zero and $T \notin \tilde{\mathcal{T}}$. Consider a profile of plans (x_1', x_2') such that

$$x_i'(t) = \begin{cases} x_i(t) & \text{if } t \notin \tilde{\mathcal{T}} \\ \hat{s}_i\big[q^N\big] & \text{if } t \in \tilde{\mathcal{T}} \end{cases}.$$

By definition, $(\sigma_i((x_1, x_2), s_i))_{i=1,2} \in \Sigma^{**}$ holds. Moreover, $q_i(x_1(t), x_2(t)) = q_i(x_1'(t), x_2'(t))$ for each i for almost all $t \in [0, T)$ and at $t = T$, and hence the ex ante payoffs under $(\sigma_i((x_1, x_2), s_i))_{i=1,2}$ and under $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ are the same, which implies that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ maximizes the ex ante payoff in Σ^*.

An analogous proof to the one for Lemma 4 can show that $(\sigma_i((x_1', x_2'), s_i))_{i=1,2}$ is a SPE. This completes the proof. *Q.E.D.*

Overall, if there is an essentially unique plan of quantities among those induced by the maximizers in Σ^{**}, then there is an essentially unique plan of quantities among those induced by the maximizers in Σ^*.

LEMMA 7—Restricting Attention to $\hat{S}$: *Fix $(x_1, x_2) \in X^2$ and suppose that $(\sigma_i((x_1, x_2), s_i^N))_{i=1,2} \in \Sigma^{**}$. Then, there exist $(\bar{x}_1, \bar{x}_2) \in X^2$ such that the following hold*:

1. *$\bar{x}_i(t) \in \hat{S}$ for each t and i.*

2. *For each* t, $q_i(\bar{x}_1(t), \bar{x}_2(t)) = q_i(x_1(t), x_2(t))$ *for each* i *and* $p(\bar{x}_1(t), \bar{x}_2(t)) = p(x_1(t), x_2(t))$.
3. $(\sigma_i((\bar{x}_1, \bar{x}_2), s_i^N))_{i=1,2} \in \Sigma^{**}$.

PROOF: To show the lemma, we first prove the following claim:

CLAIM 1: *Fix an arbitrary* $\bar{q} \in [0, q^N]$ *and* $(s_1, s_2) \in S^2$ *such that* $q_1(s_1, s_2) = q_2(s_1, s_2) = \bar{q}$ *and* $2\bar{q} = D(p(s_1, s_2))$. *The following are true*:

1. *The price, quantities, and profits under* $(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$ *are the same as those under* (s_1, s_2). *Formally, for each* $i = 1, 2$, *the following three equalities hold*:

$$p\big(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]\big) = p(s_1, s_2);$$

$$q_i\big(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]\big) = q_i(s_1, s_2);$$

$$\pi_i\big(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]\big) = \pi_i(s_1, s_2).$$

2. *The payoff after any deviation under* $(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$ *is no more than the one under* (s_1, s_2). *Formally, for each* $s_1' \in S$ *such that* $\pi_1(s_1', \hat{s}_2[\bar{q}]) \geq \pi_1(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$, *the following inequality holds for seller* 1:

$$\pi_1\big(s_1', \hat{s}_2[\bar{q}]\big) \leq \sup_{s_1'' \in S} \pi_1\big(s_1'', s_2\big).$$

The symmetric inequality holds for seller 2, *too*.

PROOF OF CLAIM 1: Part 1: The equalities on quantities directly follow from the definition of $\hat{s}_i[\bar{q}]$.

Given the equalities on quantities and the definition of the $p(\cdot, \cdot)$ function, if $\bar{q} > 0$, the equality on prices holds because $\hat{s}_1[\bar{q}](p) + \hat{s}_2[\bar{q}](p) = 2\bar{q} \leq D(p)$ for all $p \leq p(s_1, s_2)$ and $\hat{s}_1[\bar{q}](p) + \hat{s}_2[\bar{q}](p) = 2\bar{q} > D(p)$ for all $p > p(s_1, s_2)$ by the definition of $\hat{s}_i[\bar{q}]$ for each i and the assumption that D is strictly decreasing for p's such that $0 < D(p) < 2q^N$. If $\bar{q} = 0$, the equality on prices holds by the definition of the $p(\cdot, \cdot)$ function.

Finally, the equalities on profits follow because we have shown the equalities on quantities and prices.

Part 2: We prove the inequality for seller 1. A symmetric argument shows that the inequality for seller 2 holds, too.

Given $(s_1, s_2) \in S^2$, let

$$S^- = \big\{s_1' \in S | p\big(s_1', \hat{s}_2[\bar{q}]\big) < p(s_1, s_2)\big\},$$

$$S^0 = \big\{s_1' \in S | p\big(s_1', \hat{s}_2[\bar{q}]\big) = p(s_1, s_2)\big\},$$

$$S^+ = \big\{s_1' \in S | p\big(s_1', \hat{s}_2[\bar{q}]\big) > p(s_1, s_2)\big\}.$$

We will show that, for each element s_1' in each of S^- and S^0, there exists $s_1'' \in S$ such that $\pi_1(s_1', \hat{s}_2[\bar{q}]) \leq \pi_1(s_1'', s_2)$. Also, we show that $\pi_1(s_1', \hat{s}_2[\bar{q}]) < \pi_1(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}])$ for each $s_1' \in S^+$. Showing these claims suffices because $S^- \cup S^0 \cup S^+ = S$.

Case 1: Consider $s_1' \in S^-$ and let $\hat{p} = p(s_1', \hat{s}_2[\bar{q}]) \in [0, p(s_1, s_2))$.

Suppose first that $\hat{p} > c$. Take $s_1'' \in S$ such that $s_1''(p) = D(\hat{p}) - s_2(\hat{p})$ for all $p \in [0, \infty)$. Note that $p(s_1'', s_2) = \hat{p}$. This is because the definition of s_1'' and the assumptions that s_2 is non-decreasing and D is strictly decreasing for prices below $p(s_1, s_2)$ imply that $s_1''(p) + s_2(p) \le D(p)$ for all $p \le \hat{p}$ and $s_1''(p) + s_2(p) > D(p)$ for all $p > \hat{p}$.

Then, we have

$$\begin{aligned}
\pi_1\big(s_1'', s_2\big) &= (\hat{p} - c)\big(D(\hat{p}) - s_2(\hat{p})\big) \quad \big(\text{by the definition of } s_1''\big) \\
&\ge (\hat{p} - c)\big(D(\hat{p}) - q_2(s_1, s_2)\big) \quad \big(\text{by “price first” and } \hat{p} < p(s_1, s_2)\big) \\
&= (\hat{p} - c)\big(D(\hat{p}) - \bar{q}\big) \quad (\text{by the definition of } \bar{q}) \\
&= \pi_1\big(s_1', \hat{s}_2[\bar{q}]\big) \quad \big(\text{by “price first” and } \hat{p} > c > 0\big).
\end{aligned}$$

If $\hat{p} \le c$, then $\pi_1(s_1', \hat{s}_2[\bar{q}]) \le 0$. Consider $s_1'' \in S$ such that $s_1''(p) = 0$ for all $p \in [0, \infty)$. Then, $\pi_1(s_1'', s_2) = 0$.

Overall, we have shown that for each $s_1' \in S^-$, there exists $s_1'' \in S$ such that $\pi_1(s_1', \hat{s}_2[\bar{q}]) \le \pi_1(s_1'', s_2)$. Hence, we have that, for each $s_1' \in S^-$, $\pi_1(s_1', \hat{s}_2[\bar{q}]) \le \sup_{s_1'' \in S} \pi_1(s_1'', s_2)$.

Case 2: Consider $s_1' \in S^0$, that is, $p(s_1', \hat{s}_2[\bar{q}]) = p(s_1, s_2)$. First, note that

$$\begin{aligned}
\pi_1\big(s_1', \hat{s}_2[\bar{q}]\big) &\le \big(p(s_1, s_2) - c\big)\big(D\big(p(s_1, s_2)\big) - \bar{q}\big) \quad \big(\text{by “price first” and the definition of } \bar{q}\big) \\
&= \big(p(s_1, s_2) - c\big)\bar{q}.
\end{aligned}$$

Second, note that

$$\begin{aligned}
\sup_{s_1'' \in S^0} \pi_1\big(s_1'', s_2\big) \ge \pi_1(s_1, s_2) &= \big(p(s_1, s_2) - c\big)q_1(s_1, s_2) \\
&= \big(p(s_1, s_2) - c\big)\bar{q}.
\end{aligned}$$

Combining, we have that, for each $s_1' \in S^0$, $\pi_1(s_1', \hat{s}_2[q]) \le \sup_{s_1'' \in S^0} \pi_1(s_1'', s_2)$.

Case 3: Consider $s_1' \in S^+$ and let $\hat{p} > p(s_1', \hat{s}_2[\bar{q}])$. Then, we have the following:

$$\begin{aligned}
\pi_1\big(s_1', \hat{s}_2[\bar{q}]\big) &\le (\hat{p} - c)\big(D(\hat{p}) - \bar{q}\big) \quad \big(\text{by “price first”}\big) \\
&= \bar{\pi}_1\big(D(\hat{p}) - \bar{q}, \bar{q}\big) \\
&\le \bar{\pi}_1(\bar{q}, \bar{q}) \quad \left(\text{by } \bar{q} < \frac{a - c}{3b}\right) \\
&= \pi_1\big(\hat{s}_1[\bar{q}], \hat{s}_2[\bar{q}]\big).
\end{aligned}$$

Overall, we have shown the desired claim. *Q.E.D.*

Having proved the claim, we now prove the lemma.

Fix $(x_1, x_2) \in X^2$ and suppose $(\sigma_i((x_1, x_2), (s_i^N)))_{i=1,2} \in \Sigma^{**}$. Seller 1's incentive compatibility constraint at time $-t$ implies the following:

$$\begin{aligned}
&e^{-\lambda t}\pi_i\big(x_1(t), x_2(t)\big) + \int_0^t \pi_i\big(x_1(\tau), x_2(\tau)\big)\lambda e^{-\lambda\tau}\, d\tau \\
&\quad \ge e^{-\lambda t}\pi_i\big(s_1', x_2(t)\big) + \big(1 - e^{-\lambda t}\big)\pi_i\big(s_1^N, s_2^N\big)
\end{aligned}$$

for all $s_1' \in S_1$. By part 1 of Claim 1, we have

$$\begin{aligned} &e^{-\lambda t}\pi_i\big(\hat{s}_1\big[q_1\big(x_1(t), x_2(t)\big)\big], \hat{s}_2\big[q_2\big(x_1(t), x_2(t)\big)\big]\big) \\ &\quad + \int_0^t \pi_i\big(\hat{s}_1\big[q_1\big(x_1(\tau), x_2(\tau)\big)\big], \hat{s}_2\big[q_2\big(x_1(\tau), x_2(\tau)\big)\big]\big)\lambda e^{-\lambda\tau}\, d\tau \\ &\geq e^{-\lambda t}\pi_i\big(s_1', x_2(t)\big) + \big(1 - e^{-\lambda t}\big)\pi_i\big(s_1^N, s_2^N\big) \end{aligned}$$

for all $s_1' \in S$.

Then, part 2 of Claim 1 implies that

$$\begin{aligned} &e^{-\lambda t}\pi_i\big(\hat{s}_1\big[q_1\big(x_1(t), x_2(t)\big)\big], \hat{s}_2\big[q_2\big(x_1(t), x_2(t)\big)\big]\big) \\ &\quad + \int_0^t \pi_i\big(\hat{s}_1\big[q_1\big(x_1(\tau), x_2(\tau)\big)\big], \hat{s}_2\big[q_2\big(x_1(\tau), x_2(\tau)\big)\big]\big)\lambda e^{-\lambda\tau}\, d\tau \\ &\geq e^{-\lambda t}\pi_i\big(s_1', \hat{s}_2\big[q_2\big(x_1(t), x_2(t)\big)\big]\big) + \big(1 - e^{-\lambda t}\big)\pi_i\big(s_1^N, s_2^N\big) \end{aligned}$$

for all $s_1' \in S$.

The last inequality is the incentive comparability constraint for seller 1 under $(\sigma_i((\bar{x}_1, \bar{x}_2), s_i^N))_{i=1,2}$. A symmetric argument shows that the incentive comparability constraint holds for seller 2 as well. *Q.E.D.*

Lemma 7 implies that, as far as the plan of quantities is concerned, restricting attention to the following set of strategy profiles is without loss of generality:

$$\bar{\bar{\Sigma}} := \big\{\big(\sigma_i\big((x_1, x_2), s_1^N\big)\big)_{i=1,2} \in \bar{\Sigma} | \exists q : [0, T] \to \mathbb{R}_+ \text{ s.t. } x_i(t) = \hat{s}_i\big[q(t)\big]\big\}.$$

Furthermore, note that a payoff of 0 can be attained in the semi-Cournot game in a Nash equilibrium in which each seller chooses $\emptyset$.

Hence, in order to prove that the reduction works, the only thing left is to show that the gain from an instantaneous deviation given any scheme $\hat{s}_i[q]$ of the opponent in the supply-schedule game is the same as the instantaneous gain from a deviation given any quantity q of the opponent in the semi-Cournot game when q is no more than the Nash quantity. The next lemma proves this.

LEMMA 8: *For any* $q \leq \frac{a-c}{3b}$, $\sup_{s_i \in S} \pi_i(s_i, \hat{s}_i[q]) = \sup_{q' \in \mathbb{R}_+} \bar{\pi}_i(q', q)$.

PROOF OF LEMMA 8: Fix $q \leq q^N$.

First, consider a deviation inducing a price strictly less than c. In the semi-Cournot competition, given seller 2's quantity q, any deviation that induces a price strictly less than c cannot be the optimal one for seller 1 since such deviations are strictly dominated by a deviation to set the zero quantity. In the supply-schedule game, given seller 2's supply schedule $\hat{s}_2[q]$, any deviation that induces a price strictly less than c cannot be the optimal one for seller 1 since $q < D(p)$ if $p < c$.

Second, consider a deviation to induce a price strictly above $p(\hat{s}_1[q], \hat{s}_2[q])$. In the semi-Cournot competition, $\arg\max_{q'} \bar{\pi}_i(q', q) \geq q^N$ if $q \leq q^N$, which implies that the induced price under the optimal deviation is no greater than $p(\hat{s}_1[q], \hat{s}_2[q])$. In the supply-schedule game, the "price first" rule implies that a deviation to any s_1 inducing a price $\tilde{p} \geq c > 0$ results in seller 1's realized supply that is no greater than $D(\tilde{p}) - q$. The same argument as in the semi-Cournot competition then implies that the induced price under the optimal deviation is no greater than $p(\hat{s}_1[q], \hat{s}_2[q])$ in the supply-schedule game as well.

Take $\hat{p} \in (c, p(\hat{s}_1[q], \hat{s}_2[q])]$. In the supply-schedule game, consider s_1' such that $p(s_1', \hat{s}_2[q]) = \hat{p}$. Then,

$$\begin{aligned}\sup_{s_1' \in S \text{ s.t. } p(s_1', \hat{s}_2[q]) = \hat{p}} \pi_i\big(s_1', \hat{s}_2[q]\big) &= \pi_i\big(\tilde{s}_1, \hat{s}_2[q]\big) \\ &= (\hat{p} - c) \cdot \big(D(\hat{p}) - q\big) \\ &= \sup_{q' \in \mathbb{R}_+ \text{ s.t. } D(\hat{p}) = q' + q} \bar{\pi}_i\big(q', q\big),\end{aligned}$$

where we define $\tilde{s}_1$ so that $\tilde{s}_1(p) = D(\hat{p}) - q$ for all p.

Thus,

$$\begin{aligned}\sup_{s_i \in S} \pi_i\big(s_i, \hat{s}_i[q]\big) &= \sup_{\hat{p} \in (c, p(\hat{s}_1[q], \hat{s}_2[q])]} \Big(\sup_{s_1' \in S \text{ s.t. } p(s_1', \hat{s}_2[q]) = \hat{p}} \pi_i\big(s_1', \hat{s}_2[q]\big) \Big) \\ &= \sup_{\hat{p} \in (c, p(\hat{s}_1[q], \hat{s}_2[q])]} \Big(\sup_{q' \in \mathbb{R}_+ \text{ s.t. } D(\hat{p}) = q' + q} \bar{\pi}_i\big(q', q\big) \Big) = \sup_{q' \in \mathbb{R}_+} \bar{\pi}_i\big(q', q\big).\end{aligned}$$

A symmetric argument holds for seller 2. *Q.E.D.*

B.1.2. *Proofs*

PROOF OF LEMMA 1: *The "if" direction:* Take any $(q_1, q_2) \in Q^N$. We prove that there exists a Nash equilibrium (s_1, s_2) with $(q_1(s_1, s_2), q_2(s_1, s_2)) = (q_1, q_2)$. To show this, fix an arbitrary $\bar{Q} \geq \frac{a}{b}$ and consider s_i for each $i = 1, 2$ such that

$$s_i(p) = \begin{cases} q_i & \text{if } p < a - b(q_1 + q_2), \\ \bar{Q} & \text{if } a - b(q_1 + q_2) \leq p. \end{cases}$$

Note that $p(s_1, s_2) = a - b(q_1 + q_2) \geq c$. Also, the "price first" rule implies that $q_1(s_1, s_2) = q_1$.

Consider s_1' such that $p(s_1', s_2) \leq p(s_1, s_2)$. Then, either (i) $p(s_1', s_2) = p(s_1, s_2)$ and $q_1(s_1', s_2) \leq q_1$, or (ii) $q_1(s_1', s_2) \geq q_1$. In case (i),

$$\pi_1\big(s_1', s_2\big) = \big(p\big(s_1', s_2\big) - c\big) q_1\big(s_1', s_2\big) \leq \big(p(s_1, s_2) - c\big) q_1(s_1, s_2) = \pi_1(s_1, s_2).$$

In case (ii),

$$\pi_1\big(s_1', s_2\big) \leq \Big(\big(a - b\big(q_1\big(s_1', s_2\big) + q_2\big)\big) - c\Big) q_1\big(s_1', s_2\big). \tag{29}$$

Since $((a - b(x + q_2)) - c)x$ is decreasing in x when $x \geq \frac{a-c-bq_2}{2b}$ and we have $\frac{a-c-bq_2}{2b} \leq q_1 \leq q_1(s_1', s_2)$ by assumption, the right-hand side of (29) is no greater than

$$\big(p(s_1, s_2) - c\big) q_1,$$

which is equal to $\pi_1(s_1, s_2)$.

Finally, there is no s_1' such that $p(s_1', s_2) > p(s_1, s_2)$. Otherwise, there exists $p \in (p(s_1, s_2), p(s_1', s_2))$ such that $s_1'(p) + s_2(p) \leq D(p)$ by the definition of the $p(\cdot, \cdot)$ function, but this would imply $s_1'(p) + \bar{Q} \leq D(p)$, which violates the assumption on $\bar{Q}$ that $\bar{Q} \geq \frac{a}{b}$.

A symmetric argument holds for s_2, and this completes the proof for the "if" direction.

The "only if" direction:

First, take $(q_1, q_2) \in \mathbb{R}^2_+$ such that $q_1 < \frac{a-c-bq_2}{2b}$ and (s_1, s_2) such that $q_i(s_1, s_2) = q_i$ for each $i = 1, 2$. We prove that (s_1, s_2) is not a Nash equilibrium. Let

$$s'_i(p) = \begin{cases} \dfrac{a-c-bq_2}{2b} & \text{if } p < a - b(q_1+q_2), \\ \bar{Q} & \text{if } a - b(q_1+q_2) \le p. \end{cases}$$

First, note that $p(s'_1, s_2) > 0$. This is because, otherwise, $s'_1(\epsilon) + s_2(\epsilon) > D(\epsilon)$ must hold for any $\epsilon > 0$. However, since $q_1 < \frac{a-c-bq_2}{2b}$ implies $a - b(q_1+q_2) > 0$, there exists $\bar{\epsilon} > 0$ such that, for all $\epsilon \in (0, \bar{\epsilon})$, $s'_1(\epsilon) + s_2(\epsilon) \le \frac{a-c-bq_2}{2b} + q_2 = \frac{a-c+bq_2}{2b} = \frac{a-c}{2b} + \frac{q_2}{2} < \frac{a-c}{2b} + \frac{\frac{a-c}{b}}{2} = \frac{a-c}{b} < D(0)$. Since $D(\cdot)$ is continuous, it is true that there exists $\epsilon > 0$ such that $s'_1(\epsilon) + s_2(\epsilon) < D(\epsilon)$, a contradiction.

If $p(s'_1, s_2) \in (0, p(s_1, s_2))$, then $q_2(s'_1, s_2) \le s_2(p(s'_1, s_2)) \le q_2$ holds, where the first inequality follows because the realized quantity must be no more than what s_2 specifies, and the second inequality follows because of the "price first" rule. Also, $q_1(s'_1, s_2) = \frac{a-c-bq_2}{2b}$ because of the definition of s'_1 and the "price first" rule. Hence, we have

$$\begin{aligned} \pi_1(s'_1, s_2) &= \big(p(s'_1, s_2) - c\big) q_1(s'_1, s_2) \\ &= \left(a - b\left(\frac{a-c-bq_2}{2b} + q_2(s'_1, s_2)\right) - c\right)\frac{a-c-bq_2}{2b} \\ &\ge \left(a - b\left(\frac{a-c-bq_2}{2b} + q_2\right) - c\right)\frac{a-c-bq_2}{2b} \\ &> \big(a - b(q_1+q_2) - c\big) q_1 \quad \left(\text{because } q_1 \ne \frac{a-c-bq_2}{2b}\right) \\ &= \pi_1(s_1, s_2). \end{aligned}$$

If $p(s'_1, s_2) = p(s_1, s_2)$, then

$$\begin{aligned} \pi_1(s'_1, s_2) &= \big(p(s'_1, s_2) - c\big) q_1(s'_1, s_2) \\ &= \big(p(s_1, s_2) - c\big) q_1(s'_1, s_2) \\ &> \big(p(s_1, s_2) - c\big) q_1(s_1, s_2) \\ &= \pi_1(s_1, s_2), \end{aligned}$$

where the inequality follows from the definitions of (q_1, q_2) and s'_1, and the "price first" rule.

Finally, by the definition of s'_1, $p(s'_1, s_2) \le p(s_1, s_2)$. Overall, (s_1, s_2) is not a Nash equilibrium.

The case with $q_2 < \frac{a-c-bq_2}{2b}$ is perfectly symmetric.

Second, take $(q_1, q_2) \in \mathbb{R}^2_+$ such that $q_1 + q_2 > \frac{a-c}{b}$ and (s_1, s_2) such that $q_i(s_1, s_2) = q_i$ for each $i = 1, 2$. We prove that (s_1, s_2) is not a Nash equilibrium. To see this, without loss of generality let $q_1 > 0$, and observe

$$\pi_1(s_1, s_2) = \big(a - b(q_1+q_2) - c\big) q_1 < \left(a - b\frac{a-c}{b} - c\right) q_1 = 0.$$

However, for s_1' such that $s_1'(p) = 0$ for all $p \in \mathbb{R}_+$, we have $\pi_1(s_1', s_2) = 0$. Hence, (s_1, s_2) is not a Nash equilibrium. *Q.E.D.*

PROOF OF PROPOSITION 2: Solving the differential equation given in the text, we obtain

$$\begin{aligned}
&\frac{6b\big(3bq-(a-c)\big)}{(a-c-bq)^2}\,dq = \lambda\,dt \\
&\iff \quad 6\left(3\ln(a-c-bq)+\frac{2(a-c)}{a-c-bq}\right) = \lambda t + C,
\end{aligned}$$

where C is a constant. Given the initial condition $\lim_{t\downarrow 0} q(t) = \frac{a-c}{3b}$, we have

$$C = 18\left(\ln(a-c)+\ln\left(\frac{2}{3}\right)+1\right).$$

The time at which the quantity reaches the collusive quantity $q^* = \frac{a-c}{4b}$ is $t^* = \frac{1}{\lambda}(36\ln(3) - 52\ln(2) - 2)$. Manipulating, we obtain the solution presented in the statement of the proposition.

Since this plan of quantities is essentially unique (in the sense defined in Section 4.1) in the revision game of the semi-Cournot competition, it follows that the plan of quantities induced by the strategy profile in Σ^{**} (and thus in Σ^*) is essentially unique in the revision game of the supply-schedule game. *Q.E.D.*

B.1.3. *Calculating the Expected Payoff Bound*

Let $\pi^* = \frac{(a-c)^2}{8b}$ be the payoff at q^*. The payoff at $q(0)$ is denoted $\pi^N = \frac{(a-c)^2}{9b}$. The expected payoff can be bounded as follows:

$$\begin{aligned}
&e^{-\lambda t^*}\big(a-c-2bx^*\big)x^* + \int_0^{t^*}\big(a-c-2bx(\tau)\big)x(\tau)\lambda e^{-\lambda\tau}\,d\tau \\
&\geq e^{-\lambda t^*}\big(a-c-2bx^*\big)x^* + \big(1-e^{-\lambda t^*}\big)\frac{(a-c)^2}{9b} \\
&\geq e^{-(36\ln(3)-52\ln(2)-2)}\frac{(a-c)^2}{8b} + \big(1-e^{-(36\ln(3)-52\ln(2)-2)}\big)\frac{(a-c)^2}{9b} \\
&= e^{-(36\ln(3)-52\ln(2)-2)}\pi^* + \big(1-e^{-(36\ln(3)-52\ln(2)-2)}\big)\frac{8}{9}\pi^* \\
&= \left(e^{-(36\ln(3)-52\ln(2)-2)} + \frac{8}{9}\big(1-e^{-(36\ln(3)-52\ln(2)-2)}\big)\right)\pi^* \\
&= \left(0.88683650092 + \frac{8}{9}(1-0.88683650092)\right)\pi^* \\
&= 0.98742627788\pi^*.
\end{aligned}$$

B.2. *A Proof and the Detail for Remark 4 for Section 4.2*

B.2.1. *Proof of Proposition 3*

PROOF: The first-order condition is

$$0 = \frac{\partial \pi_A}{\partial x_A} = \frac{\delta}{2}\big((1-x_A)+w\big) - \frac{1+\delta(x_A-x_B)}{2} - \frac{\delta}{2}\gamma(1-x_B).$$

This implies that, if x_A is a best response to x_B, then

$$x_A = \begin{cases} 0 \quad \text{if } x_B \le \dfrac{\frac{1}{\delta}+\gamma-w-1}{1+\gamma}, \\ \dfrac{(1+\gamma)x_B-\gamma+w+\frac{\delta-1}{\delta}}{2} \in (0,1] \\ \qquad \text{if } x_B \in \left(\dfrac{\frac{1}{\delta}+\gamma-w-1}{1+\gamma}, \dfrac{\frac{1}{\delta}+\gamma-w+1}{1+\gamma}\right], \\ 1 \quad \text{if } x_B > \dfrac{\frac{1}{\delta}+\gamma-w+1}{1+\gamma}. \end{cases} \tag{30}$$

The symmetric expression holds for B's best response to an arbitrary x_A. This enables us to compute the unique Nash equilibrium of the component game as in (11).

First, suppose that $w \le \frac{(1-\delta)+\delta\gamma}{\delta}$. Then, the Nash equilibrium is the action profile that maximizes each candidate's payoff among symmetric action profiles. Thus, there is a unique optimal trigger strategy plan, and it is the one in which the Nash equilibrium action profile $(0,0)$ is played on (and off) the path.

Consequently, in what follows, we consider the case $\frac{(1-\delta)+\delta\gamma}{\delta} < w$. Under this assumption, we first solve for the optimal grim trigger strategy plan assuming that $\delta = 1$. Then, using the result for the case with $\delta = 1$, we solve for the optimal grim trigger strategy plan for the case with $\delta < 1$. We denote the optimal plan under parameter δ by $x^\delta(\cdot)$.

A. The case with $\delta = 1$:

First, we assume $\delta = 1$ and solve for x^1. Let us compute $d(x)$, $\pi(x)$, and π^N. By substituting x into x_A and x_B in (10), we have

$$\pi(x) = \frac{1}{2}\big((1+\gamma)(1-x)+w\big).$$

Thus, substituting (11) and $\delta = 1$ into this,

$$\pi^N = \begin{cases} \dfrac{1}{2}\left((1+\gamma)\dfrac{1-w}{1-\gamma}+w\right) & \text{if } w \le \dfrac{1}{\delta}, \\ \dfrac{w}{2} & \text{if } \dfrac{1}{\delta} < w. \end{cases}$$

A-1. The case with $\frac{(1-\delta)+\delta\gamma}{\delta} < w \le \frac{1}{\delta}$:

Note that, in this case, $[0,1] \subseteq (\frac{\frac{1}{\delta}+\gamma-w-1}{1+\gamma}, \frac{\frac{1}{\delta}+\gamma-w+1}{1+\gamma}]$, and thus a best response to any $x_B \in [0,1]$ is $x_A = \frac{(1+\gamma)x_B-\gamma+w+\frac{\delta-1}{\delta}}{2}$. Substituting this into (10), setting $x = x^B$ and $\delta = 1$, and rearranging, for every $x \in [0,1]$,

$$d(x) + \pi(x) = \gamma(1-x) + \frac{1}{2}\left(\frac{(1-\gamma)(1-x)+w+1}{2}\right)^2.$$

Now, recall that our optimal plan is given by

$$\frac{dx^1}{dt} = \lambda \frac{d(x^1) + \pi(x^1) - \pi^N}{d'(x^1)}.$$

Hence, by substituting, we obtain

$$\frac{dx^1}{dt} = -\lambda \frac{(1-\gamma)^2(1-x^1) + (1-\gamma)w + 3 + 5\gamma}{2(1-\gamma)^2}.$$

This implies

$$\int \lambda\, dt = -\int \frac{2(1-\gamma)^2}{(1-\gamma)^2(1-x^1) + (1-\gamma)w + 3 + 5\gamma}\, dx^1,$$

which implies

$$\lambda t + C = 2\ln\big((1-\gamma)^2(1-x^1) + (1-\gamma)w + 3 + 5\gamma\big)$$

for some constant C. To solve for C, substitute (11) and $t = 0$ into this to get $C = 2\ln(4 + 4\gamma)$. Hence, we have

$$e^{\frac{\lambda}{2}t} = \frac{(1-\gamma)^2(1-x^1(t)) + (1-\gamma)w + 3 + 5\gamma}{4+4\gamma},$$

or

$$x^1(t) = \frac{-e^{\frac{\lambda}{2}t}(4+4\gamma) + \big((1-\gamma)w + 4 + 3\gamma + \gamma^2\big)}{(1-\gamma)^2}. \tag{31}$$

A-2. The case with $\frac{1}{\delta} < w$:

Second, suppose that $\frac{1}{\delta} < w$. For each $w > \frac{1}{\delta}$, we have $\frac{\frac{1}{\delta}+\gamma-w+1}{1+\gamma} < 1$. Hence, (30) implies that there exists $\epsilon > 0$ such that, for all $x \in (1-\epsilon, 1]$, action 1 is the unique best response to x. Thus, by substituting $x_A = 1$ into (10), setting $x_B = x$ and $\delta = 1$, we obtain

$$d(x) + \pi(x) = \frac{2-x}{2}w + \frac{x}{2}\gamma(1-x)$$

for $x \in (1-\epsilon, 1]$. This implies that $\frac{d(x)}{\pi(x)-\pi^N} = \frac{w-1+\gamma(x-1)}{1+\gamma}$ for all $x \in (1-\epsilon, 1]$. This converges to $\frac{w-1}{1+\gamma}$ as $x \to 1$, and it is strictly positive because $\frac{1}{\delta} < w$. Theorem 4 then implies that there is a unique trigger strategy equilibrium, and in that equilibrium, candidates play the Nash action all the time.

B. The case with general δ:

Now, we consider the case with $\delta < 1$ and solve for x^δ. To deal with this case, it is useful to introduce a change of a variable for each $i = A, B$ as follows:

$$y_i = 1 - \delta(1 - x_i).$$

Note that, since $x_i \in [0, 1]$, we have $y_i \in [1 - \delta, 1]$. Moreover,

$$y_i - y_j = \delta(x_i - x_j) \quad \text{and} \quad 1 - x_i = \frac{1}{\delta}(1 - y_i)$$

hold. Hence, the payoff in (10) can be rewritten as

$$\begin{aligned}\pi_A(x_A, x_B) &= \frac{1 + (y_A - y_B)}{2}\left(\frac{1}{\delta}(1 - y_A) + w\right) + \frac{1 + (y_B - y_A)}{2} \cdot \gamma \cdot \frac{1}{\delta}(1 - y_B) \\ &= \frac{1}{\delta}\left[\frac{1 + (y_A - y_B)}{2}\big((1 - y_A) + \delta w\big) + \frac{1 + (y_B - y_A)}{2} \cdot \gamma(1 - y_B)\right].\end{aligned}$$

Note that this expression is proportional to (10) in which we substitute y_i into x_i for each $i = A, B$ and δw into w.[3] Hence, by (31), the optimal trigger strategy plan under general δ satisfies

$$y(t) = \begin{cases} \dfrac{-e^{\frac{\lambda}{2}t}(4 + 4\gamma) + \big((1 - \gamma)\delta w + 4 + 3\gamma + \gamma^2\big)}{(1 - \gamma)^2} & \text{if } w \le \dfrac{1}{\delta}, \\ 1 & \text{if } \dfrac{1}{\delta} < w. \end{cases}$$

Hence,

$$x^\delta(t) = \frac{y(t) - (1 - \delta)}{\delta} = \begin{cases} \dfrac{-e^{\frac{\lambda}{2}t}(4 + 4\gamma) + \delta(1 - \gamma)(w + 1 - \gamma) + 3 + 5\gamma}{\delta(1 - \gamma)^2} \\ \quad = x^N - \dfrac{\big(e^{\frac{\lambda}{2}t} - 1\big)(4 + 4\gamma)}{\delta(1 - \gamma)^2} & \text{if } w \le \dfrac{1}{\delta}, \\ 1 & \text{if } \dfrac{1}{\delta} < w. \end{cases}$$

Solving $x^\delta(t^*) = 0$, we obtain

$$t^* = \frac{2}{\lambda}\ln\left(\frac{\delta(1 - \gamma)(w + 1 - \gamma) + 3 + 5\gamma}{4 + 4\gamma}\right). \qquad \textit{Q.E.D.}$$

APPENDIX C: APPENDIX FOR SECTION 5

C.1. *Proofs for Section 5*

C.1.1. *Proof of Theorem 3*

PROOF: Take $\epsilon > 0$ for Assumption (*) and $k \in (0, 1)$ for condition (13) to hold.

[3]This and $\frac{dx_i}{dt}\delta = \frac{dy_i}{dt}$ imply the differential equation for general δ presented in Section 4.2.

Note first that, by condition (13) and Assumption (*)-3, we can find $\bar{a} \in (a^N, a^N + \epsilon]$ such that, for all $a \in [a^N, \bar{a}]$, $(d(a))^k \leq \pi(a) - \pi^N$ holds.

Next, we introduce a generalized inverse of function d that is measurable. We will construct a non-trivial equilibrium plan from this function. Note that, by definition, $d(a) \geq 0$ for all a and $d(a) = 0$ means that a is a symmetric Nash equilibrium. Since we are assuming that a^N is the unique symmetric Nash equilibrium, $d > 0$ on $(a^N, \overline{a}]$. Our goal here is to find a measurable function $b : [0, d(\overline{a})] \to [a^N, \bar{a}]$ such that $d(b(\delta)) = \delta$ for each $\delta \in [0, d(\overline{a})]$. If d^{-1} exists in the given domain (i.e., if d is increasing on $[a^N, \bar{a}]$), then we let $b = d^{-1}$. More generally, we construct b as follows. First, define a function on $[a^N, \overline{a}]$ by

$$\overline{d}(a) := \max_{a' \in [a^N, a]} d(a').$$

This is well-defined because the function d is continuous on a compact set $[a^N, a]$. By construction, $\overline{d}$ is non-decreasing, and it is continuous by Berge's Theorem of the Maximum.[4] By construction, $\overline{d}(0) = 0$ and $\overline{d}(\overline{a}) \geq d(\overline{a})$. Hence, the continuity of $\overline{d}$ implies that, for any $\delta \in [0, d(\overline{a})]$, there is some a_δ such that $\delta = \overline{d}(a_\delta)$. By the definition of $\overline{d}$, there must be some a^*_δ such that $\delta = \overline{d}(a_\delta) = d(a^*_\delta)$ (i.e., a^*_δ maximizes d on $[a^N, a_\delta]$). Define $b(\delta)$ to be a^*_δ.[5] By construction, b is an increasing function and therefore measurable.[6]

Now let $\hat{\epsilon} := \min\{d(\bar{a})^{\frac{1-s}{2}}, \frac{\lambda(1-s)}{s+1}\}$. We are going to show that a trigger strategy plan

$$x(t) = \begin{cases} b\left(t^{\frac{2}{1-k}}\right) & \text{if } t < \hat{\epsilon}, \\ b\left(\hat{\epsilon}^{\frac{2}{1-k}}\right) & \text{if } t \geq \hat{\epsilon}, \end{cases} \tag{32}$$

satisfies the incentive constraint

$$\int_0^t \left(\pi\left(x(\tau)\right) - \pi^N\right)\lambda e^{-\lambda\tau}\, d\tau \geq d\left(x(t)\right)e^{-\lambda t} \tag{33}$$

for all $t \in [0, T]$. First, we show that the plan $x(t)$ is well-defined. Recall that $\hat{\epsilon}$ was defined to be less than $d(\bar{a})^{\frac{1-k}{2}}$, and therefore, for all $t \leq \hat{\epsilon}$, we have $t^{\frac{2}{1-k}} \leq \hat{\epsilon}^{\frac{2}{1-k}} \leq d(\bar{a})$. Hence, $t^{\frac{2}{1-k}}$ (for $t \leq \hat{\epsilon}$) is in the domain of b (i.e., $[0, d(\overline{a})]$), and therefore $x(t)$ given by (32) is indeed well-defined. Second, since b is measurable, the integral in the above incentive constraint is well-defined. Third, we show that the inequality in the incentive constraint (33) holds. To see this, first consider the case $t \leq \hat{\epsilon}$. We have

$$\begin{aligned} \int_0^t \left(\pi\left(x(\tau)\right) - \pi^N\right)\lambda e^{-\lambda\tau}\, d\tau &\geq \int_0^t \left(d\left(x(\tau)\right)\right)^k \lambda e^{-\lambda\tau}\, d\tau = \int_0^t \tau^{\frac{2k}{1-k}} \lambda e^{-\lambda\tau}\, d\tau \\ &> \lambda e^{-\lambda t} \int_0^t \tau^{\frac{2k}{1-k}}\, d\tau \end{aligned}$$

[4]The correspondence that maps a to $[a^N, a]$ is both upper and lower semicontinuous, and d is continuous. Hence, the conditions for Berge's theorem are satisfied.

[5]If a^*_δ is not unique, choose any one.

[6]Suppose b is not increasing and there are $\delta < \delta'$ such that $b(\delta) \geq b(\delta')$ $(\geq a^N)$. By the construction of b, there is some a_δ such that $b(\delta) \in \arg\max_{a' \in [a^N, a_\delta]} d(a')$. This implies that $d(b(\delta)) \geq d(a)$ for all $a \in [a^N, b(\delta)]$, and in particular for $a = b(\delta')$. Thus, we obtain $\delta = d(b(\delta)) \geq d(b(\delta')) = \delta'$, which contradicts our premise $\delta < \delta'$.

$$
\begin{aligned}
&= \lambda e^{-\lambda t} \frac{1}{\dfrac{2k}{1-k}+1} t^{\frac{2k}{1-k}+1} = \left(\frac{\lambda(1-k)}{k+1} t^{-1}\right) t^{\frac{2}{1-k}} e^{-\lambda t} \\
&\geq t^{\frac{2}{1-k}} e^{-\lambda t} = d\bigl(x(t)\bigr) e^{-\lambda t}.
\end{aligned}
$$

The first inequality follows from (i) $x(\tau) = b(\tau^{\frac{2}{1-k}}) \in [a^N, \bar{a}]$ (because the range of function b is $[a^N, \bar{a}]$) and (ii) $\pi(a) - \pi^N \geq (d(a))^k$ for all $a \in [a^N, \bar{a}]$ (as we have shown at the beginning of the proof). The last inequality follows from $t \leq \hat{\epsilon} \leq \frac{\lambda(1-k)}{k+1}$ (by the definition of $\hat{\epsilon}$).

Next, consider the case $t > \hat{\epsilon}$. We have

$$
\begin{aligned}
\int_0^t \bigl(\pi\bigl(x(\tau)\bigr) - \pi^N\bigr)\lambda e^{-\lambda\tau}\, d\tau &\geq \int_0^{\hat{\epsilon}} \bigl(\pi\bigl(x(\tau)\bigr) - \pi^N\bigr)\lambda e^{-\lambda\tau}\, d\tau \\
&\geq \hat{\epsilon}^{\frac{2}{1-k}} e^{-\lambda\hat{\epsilon}} \geq \hat{\epsilon}^{\frac{2}{1-k}} e^{-\lambda t} = d\bigl(x(t)\bigr) e^{-\lambda t}.
\end{aligned}
$$

The first inequality follows from $\pi(x(\tau)) - \pi^N = \pi(b(\hat{\epsilon}^{\frac{2}{1-k}})) - \pi^N \geq 0$ for all $\tau > \hat{\epsilon}$ because (i) the range of b is $[a^N, \overline{a}]$ and (ii) $\pi(a) - \pi^N \geq 0$ for all $a \in [a^N, \overline{a}]$ by Assumption (*)-3. The second inequality follows from the third inequality for the case of $t \leq \hat{\epsilon}$.

Hence, the non-trivial plan (32) satisfies the incentive constraint (33) for all $t \in [0, T]$. This completes the proof. *Q.E.D.*

C.1.2. *Proof of Theorem 4*

First, we introduce notation to define general strategies in the revision game. Let a history h_t at time $t \in [0, T)$ be a description of the current remaining time, the action profile at time $-T$, and a sequence of pairs of the remaining time and the action profile chosen at the past opportunities, as follows:

$$
h_t = \bigl(t, a^T, \bigl(t^k, a^k\bigr)_{k=1}^n\bigr)
$$

for some nonnegative integer n, where $t^k \in (t, T)$ for any k, and $t^{k-1} > t^k$ for any integer k no less than 2 and no more than n. Note that the description of h_t does not include the information about the action profile taken at time $-t$. Let H_t be the set of all such histories. The set of histories at time $-T$, H_T, is a singleton set consisting of a null history. Let $H = \bigcup_{t\in[0,T]} H_t$. Player i's (pure) strategy is defined as a mapping $\sigma_i : H \to A_i$. We define (pure-strategy) SPE in the standard manner.

PROOF: First, let us introduce a few notations. Denote by "$\tilde{h}_t = h_t$" the event under which the history at time $-t$ is $h_t \in H_t$. We also denote by $h_t^+ = (h_t, a)$ a pair of a history at $-t$ and an action profile taken at $-t$. Denote by "$\tilde{h}_t^+ = (h_t, a)$" the event under which the history at time $-t$ is $h_t \in H_t$ and players take the action profile a.

Now, fix a SPE σ. Step 1 shows that, if players play an action profile a' under some history at some time $-t$ under σ, then $\pi_i(a') = \pi_i(a^N)$ holds for each player i. Then we show in Step 2 that only a^N can be played under any history under σ.

Step 1: Only the Nash payoff is possible under σ. Suppose that at time $-t \in [-T, 0]$, it is the case that for every time $-s > -t$, if an action profile $a' \in A$ is taken in some SPE, then $\pi_i(a') = \pi_i(a^N)$ holds. We will show that for any i, $\pi_i(\sigma(h_{t+\epsilon})) = \pi_i(a^N)$ for any $h_{t+\epsilon} \in H_{t+\epsilon}$ if $\epsilon > 0$ is sufficiently small.

Step 1-1: Defining $\mathcal{C}$, $\bar{\mathcal{D}}$, and $\underline{\mathcal{D}}$. Fix $\epsilon \geq 0$ and take an arbitrary history $h_{t+\epsilon} \in H_{t+\epsilon}$. Let $\mathcal{C}$ be the continuation payoff from following σ_i at history $h_{t+\epsilon}$, and $\bar{\mathcal{D}}$ be the supremum continuation payoff from a deviation. Note that $\mathcal{C}$ can be written as follows:

$$\mathcal{C} = \underbrace{\left(1-e^{-\lambda t}\right)}_{\text{Prob of at least one arrival in }(-t,0]} \times \pi_i^N + \underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in }(-t,0]} \times \Bigg[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in }(-t-\epsilon,t]} \pi_i\big(\sigma(h_{t+\epsilon})\big) + \int_0^\epsilon \underbrace{\mathbb{E}_\sigma\big[\pi_i\big(\sigma(h_{t+s})\big) | \tilde{h}^+_{t+\epsilon} = \big(h_{t+\epsilon}, \sigma(h_{t+\epsilon})\big)\big]}_{\text{Payoff when the final arrival is at }-t-s} \lambda e^{-\lambda s}\, ds \Bigg].$$

The incentive compatibility condition for player i at history $h_{t+\epsilon}$ can be expressed as

$$\bar{\mathcal{D}} \leq \mathcal{C}.$$

In Step 1-2, we show that this incentive compatibility condition implies

$$\underline{\mathcal{D}} \leq \mathcal{C},$$

where

$$\underline{\mathcal{D}} := \underbrace{\left(1-e^{-\lambda t}\right)}_{\text{Prob of at least one arrival in }(-t,0]} \times \pi_i^N + \underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in }(-t,0]} \times \Bigg[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in }(-t-\epsilon,-t]} \times \big(\pi_i\big(\sigma(h_{t+\epsilon})\big) + d\big(\sigma(h_{t+\epsilon})\big)\big) + \int_0^\epsilon \underbrace{\inf_{a\in A} \mathbb{E}_\sigma\big[\pi_i\big(\sigma(h_{t+s})\big) | \tilde{h}^+_{t+\epsilon} = (h_{t+\epsilon}, a)\big]}_{\text{Infimum payoff when the final arrival is at }-t-s} \lambda e^{-\lambda s}\, ds \Bigg].$$

Step 1-2: Showing $\bar{\mathcal{D}} \leq \mathcal{C} \implies \underline{\mathcal{D}} \leq \mathcal{C}$. To prove that $\bar{\mathcal{D}} \leq \mathcal{C}$ implies $\underline{\mathcal{D}} \leq \mathcal{C}$, it suffices to prove that $\underline{\mathcal{D}} \leq \bar{\mathcal{D}}$. To see why this inequality holds, define i's expected continuation payoff from deviating to $a_i \in A_i$ at history $h_{t+\epsilon}$ and then following σ_i thereafter:

$$\mathcal{D}(a_i) := \underbrace{\left(1-e^{-\lambda t}\right)}_{\text{Prob of at least one arrival in }(-t,0]} \times \pi_i^N + \underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in }(-t,0]} \times \Bigg[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in }(-t-\epsilon,-t]} \times \big(\pi_i\big(a_i, \sigma(h_{t+\epsilon})\big)\big) + \int_0^\epsilon \underbrace{\mathbb{E}_\sigma\big[\pi_i\big(\sigma(h_{t+s})\big) | \tilde{h}^+_{t+\epsilon} = \big(h_{t+\epsilon}, \big(a_i, \sigma_{-i}(h_{t+\epsilon})\big)\big)\big]}_{\text{Payoff when the final arrival is at }-t-s} \lambda e^{-\lambda s}\, ds \Bigg].$$

By the definition of $d_i(\cdot)$, there must exist a sequence $\{a_i^k\}_{k=1}^\infty$ such that $\pi_i(a_i^k, \sigma_{-i}(h_{t+\epsilon})) \to \pi_i(\sigma(h_{t+\epsilon})) + d_i(\sigma(h_{t+\epsilon}))$ as $k \to \infty$. Hence, for any $\xi > 0$, there exists $K_\xi < \infty$ such that, for all $k > K_\xi$, $\pi_i(a_i^k, \sigma_{-i}(h_{t+\epsilon})) \geq \pi_i(\sigma(h_{t+\epsilon})) + d_i(\sigma(h_{t+\epsilon})) - \xi$.

Therefore, for any $\xi > 0$, $k > K_\xi$, we have

$$\begin{aligned}\mathcal{D}(a_i^k) \geq\ & \underbrace{(1-e^{-\lambda t})}_{\text{Prob of at least one arrival in } (-t,0]} \times \pi_i^N \\ & + \underbrace{e^{-\lambda t}}_{\text{Prob of no arrival in } (-t,0]} \times \Bigl[\underbrace{e^{-\lambda \epsilon}}_{\text{Prob of no arrival in } (-t-\epsilon,-t]} \times \bigl(\pi_i(\sigma(h_{t+\epsilon})) + d_i(\sigma(h_{t+\epsilon})) - \xi\bigr) \\ & + \int_0^\epsilon \underbrace{\mathbb{E}_\sigma\bigl[\pi_i(\sigma(h_{t+s}))|\tilde{h}_{t+\epsilon}^+ = (h_{t+\epsilon}, (a_i^k, \sigma_{-i}(h_{t+\epsilon})))\bigr]}_{\text{Payoff when the final arrival is at } -t-s} \lambda e^{-\lambda s}\, ds \Bigr]. \end{aligned} \tag{34}$$

By the definition of $\underline{\mathcal{D}}$, the right-hand side of (34) is no less than $\underline{\mathcal{D}} - e^{-\lambda(t+\epsilon)}\xi$. Hence, we have

$$\underline{\mathcal{D}} - e^{-\lambda(t+\epsilon)}\xi \leq \mathcal{D}(a_i^k) \tag{35}$$

for any $\xi > 0$ and $k > K_\xi$.

Note also that, for any k, deviating to a_i^k and following σ_i thereafter is a feasible deviation. Thus, for any k, we have

$$\mathcal{D}(a_i^k) \leq \bar{\mathcal{D}}. \tag{36}$$

Conditions (35) and (36) imply

$$\underline{\mathcal{D}} - e^{-\lambda(t+\epsilon)}\xi \leq \bar{\mathcal{D}}$$

for any $\xi > 0$. Thus, we obtain

$$\underline{\mathcal{D}} \leq \bar{\mathcal{D}}.$$

Hence, the incentive compatibility condition ($\bar{\mathcal{D}} \leq \mathcal{C}$) implies $\underline{\mathcal{D}} \leq \mathcal{C}$.

Step 1-3: Bounding $|\pi_i(\sigma(h_{t+\epsilon})) - \pi^N|$.

Now, manipulating $\underline{\mathcal{D}} \leq \mathcal{C}$, we obtain

$$\begin{aligned} d_i(a(h_{t+\epsilon}, \sigma)) \leq e^{\lambda\epsilon} \int_0^\epsilon \Bigl(& \mathbb{E}_\sigma\bigl[\pi_i(\sigma(h_{t+s}))|\tilde{h}_{t+\epsilon} = h_{t+\epsilon}\bigr] \\ & - \inf_{a \in A} \mathbb{E}_\sigma\bigl[\pi_i(\sigma(h_{t+s}))|\tilde{h}_{t+\epsilon}^+ = (h_{t+\epsilon}, a)\bigr]\Bigr) \lambda e^{-\lambda s}\, ds. \end{aligned} \tag{37}$$

If $|\pi_i(\sigma(h_{t+s})) - \pi^N| \leq M$ for all $s \in [0, \epsilon]$, (37) implies

$$d_i(\sigma(h_{t+\epsilon})) \leq e^{\lambda\epsilon} \int_0^\epsilon 2M\lambda e^{-\lambda s}\, ds,$$

where the right-hand side is no more than $2M\lambda\epsilon e^{\lambda\epsilon}$. This and condition (14) imply

$$\bigl|\pi_i(\sigma(h_{t+\epsilon})) - \pi^N\bigr| \leq \frac{2\lambda}{m}\epsilon e^{\lambda\epsilon} M,$$

where

$$m = \inf_{a \in A \setminus \{a^N\}} \frac{d_i(a)}{|\pi_i(a) - \pi_i^N|} > 0$$

is a positive number implied by condition (14). The same argument can be used to show that, for any $s \in [0, \epsilon]$, for any $h_{t+s} \in H_{t+s}$,

$$\left|\pi_i\big(\sigma(h_{t+s})\big) - \pi^N\right| \le \frac{2\lambda}{m} s e^{\lambda s} M,$$

where the right-hand side is no more than $\frac{2\lambda}{m}\epsilon e^{\lambda\epsilon}M$. Hence, we conclude that if $|\pi_i(\sigma(h_{t+s})) - \pi^N| \le M$ for all $s \in [0, \epsilon]$, then $|\pi_i(\sigma(h_{t+s})) - \pi^N| \le \frac{2\lambda}{m}\epsilon e^{\lambda\epsilon}M$ for all $s \in [0, \epsilon]$.

Since $|\pi_i(\sigma(h_{t+s})) - \pi^N| \le \bar{\pi}_i - \underline{\pi}_i$ for all $s \in [0, \epsilon]$, this implies that for any positive integer n, we have

$$\left|\pi_i\big(\sigma(h_{t+\epsilon})\big) - \pi^N\right| \le \left(\frac{2\lambda}{m}\epsilon e^{\lambda\epsilon}\right)^n (\bar{\pi}_i - \underline{\pi}_i).$$

Notice that there exists $\bar{\epsilon} > 0$ such that, for any $\epsilon \in (0, \bar{\epsilon})$, $\frac{2\lambda}{m}\epsilon e^{\lambda} < 1$ holds. Hence, there exists $\bar{\epsilon} > 0$ such that, for any $\epsilon \in (0, \bar{\epsilon})$, the only action profile $\sigma(h_{t+\epsilon})$ that satisfies the above equality for all n is $\pi_i(\sigma(h_{t+\epsilon})) = \pi_i(a^N)$.

Hence, for every time $-t \in [-T, 0]$, in any SPE, if an action profile a' is taken under some history at $-t$, then for each player i, we have $\pi_i(a') = \pi_i(a^N)$.

Step 2: Only Nash action is possible under σ. Now, suppose that under σ, there exists some t and $h_t \in H_t$ such that $\sigma(h_t) \neq a^N$. Then, player i's incentive compatibility condition at h_t can be written as follows:

$$e^{-\lambda t}\Big(\pi_i\big(\sigma(h_t)\big) + d_i\big(\sigma(h_t)\big)\Big) + \big(1 - e^{-\lambda t}\big)\pi_i^N \le \pi^N,$$

which is equivalent to $d_i(\sigma(h_t)) \le 0$. However, since a^N is a unique Nash equilibrium and $\sigma(h_t) \neq a^N$, there exists i such that $d_i(\sigma(h_t)) > 0$. This is a contradiction. Hence, we conclude that for any $t \in [0, T]$, for any $h_t \in H_t$, we have $\sigma(h_t) = a^N$. *Q.E.D.*

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