

SUPPLEMENT TO “MECHANISM DESIGN WITH AFTERMARKETS: CUTOFF MECHANISMS”

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PIOTR DWORCZAK

Department of Economics, Northwestern University

APPENDIX A: IMPLEMENTABILITY FOR ALL DISTRIBUTIONS AND AFTERMARKETS

IN THIS APPENDIX, I extend and discuss the characterization of cutoff rules from Theorem 1. In particular, I show that cutoff rules are uniquely pinned down by requiring implementability for a sufficiently large set of aftermarkets and prior distributions.

I let \mathcal{A} and \mathcal{F} denote an abstract set of possible aftermarkets and prior distributions, respectively. For example, \mathcal{A} may include various versions of a post-mechanism game differing in parameters of the bargaining protocol and characteristics of third-party players, or different equilibria of the same aftermarket game. I look at the case $N = 1$ to simplify exposition (hence drop the subscripts).

DEFINITION 7—Flexibility: A mechanism frame (x, π) is flexible with respect to $(\mathcal{F}, \mathcal{A})$ if (x, π) is implementable for any prior distribution $f \in \mathcal{F}$ and any aftermarket $A \in \mathcal{A}$.

DEFINITION 8—Richness: The pair $(\mathcal{F}, \mathcal{A})$ satisfies *Richness* if, for any mechanism frame (x, π) and $\theta > \hat{\theta}$, there exists a prior distribution $f \in \mathcal{F}$ and an aftermarket $A \in \mathcal{A}$ such that

$$\pi(s|\theta)x(\theta) < \pi(s|\hat{\theta})x(\hat{\theta}) \implies u(\theta; f^s) > u(\hat{\theta}; f^s), \quad (\text{A.1})$$

$$\pi(s|\theta)x(\theta) > \pi(s|\hat{\theta})x(\hat{\theta}) \implies u(\theta; f^s) = u(\hat{\theta}; f^s). \quad (\text{A.2})$$

PROPOSITION 5: *Suppose that a mechanism frame (x, π) is flexible with respect to $(\mathcal{F}, \mathcal{A})$ that satisfies the Richness condition. Then, (x, π) is a cutoff rule.*

It is easy to observe that the set of all prior distributions and all monotone aftermarkets trivially satisfies the Richness condition. Thus, Proposition 5 implies one direction of Theorem 1.

PROOF: The proof is identical to the proof of (the converse part of) Theorem 1. The Richness condition is exactly enough to guarantee existence of a prior f and an aftermarket A that make equation (3.4) hold. *Q.E.D.*

The proof of the proposition is simple because the Richness condition is tailored to the result. The difficulty often lies in proving that a certain set of priors and aftermarkets satisfy the Richness condition. I go through such an example next. The example illustrates the fact that the set \mathcal{A} need not be very large if \mathcal{F} is large.

Piotr Dworczak: <https://sites.northwestern.edu/dworczak/>

EXAMPLE 9—Resale: Consider the resale aftermarket from Example 1 assuming for now that $\lambda = 1$ (the aftermarket happens with probability 1), and the third party has a constant value v larger than the highest type of the agent and makes a take-it-or-leave-it offer with indifference broken in the agent’s favor (note that $|\mathcal{A}| = 1$). Let \mathcal{F} be the set of all type distributions with binary support.

I prove that $(\mathcal{F}, \mathcal{A})$ satisfies Richness. Fix any mechanism frame (x, π) and $\theta > \hat{\theta}$. Consider a distribution with pmf f supported on the set $\{\hat{\theta}, \theta\}$. The optimal price offered by the third party is either $\hat{\theta}$ or θ . Following a signal realization s , the third party Bayes-updates beliefs (see equation (2.1)), and offers price $\hat{\theta}$ if

$$(\theta - \hat{\theta})\pi(s|\hat{\theta})x(\hat{\theta})f(\hat{\theta}) > (v - \theta)\pi(s|\theta)x(\theta)f(\theta). \quad (\text{A.3})$$

Price θ is uniquely optimal following signal realizations s under which the opposite strict inequality holds. Define f as the unique pmf supported on $\{\hat{\theta}, \theta\}$ such that $f(\hat{\theta})/f(\theta) = (v - \theta)/(\theta - \hat{\theta})$. That is, in the absence of additional information, the third party is indifferent between offering price θ and $\hat{\theta}$.

Suppose that the premise of condition (A.1) holds: $\pi(s|\theta)x(\theta) < \pi(s|\hat{\theta})x(\hat{\theta})$. Then, by choice of f , condition (A.3) must hold, and therefore the price $\hat{\theta}$ is uniquely optimal for the third party. It follows that type θ rejects the offer and receives $u(\theta; f^s) = \theta$, while type $\hat{\theta}$ accepts the offer and receives $u(\hat{\theta}; f^s) = \hat{\theta}$. Thus, (A.1) holds. Now suppose that the premise of condition (A.2) holds: $\pi(s|\theta)x(\theta) > \pi(s|\hat{\theta})x(\hat{\theta})$. Then, price θ is uniquely optimal for the third party, and both types resell, getting utility $\theta = u(\theta; f^s) = u(\hat{\theta}; f^s)$. Thus, (A.2) holds.

Finally, notice that when $(\mathcal{F}, \mathcal{A})$ satisfies Richness, then all supersets of \mathcal{F} and \mathcal{A} also satisfy Richness. This means that the set of aftermarkets described by Example 1 with no restrictions on parameters satisfies Richness.

The example serves as an illustration for the intuition behind the Richness condition. Aftermarkets differ in the sensitivity of induced payoffs to the information revealed by the mechanism. The Richness condition requires that, among possible priors and aftermarkets, we can always find some that make payoffs particularly sensitive to information. The premise in condition (A.1) can be interpreted as “bad news” about the agent’s type—after observing a signal realization s that satisfies the left-hand side inequality, the posterior probability of the lower type $\hat{\theta}$ increases. Under some prior distribution f and aftermarket \mathcal{A} , the expected payoff of the higher type θ has to strictly exceed the expected payoff of the lower type $\hat{\theta}$ following “bad news.” On the other hand, when the mechanism sends “good news” (condition A.2), the expected payoffs of the two types should be equal. In Example 9, for any two types θ and $\hat{\theta}$, there exists a prior f under which the third party is indifferent between a high and a low price in the aftermarket. Therefore, any “bad news” (a signal realization that is more likely under the low type) will tilt the price to be low, leading to a gap between the payoffs of the high and the low type. On the other hand, any “good news” will tilt the price to be high, in which case both types resell and enjoy the same payoff.

APPENDIX B: MISSING PROOFS

B.1. Proof of Lemma 1

Consider the problem of maximizing

$$\sum_{\theta \in \Theta} \sum_{s \in \mathcal{S}} V(\theta; f^s) \pi(s|\theta) x(\theta) f(\theta)$$

over π subject to (x, π) being a cutoff rule. By definition of a cutoff rule, there exists a function $\gamma : C \rightarrow \Delta(\mathcal{S})$ such that $\pi(s|\theta) x(\theta) = \sum_{c \leq \theta} \gamma(s|c) \Delta x(c)$. Thus, the problem becomes

$$\begin{aligned} & \max_{\gamma} \sum_{\theta \in \Theta} \sum_{s \in \mathcal{S}} V(\theta; f^s) \sum_{c \leq \theta} \gamma(s|c) \Delta x(c) f(\theta) \\ &= \max_{\gamma} \sum_{s \in \mathcal{S}} \underbrace{\left(\sum_c \gamma(s|c) \Delta x(c) \right)}_{s_s} \sum_{\theta \in \Theta} V(\theta; f^s) \underbrace{\left(\frac{\sum_{c \leq \theta} \gamma(s|c) \Delta x(c)}{\sum_c \gamma(s|c) \Delta x(c)} \right)}_{y^s(\theta)} f(\theta). \quad (\text{B.1}) \end{aligned}$$

In the above expression, s_s is the unconditional probability of sending signal s , and the remaining expression is equal to $\mathcal{V}(y^s)$, as defined in (4.2), where y^s is the cdf of the cutoff conditional on signal s . Thus, the objective function can be written as $\mathbb{E}_{s \sim c} \mathcal{V}(y^s)$. To confirm that \mathcal{V} depends solely on the conditional distribution of the cutoff, note that $f^s = f^{y^s}$ by (2.1) and (4.1), so that

$$\mathcal{V}(y^s) = \mathbb{E}_{\tilde{c} \sim y^s} \left[\sum_{\theta \in \Theta} V(\theta; f^{y^s}) \mathbf{1}_{\{\theta \geq \tilde{c}\}} f(\theta) \right].$$

Thus, the problem is mathematically equivalent to the Bayesian persuasion problem of [Kamenica and Gentzkow \(2011\)](#). Instead of optimizing over distributions ς of signals, we can optimize over distributions of distributions $\varrho \in \Delta(\Delta(C))$ subject to a Bayes-plausibility constraint. This yields equations (4.3) and (4.4).

B.2. Proof of Theorem 3

Given (x, π) , its *reduced form* under distribution f , denoted (x^f, π^f) , is defined by

$$\begin{aligned} x_i^f(\theta_i) &= \sum_{\theta_{-i} \in \Theta_{-i}} x_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}), \\ \pi_i^f(s|\theta_i) x_i^f(\theta_i) &= \sum_{\theta_{-i} \in \Theta_{-i}} \pi_i(s|\theta_i, \theta_{-i}) x_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}), \end{aligned}$$

for all $s \in \mathcal{S}_i$, $\theta_i \in \Theta_i$, and $i \in \mathcal{N}$.

The designer's and the agents' expected payoffs, as well as the posterior beliefs f_i^s , depend only on the reduced form of a mechanism (see equations (2.1) and (2.2)). However, the definition of a cutoff rule relies on properties of i 's allocation and disclosure rule that hold conditional on any given profile of other agents' reports θ_{-i} . To work with reduced forms, I establish which reduced forms correspond to cutoff rules.

LEMMA 2: A pair $(\bar{x}, \bar{\pi})$, where $\bar{x}_i : \Theta_i \rightarrow [0, 1]$ and $\bar{\pi}_i : \Theta_i \rightarrow \Delta(\mathcal{S}_i)$, for all i , is a reduced form of a cutoff rule under prior distribution f if and only if:

1. the interim allocation rule $\bar{x}_i(\theta_i)$ is non-decreasing in θ_i , for all $i \in \mathcal{N}$;
2. the interim signal function $\bar{\pi}_i$ can be represented as

$$\bar{\pi}_i(s|\theta_i)\bar{x}_i(\theta_i) = \sum_{c \leq \theta_i} \gamma_i(s|c)\Delta\bar{x}_i(c), \quad (\text{B.2})$$

for some signal function $\gamma_i : C_i \rightarrow \Delta(\mathcal{S}_i)$, for all $i \in \mathcal{N}$, θ_i , and $s \in \mathcal{S}_i$;

3. interim expected allocation rules are jointly feasible under f :

$$\sum_{i \in \mathcal{N}} \sum_{\theta_i > \tau_i} \bar{x}_i(\theta_i)f_i(\theta_i) \leq 1 - \prod_{i \in \mathcal{N}} F_i(\tau_i), \quad \forall \tau \in \mathbb{R}^{\mathcal{N}}. \quad (\text{M-B})$$

PROOF OF LEMMA 2: “Only if”: In this part of the proof, I show that a reduced form (x^f, π^f) of any cutoff rule (x, π) satisfies conditions 1–3. Condition 1 holds because $x_i(\theta_i, \theta_{-i})$ is non-decreasing in θ_i for every θ_{-i} , and thus also when expectation is taken with respect to θ_{-i} . Condition 3 must hold whenever x is feasible, $\sum_{i \in \mathcal{N}} x_i(\theta) \leq 1$ for all θ ; indeed [Border \(2007\)](#) (Theorem 3) and [Mierendorff \(2011\)](#) (Theorems 2 and 3) showed that the interim expected allocation rules must satisfy the generalized (asymmetric) Matthews–Border (M-B) in this case. Finally, to show that condition 2 holds as well, notice that since $\pi_i(s|\theta_i, \theta_{-i})x_i(\theta_i, \theta_{-i})$ is non-decreasing in θ_i for each θ_{-i} (by definition of cutoff rules), $\pi_i^f(s|\theta_i)x_i^f(\theta_i)$ is also non-decreasing in θ_i , for any $s \in \mathcal{S}_i$. A reduced form can be formally treated as a single-agent mechanism since x_i^f and π_i^f are mappings from individual type spaces Θ_i into allocations and signals, respectively. It follows from Proposition 1 that (x_i^f, π_i^f) , viewed as a single-agent mechanism, is a cutoff rule, and in particular satisfies condition 2 of Lemma 2.

“If”: Given a reduced form $(\bar{x}, \bar{\pi})$ satisfying conditions 1–3, I will prove existence of a cutoff rule (x, π) such that $(x^f, \pi^f) = (\bar{x}, \bar{\pi})$. By Theorems 2 and 3 in [Mierendorff \(2011\)](#), condition (M-B) (along with the fact that each \bar{x}_i is monotone) implies that there exists a joint allocation rule x such that $x^f = \bar{x}$. Define $\pi : \Theta \rightarrow \prod_{i \in \mathcal{N}} \Delta(\mathcal{S}_i)$ by $\pi_i(s|\theta_i, \theta_{-i}) = \bar{\pi}_i(s|\theta_i)$, for all $s \in \mathcal{S}_i$, $\theta_i \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, $i \in \mathcal{N}$. Then, (x, π) is a well-defined mechanism frame such that $(x^f, \pi^f) = (\bar{x}, \bar{\pi})$. The goal is to modify (x, π) in order to obtain a cutoff rule (x^*, π^*) that induces the same reduced form. Intuitively, this modification is closely analogous to how a Bayesian IC mechanism can be modified to produce a payoff-equivalent dominant-strategy IC mechanism, in an approach pioneered by [Manelli and Vincent \(2010\)](#) and developed further by [Gershkov, Goeree, Kushnir, Moldovanu, and Shi \(2013\)](#).¹

To apply the techniques of [Gershkov et al. \(2013\)](#), I introduce the following notation. Let $\mathcal{K} = (\mathcal{N} \cup \{0\}) \times (\bigcup_i \mathcal{S}_i)$ be the set of social alternatives, where an outcome $k = (i, s)$ is interpreted as player i getting the object ($i = 0$ denotes the mechanism designer) and signal s being sent. An allocation rule in this setting is defined as an element of the set $\mathcal{Y} = \{y^{i,s} : y^{i,s}(\theta) \geq 0, \forall (i, s) \in \mathcal{K}, \sum_{i \in \mathcal{N}, s \in \mathcal{S}_i} y^{i,s}(\theta) \leq 1, \forall \theta\}$, where $\{y^{i,s}\}$ is a shorthand notation for $\{y^{i,s} : i \in \mathcal{N} \cup \{0\}, s \in \mathcal{S}_i\}$. That is, $y^{i,s}(\theta)$ is the probability of implementing outcome (i, s) conditional on type profile θ . Define an allocation rule

¹I use the proof technique of [Gershkov et al. \(2013\)](#) but not their main result which is stated in terms of interim expected utilities: Because in my problem the allocation rule is monotone not only for every $i \in \mathcal{N}$ but also for all $s \in \mathcal{S}_i$, I am able to show the equivalence in terms of interim expected allocations.

$x^{i,s}(\boldsymbol{\theta}) = \pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i})$, for all $i \in \mathcal{N}$, and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, as the probability that outcome $\{i, s\}$ is implemented in the mechanism frame $(\mathbf{x}, \boldsymbol{\pi})$ (where x^0 is defined as the residual probability). Clearly, $\{x^{i,s}\} \in \mathcal{Y}$. The following result follows directly from Lemma 3 in Gershkov et al. (2013).

LEMMA 3—Gershkov et al. (2013): *Suppose that for allocation $\{x^{i,s}\}$, $\sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} x^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i})$ is non-decreasing in θ_i , for all $i \in \mathcal{N}$, $s \in \mathcal{S}_i$. Define $\{y^{i,s}\}$ as the solution to the problem:*

$$\min_{\{y^{i,s}\} \in \mathcal{D}} \sum_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}, s \in \mathcal{S}_i} (y^{i,s}(\boldsymbol{\theta}))^2, \quad \text{where}$$

$$\mathcal{D} = \left\{ \{y^{i,s}\} \in \mathcal{Y} : \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} y^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} x^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}), \forall i, \theta_i, s \right\}.$$

Then, $y^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in θ_i , for all $\boldsymbol{\theta}_{-i}$, and all $i \in \mathcal{N}$, $s \in \mathcal{S}_i$.

The allocation function $\{x^{i,s}\}$ satisfies the assumption of Lemma 3 because

$$\sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} x^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} \bar{\pi}_i(s|\theta_i)x_i(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}) = \bar{\pi}_i(s|\theta_i)\bar{x}_i(\theta_i),$$

and the last expression is non-decreasing in θ_i because $(\bar{\mathbf{x}}, \bar{\boldsymbol{\pi}})$ satisfies condition 2 in Lemma 2 (which implies monotonicity). Given the allocation $\{y^{i,s}\}$ produced from $\{x^{i,s}\}$ by Lemma 3, I now define a mechanism $(\mathbf{x}^*, \boldsymbol{\pi}^*)$ by

$$x_i^*(\boldsymbol{\theta}) = \sum_{s \in \mathcal{S}_i} y^{i,s}(\boldsymbol{\theta}),$$

$$\pi_i^*(s|\boldsymbol{\theta}) = \frac{y^{i,s}(\boldsymbol{\theta})}{x_i^*(\boldsymbol{\theta})},$$

with $\pi_i^*(s|\boldsymbol{\theta})$ defined in an arbitrary way for $\boldsymbol{\theta}$ such that $x_i^*(\boldsymbol{\theta}) = 0$.

To show that $(\mathbf{x}^*, \boldsymbol{\pi}^*)$ is a cutoff rule, it is enough to invoke Proposition 1 found in Section 3—because $\pi_i^*(s|\theta_i, \boldsymbol{\theta}_{-i})x_i^*(\theta_i, \boldsymbol{\theta}_{-i}) \equiv y^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in θ_i , for all $s \in \mathcal{S}_i$ and $\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$, it must be a cutoff rule.

Finally, $(\mathbf{x}^{*f}, \boldsymbol{\pi}^{*f}) = (\bar{\mathbf{x}}, \bar{\boldsymbol{\pi}})$ follows from the fact that $\{y^{i,s}\} \in \mathcal{D}$:

$$\begin{aligned} & \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} \pi_i^*(s|\theta_i, \boldsymbol{\theta}_{-i})x_i^*(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}) \\ &= \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} y^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}) \\ &= \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} x_i^{i,s}(\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}) = \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} x_i(\theta_i, \boldsymbol{\theta}_{-i})\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})f_{-i}(\boldsymbol{\theta}_{-i}), \end{aligned}$$

for all i, s , and θ_i . The same calculation can be done for \mathbf{x}^* by summing over s . *Q.E.D.*

I call a reduced-form mechanism frame satisfying conditions 1–3 of Lemma 2 a *reduced-form cutoff rule*. By Lemma 2, optimization over cutoff mechanisms can be performed in

the space of reduced-form cutoff mechanisms. For a fixed allocation \mathbf{x} and distribution f , a reduced-form cutoff mechanism for agent i is formally equivalent to a single-agent cutoff mechanism from Section 4.1. Moreover, the disclosure problem for any agent i can be solved independently from the disclosure problems for all other agents $j \neq i$ because ex post there is only one participant in the aftermarket. Thus, we can use the proof of Lemma 1 to establish the following result.

LEMMA 4: *For every non-decreasing allocation rule \mathbf{x} , the problem of maximizing (2.2) over $\boldsymbol{\pi}$ subject to $(\mathbf{x}, \boldsymbol{\pi})$ being a cutoff rule is equivalent to solving, for every $i \in \mathcal{N}$,*

$$\max_{\varrho_i \in \Delta(\Delta(C_i))} \mathbb{E}_{y_i \sim \varrho_i} \mathcal{V}_i(y_i) \quad (\text{B.3})$$

subject to

$$\mathbb{E}_{y_i \sim \varrho_i} y_i(\theta_i) = x_i^f(\theta_i), \quad \forall \theta_i \in \Theta_i. \quad (\text{B.4})$$

Applying Corollary 2 in [Kamenica and Gentzkow \(2011\)](#), I obtain the concave-closure characterization of the optimal payoff.

COROLLARY 4: *The maximal expected payoff to the mechanism designer in the problem (B.3)–(B.4) is equal to*

$$\sum_{i \in \mathcal{N}} \text{co} \mathcal{V}_i(x_i^f) \equiv \sum_{i \in \mathcal{N}} \sup\{\nu : (x_i^f, \nu) \in \text{CH}(\text{graph}(\mathcal{V}_i))\},$$

where $\text{graph}(\mathcal{V}_i) \equiv \{(\hat{x}, \hat{\nu}) \in \mathcal{X}_i \times \mathbb{R} : \hat{\nu} = \mathcal{V}_i(\hat{x})\}$.

Theorem 3 follows directly from Lemma 2, Lemma 4, and Corollary 4.

B.3. Supplementary Material for Example 5

I prove that, in Example 5, it is indeed optimal to let high types trade with maximal probability, that is, $\bar{x}(h) = (2/N)(1 - 1/2^N)$ in the optimal cutoff mechanism. By Theorem 3, and because symmetric mechanisms are without loss of optimality, the optimization problem is given by

$$\begin{aligned} \max_{\bar{x}(l), \bar{x}(h)} \quad & \bar{x}(h) \text{co} \mathcal{V}_1 \left(1 - \frac{\bar{x}(l)}{\bar{x}(h)} \right) \\ \text{s.t.} \quad & \bar{x}(h) \leq (2/N)(1 - 1/2^N), \\ & \bar{x}(l) + \bar{x}(h) \leq 2/N, \quad \bar{x}(l) \leq \bar{x}(h). \end{aligned} \quad (\text{B.5})$$

Towards a contradiction, assume that $\bar{x}(h) < (2/N)(1 - 1/2^N)$ in any optimal mechanism. When $\bar{x}(h) < (2/N)(1 - 1/2^N)$, it must be that $\bar{x}(l) + \bar{x}(h) = 2/N$. Indeed, if both inequalities were strict, then the designer could increase the expected surplus by choosing $(1 + \epsilon)\bar{x}$ instead of \bar{x} for some small enough $\epsilon > 0$. Thus, substituting the binding constraint, the optimal mechanism with $\bar{x}(h) < (2/N)(1 - 1/2^N)$ must solve

$$\max_{\bar{x}(h)} \bar{x}(h) \text{co} \mathcal{V}_1 \left(2 - \frac{2}{N\bar{x}(h)} \right) \quad \text{s.t.} \quad \frac{1}{N} \leq \bar{x}(h) \leq \frac{2}{N}(1 - 1/2^N). \quad (\text{B.6})$$

By direct calculation (using the assumption $f(l) = f(h) = 1/2$),

$$\text{co}\mathcal{V}_1(\alpha) = \frac{1}{2} \begin{cases} \frac{\alpha}{\alpha^*} v(2 - \alpha^*) + \left(1 - \frac{\alpha}{\alpha^*}\right)(v + h), & \alpha < \alpha^*, \\ v(2 - \alpha), & \alpha \geq \alpha^*, \end{cases}$$

and thus,

$$x\text{co}\mathcal{V}_1\left(2 - \frac{2}{Nx}\right) = \begin{cases} x(v - h) \frac{2 - \alpha^*}{2\alpha^*} - \frac{1}{N\alpha^*}(v(1 - \alpha^*) - h), & x < \frac{2}{N} \frac{1}{2 - \alpha^*}, \\ \frac{v}{N}, & x \geq \frac{2}{N} \frac{1}{2 - \alpha^*}. \end{cases} \quad (\text{B.7})$$

This shows that the objective function in (B.6) is non-decreasing in $\bar{x}(h)$; thus, it is optimal to set $\bar{x}(h)$ to its upper bound $(2/N)(1 - 1/2^N)$, a contradiction.

B.4. Proof of Proposition 2 and Supplementary Material for Section 4.3

In this appendix, I formalize the result stated in Section 4.3 about feasible distributions of posterior beliefs over the winner's type induced by cutoff mechanisms, and prove Proposition 2.

For a fixed (interim expected) allocation rule \bar{x}_i , I call $f_i^{\bar{x}_i}$, defined by (4.1), the *no-communication posterior*:

$$f_i^{\bar{x}_i}(\theta) = \frac{\bar{x}_i(\theta) f_i(\theta)}{\sum_{\tau \in \Theta_i} \bar{x}_i(\tau) f_i(\tau)}, \quad \forall \theta \in \Theta_i.$$

The no-communication posterior is the belief over the type of the winner held by the third party when the interim allocation rule is \bar{x}_i , and the mechanism reveals no information (other than the identity of the winner). Recall that a pmf g *likelihood-ratio dominates* a full-support pmf f (denoted $g \succ^{\text{LR}} f$) if $g(\theta)/f(\theta)$ is non-decreasing.

LEMMA 5: *A distribution of beliefs $\eta_i \in \Delta(\Delta(\Theta_i))$ over i 's type conditional on i being the winner is induced by a cutoff mechanism with interim allocation rule \bar{x}_i if and only if*

$$\bar{f}_i \succ^{\text{LR}} f_i, \quad \forall \bar{f}_i \in \text{supp}(\eta_i) \quad (\text{B.8})$$

and

$$\mathbb{E}_{\bar{f}_i \sim \eta_i} \bar{f}_i(\theta) \equiv \int_{\text{supp}(\eta_i)} \bar{f}_i(\theta) d\eta_i(\bar{f}_i) = f_i^{\bar{x}_i}(\theta), \quad \forall \theta \in \Theta_i. \quad (\text{B.9})$$

Condition (B.9) is the standard Bayes-plausibility constraint, except that posterior beliefs must average out to the no-communication posterior, instead of the prior. This is because the distribution of beliefs is conditional on agent i being the winner. Condition (B.8) is an additional constraint on posterior belief—each posterior must LR dominate the prior.

PROOF OF LEMMA 5: Because the lemma is stated for some fixed i , I drop the subscript i to simplify notation. I first show that every ex ante (unconditional) distribution $\varrho \in \Delta(\Delta(C))$ of beliefs over the cutoff for some agent i that is feasible under allocation \bar{x} defines a posterior (conditional) distribution $\eta \in \Delta(\Delta(\Theta))$ of beliefs over i 's type conditional on i being the winner that satisfies conditions (B.8)–(B.9).

Because ϱ is a feasible distribution of beliefs over the cutoff, it satisfies the Bayes-plausibility condition (see equations (4.4) and (B.4)) which states that

$$\mathbb{E}_{y \sim \varrho} y(\theta) = \bar{x}(\theta), \quad \forall \theta \in \Theta. \quad (\text{B.10})$$

For every $y \in \text{supp}(\varrho)$, let f^y , defined by (4.1), be the corresponding posterior belief over the type of the winner. Each f^y satisfies condition (B.8) because y is a non-decreasing function. Given the ex ante distribution ϱ for agent i , define the posterior distribution $\bar{\varrho}$ conditional on i being the winner:

$$\bar{\varrho}(\mathcal{G}) = \frac{\int_{\mathcal{G}} \left(\sum_{\theta} y(\theta) f(\theta) \right) d\varrho(y)}{\int_{\text{supp}(\varrho)} \left(\sum_{\theta} y(\theta) f(\theta) \right) d\varrho(y)} \quad \text{for any measurable } \mathcal{G} \subseteq \Delta(\Delta(C)). \quad (\text{B.11})$$

Conditional on i becoming the winner, there is higher probability that the cutoff for i was drawn from a distribution that puts relatively more mass on low cutoff realizations. That is why the posterior distribution $\bar{\varrho}$ puts more weight on distributions y that allocate the good with higher probability. Define the corresponding posterior distribution $\eta \in \Delta(\Delta(\Theta))$ of beliefs over the type of the winner by

$$\eta(\mathcal{F}) = \bar{\varrho}(\{y \in \Delta(\Delta(C)) : f^y \in \mathcal{F}\}) \quad \text{for any measurable } \mathcal{F} \subseteq \Delta(\Delta(\Theta)).$$

To show that condition (B.9) holds, note that because ϱ is a feasible ex ante distribution, it satisfies condition (B.10), and hence

$$\int_{\text{supp}(\varrho)} \left(\sum_{\theta} \hat{y}(\theta) f(\theta) \right) d\varrho(\hat{y}) = \sum_{\theta} \left(\int_{\text{supp}(\varrho)} \hat{y}(\theta) d\varrho(\hat{y}) \right) f(\theta) = \sum_{\theta} \bar{x}(\theta) f(\theta).$$

Then, we have

$$\begin{aligned} \int_{\text{supp}(\eta)} \bar{f}(\theta) d\eta(\bar{f}) &= \int_{\text{supp}(\varrho)} f^y(\theta) d\bar{\varrho}(y) \\ &= \int_{\text{supp}(\varrho)} \frac{y(\theta) f(\theta)}{\sum_{\theta} y(\tau) f(\tau)} \frac{\sum_{\theta} y(\tau) f(\tau)}{\int_{\text{supp}(\varrho)} \left(\sum_{\theta} \hat{y}(\tau) f(\tau) \right) d\varrho(\hat{y})} d\varrho(y) \\ &= \frac{\left(\int_{\text{supp}(\varrho)} y(\theta) d\varrho(y) \right) f(\theta)}{\sum_{\theta} \bar{x}(\tau) f(\tau)} = \frac{\bar{x}(\theta) f(\theta)}{\sum_{\theta} \bar{x}(\tau) f(\tau)} = f^{\bar{x}}(\theta), \end{aligned}$$

which is condition (B.9).

To show the opposite direction, start with a conditional distribution of posterior beliefs over the winner's type $\eta \in \Delta(\Delta(\Theta))$, satisfying conditions (B.8) and (B.9) for a non-decreasing allocation rule \bar{x} . I will define a feasible ex ante (unconditional) distribution of beliefs over the cutoff $\varrho \in \Delta(\Delta(C))$ such that ϱ induces η .

First, for each $\bar{f} \in \text{supp}(\eta)$, define

$$y^{\bar{f}}(\theta) := \left(\bar{x}(\bar{\theta}) \frac{f(\bar{\theta})}{\bar{f}(\bar{\theta})} \right) \frac{\bar{f}(\theta)}{f(\theta)}, \quad \forall \theta \in \Theta, \quad (\text{B.12})$$

where $\bar{\theta} = \max\{\Theta\}$. Because \bar{f} likelihood-ratio dominates f , the function $y^{\bar{f}}(\theta)$ is non-decreasing and bounded above by 1 on Θ . Thus, it defines a non-decreasing allocation rule, and hence also a cdf of the corresponding distribution of the cutoff (after extending it to C). Define a distribution $\varrho \in \Delta(\Delta(C))$ supported on $\{y^{\bar{f}} : \bar{f} \in \text{supp}(\eta)\}$ and defined by

$$\varrho(\{y^{\bar{f}} : \bar{f} \in \mathcal{F}\}) = \frac{\int_{\mathcal{F}} \bar{f}(\bar{\theta}) d\eta(\bar{f})}{\int_{\text{supp}(\eta)} \bar{f}(\bar{\theta}) d\eta(\bar{f})} \quad \text{for any measurable } \mathcal{F} \subseteq \Delta(\Delta(\Theta)). \quad (\text{B.13})$$

With this, I have to verify that ϱ is feasible, that is, it satisfies (B.10), and induces η . We have

$$\begin{aligned} \int_{\text{supp}(\varrho)} y^{\bar{f}}(\theta) d\varrho(y^{\bar{f}}) &= \int_{\text{supp}(\eta)} \left(\bar{x}(\bar{\theta}) \frac{f(\bar{\theta})}{\bar{f}(\bar{\theta})} \right) \frac{\bar{f}(\theta)}{f(\theta)} \frac{\bar{f}(\bar{\theta})}{\int_{\text{supp}(\eta)} \bar{f}(\bar{\theta}) d\eta(\bar{f})} d\eta(\bar{f}) \\ &= \int_{\text{supp}(\eta)} \frac{\bar{f}(\theta)}{f(\theta)} \left(\sum_{\tau \in \Theta} \bar{x}(\tau) f(\tau) \right) d\eta(\bar{f}) \\ &= f^{\bar{x}}(\theta) \frac{1}{f(\theta)} \left(\sum_{\tau \in \Theta} \bar{x}(\tau) f(\tau) \right) = \bar{x}(\theta), \end{aligned} \quad (\text{B.14})$$

where I have used condition (B.9) twice.

To show that ϱ induces η , note that $f^{y^{\bar{f}}} = \bar{f}$. Moreover, using (B.11) and (B.13), the posterior distribution (conditional on the agent being the winner) over $y^{\bar{f}}$ is given by, for any measurable $\mathcal{F} \in \Delta(\Delta(\Theta))$,

$$\bar{\varrho}(\{y^{\bar{f}} : \bar{f} \in \mathcal{F}\}) = \frac{\int_{\mathcal{F}} \left(\sum_{\theta} y^{\bar{f}}(\theta) f(\theta) \right) \bar{f}(\bar{\theta}) d\eta(\bar{f})}{\int_{\text{supp}(\eta)} \left(\sum_{\theta} y^{\bar{f}}(\theta) f(\theta) \right) \bar{f}(\bar{\theta}) d\eta(\bar{f})} = \int_{\mathcal{F}} d\eta(\bar{f}) = \eta(\mathcal{F}),$$

which shows that ϱ induces the posterior distribution η over the winner's type. *Q.E.D.*

B.4.1. Proof of Proposition 2

The proof follows from Lemma 4 (see Appendix B.2) and Lemma 5. Fixing an agent i , I drop the subscripts i to simplify notation. Starting from the objective function (B.3),

interim allocation rule \bar{x} , and a feasible ex ante distribution ϱ of beliefs over i 's cutoff,

$$\begin{aligned} \mathbb{E}_{y \sim \varrho} \mathcal{V}(y) &\stackrel{(1)}{=} \int_{\text{supp}(\eta)} \mathcal{V}(y^{\bar{f}}) \frac{\bar{f}(\bar{\theta})}{\int_{\text{supp}(\eta)} \hat{f}(\bar{\theta}) d\eta(\bar{f})} d\eta(\bar{f}) \\ &\stackrel{(2)}{=} \int_{\text{supp}(\eta)} \left(\sum_{\theta \in \Theta} V(\theta; f^{y^{\bar{f}}}) y^{\bar{f}}(\theta) f(\theta) \right) \frac{\bar{f}(\bar{\theta})}{f^{\bar{x}}(\bar{\theta})} d\eta(\bar{f}) \\ &\stackrel{(3)}{=} \left(\sum_{\theta \in \Theta} \bar{x}(\theta) f(\theta) \right) \int_{\text{supp}(\eta)} \underbrace{\sum_{\theta \in \Theta} V(\theta; \bar{f}) \bar{f}(\theta)}_{\mathcal{W}(\bar{f})} d\eta(\bar{f}), \end{aligned}$$

where (1) follows from the proof of Lemma 5 and specifically (B.13), where $y^{\bar{f}}$ is defined in (B.12), (2) uses definitions (4.1) and (4.2), and (3) uses the definition of \mathcal{W} and $f^{\bar{x}}$, in particular $f^{y^{\bar{f}}} = \bar{f}$. This proves that the objective function can be written as

$$\left(\sum_{\theta \in \Theta} \bar{x}(\theta) f(\theta) \right) \mathbb{E}_{\bar{f} \sim \eta} \mathcal{W}(\bar{f}),$$

where feasible η satisfy conditions (B.8) and (B.9), by Lemma 5. Given this representation of the objective function and Lemma 5, the concave closure characterization follows immediately.

B.5. Proof of Theorem 4

Consider a regular mechanism frame (x, π) that is ExD implementable. We can assume without loss of generality that any two distinct signal realizations $s, \hat{s} \in \mathcal{S}$ induce posterior beliefs f^s and $f^{\hat{s}}$ that lead to different payoffs for some type of the agent: There exists θ such that $u(\theta; f^s) \neq u(\theta; f^{\hat{s}})$. If this was not the case, we could merge signals s and \hat{s} without affecting any of the payoffs. Indeed, this follows from the assumption that whenever $u(\theta; \bar{f}) = u(\theta; \bar{g})$ for all $\theta \in \Theta$, then any convex combination of \bar{f} and \bar{g} yields the same aftermarket payoff to the agent and the designer. By the regularity of the mechanism frame, we can order the finite set \mathcal{S} by $>$ in such a way that for any $s, \hat{s} \in \mathcal{S}$ (sent with positive probability), $s > \hat{s}$ implies that $f^s \succeq^{\text{LR}} f^{\hat{s}}$.

An ExD-IC mechanism can be represented as a randomization over mechanisms that are deterministic and IC: For some measurable space Θ_0 and cdf $F_0 \in \Delta(\Theta_0)$,

$$\pi(s|\theta)x(\theta) = \int_{\Theta_0} \hat{\pi}(s|\theta; \theta_0) \hat{x}(\theta; \theta_0) dF_0(\theta_0), \quad (\text{B.15})$$

where $(\hat{x}(\cdot; \theta_0), \hat{\pi}(\cdot; \theta_0))$ is a deterministic implementable mechanism frame for any $\theta_0 \in \Theta_0$. The following lemma establishes key properties of deterministic IC mechanisms under a strictly monotone and submodular aftermarket.

LEMMA 6: *If the aftermarket is strictly monotone, then $\hat{x}(\theta; \theta_0) = \mathbf{1}_{\{\theta \geq \theta^*(\theta_0)\}}$ for some $\theta^* : \Theta_0 \rightarrow \Theta$. If additionally the aftermarket is strictly submodular, then $\hat{\pi}(s|\theta; \theta_0) = \mathbf{1}_{\{s = s^*(\theta, \theta_0)\}}$ for some function $s^* : \Theta \times \Theta_0 \rightarrow \mathcal{S}$ that is non-increasing in θ (higher types receive beliefs that are ranked lower in the LR order).*

PROOF OF LEMMA 6: Fix $\theta_0 \in \Theta_0$; I will suppress θ_0 from the notation and use $(\hat{x}, \hat{\pi})$ to denote the corresponding deterministic IC mechanism frame. Because for any $\theta \in \Theta$, $\hat{\pi}(s|\theta) \in \{0, 1\}$, I will write $s(\theta)$ for the (unique) signal sent when the agent reports type θ . Incentive-compatibility implies that, for any $\theta, \hat{\theta} \in \Theta$,

$$u(\theta; f^{s(\theta)})\hat{x}(\theta) - u(\theta; f^{s(\hat{\theta})})\hat{x}(\hat{\theta}) \geq u(\hat{\theta}; f^{s(\theta)})\hat{x}(\theta) - u(\hat{\theta}; f^{s(\hat{\theta})})\hat{x}(\hat{\theta}). \quad (\text{B.16})$$

First, consider two types $\theta, \hat{\theta}$ such that $\hat{x}(\theta) = 1$ but $\hat{x}(\hat{\theta}) = 0$. Equation (B.16) implies $u(\theta; f^{s(\theta)}) \geq u(\hat{\theta}; f^{s(\theta)})$. By strict monotonicity of the aftermarket, $u(\theta; \bar{f})$ is strictly increasing in θ for any $\bar{f} \in \Delta(\Theta)$, so to avoid a contradiction we must have $\theta > \hat{\theta}$. Thus, the allocation rule \hat{x} is non-decreasing. Because $\hat{x}(\theta) \in \{0, 1\}$ for any $\theta \in \Theta$, there must exist some θ^* such that $\hat{x}(\theta) = \mathbf{1}_{\{\theta \geq \theta^*\}}$. Because θ_0 was arbitrary, this proves the first part of Lemma 6.

Next, consider two types θ and $\hat{\theta}$ that receive the object under \hat{x} , with $\theta > \hat{\theta}$. Equation (B.16) implies $u(\theta; f^{s(\theta)}) - u(\theta; f^{s(\hat{\theta})}) \geq u(\hat{\theta}; f^{s(\theta)}) - u(\hat{\theta}; f^{s(\hat{\theta})})$. We must have $s(\theta) \leq s(\hat{\theta})$ as otherwise we obtain a contradiction with strict submodularity of the aftermarket which states that if $s(\theta) > s(\hat{\theta})$, then $u(\tau; f^{s(\theta)}) - u(\tau; f^{s(\hat{\theta})})$ is strictly decreasing in τ . Because θ_0 was arbitrary, this proves the second part of Lemma 6. *Q.E.D.*

By the representation (B.15) and Lemma 6,

$$\pi(s|\theta)x(\theta) = \int_{\Theta_0} \mathbf{1}_{\{s=s^*(\theta, \theta_0)\}} \mathbf{1}_{\{\theta \geq \theta^*(\theta_0)\}} dF_0(\theta_0),$$

with $s^* : \Theta \times \Theta_0 \rightarrow \mathcal{S}$ non-increasing in the first argument θ . For any $r \in \mathbb{R}$, $\theta > \hat{\theta}$, we have

$$\sum_{s \in \mathcal{S}: s \leq r} \pi(s|\theta)x(\theta) \geq \sum_{s \in \mathcal{S}: s \leq r} \pi(s|\hat{\theta})x(\hat{\theta}) \quad (\text{B.17})$$

because

$$\sum_{s \in \mathcal{S}: s \leq r} [\mathbf{1}_{\{s=s^*(\theta, \theta_0)\}} - \mathbf{1}_{\{s=s^*(\hat{\theta}, \theta_0)\}}] \geq 0$$

by the fact that s^* is non-increasing in θ .

Recall that for $s, \hat{s} \in \mathcal{S}$ such that $s > \hat{s}$, f^s LR dominates $f^{\hat{s}}$. This means that

$$\frac{f^s(\theta)}{f^{\hat{s}}(\theta)} = \frac{\pi(s|\theta)}{\pi(\hat{s}|\theta)} \phi(s, \hat{s}) \quad \text{is non-decreasing in } \theta, \quad (\text{B.18})$$

where $\phi(s, \hat{s})$ is a term that does not depend on θ . Towards a contradiction, suppose that (x, π) is not a cutoff rule. Then, by Proposition 1, for some $r \in \mathcal{S}$ and $\theta > \hat{\theta}$, we have $\pi(r|\hat{\theta})x(\hat{\theta}) > \pi(r|\theta)x(\theta)$. For any $s < r$, by (B.18),

$$\frac{\pi(s|\theta)x(\theta)}{\pi(s|\hat{\theta})x(\hat{\theta})} \leq \frac{\pi(r|\theta)x(\theta)}{\pi(r|\hat{\theta})x(\hat{\theta})}.$$

Because

$$\frac{\pi(r|\theta)x(\theta)}{\pi(r|\hat{\theta})x(\hat{\theta})} < 1,$$

it follows that for all $s \leq r$, we have $\pi(s|\theta)x(\theta) < \pi(s|\hat{\theta})x(\hat{\theta})$, so that also

$$\sum_{s \in \mathcal{S}: s \leq r} \pi(s|\theta)x(\theta) < \sum_{s \in \mathcal{S}: s \leq r} \pi(s|\hat{\theta})x(\hat{\theta}).$$

This is a contradiction with equation (B.17) that holds for all r and $\theta > \hat{\theta}$.

B.6. Proof of Theorem 4'

Strict monotonicity and strict submodularity were only used in the proof of Lemma 6, so I only have to prove that the conclusion of Lemma 6 holds when the aftermarket is monotone and submodular (not necessarily strictly) and the deterministic mechanism frame $(\hat{x}, \hat{\pi})$ is strictly implementable.

Using the same notation as in the proof of Lemma 6, note that equation (B.16) still holds. Consider two types $\theta, \hat{\theta}$ such that $\hat{x}(\theta) = 1$ but $\hat{x}(\hat{\theta}) = 0$. Equation (B.16) implies $u(\theta; f^{s(\theta)}) \geq u(\hat{\theta}; f^{s(\theta)})$. By monotonicity of the aftermarket, we must have either $\theta \geq \hat{\theta}$, or $u(\theta; f^{s(\theta)}) = u(\hat{\theta}; f^{s(\theta)})$. I will show that the latter case contradicts strict implementability. Indeed, in this case, the transfer implementing the outcome $\hat{x}(\theta) = 1$ and $\hat{x}(\hat{\theta}) = 0$ must make both types θ and $\hat{\theta}$ indifferent between reporting θ and $\hat{\theta}$, despite the fact that they receive different allocations. This means that $\theta \geq \hat{\theta}$, and the rest of the proof of the first part of Lemma 6 is unchanged.

Next, consider two types θ and $\hat{\theta}$ that receive the object under \hat{x} , with $\theta > \hat{\theta}$. Equation (B.16) implies $u(\theta; f^{s(\theta)}) - u(\theta; f^{s(\hat{\theta})}) \geq u(\hat{\theta}; f^{s(\theta)}) - u(\hat{\theta}; f^{s(\hat{\theta})})$. By submodularity of the aftermarket, we must have either $s(\theta) \leq s(\hat{\theta})$ or $u(\theta; f^{s(\theta)}) - u(\theta; f^{s(\hat{\theta})}) = u(\hat{\theta}; f^{s(\theta)}) - u(\hat{\theta}; f^{s(\hat{\theta})})$. In the latter case, both types θ and $\hat{\theta}$ must be indifferent between reporting θ and $\hat{\theta}$; indeed, since the implementing transfer \hat{t} must satisfy $u(\theta; f^{s(\theta)}) - u(\hat{\theta}; f^{s(\theta)}) = \hat{t}(\theta) - \hat{t}(\hat{\theta}) = u(\theta; f^{s(\hat{\theta})}) - u(\hat{\theta}; f^{s(\hat{\theta})})$, we can conclude that $u(\theta; f^{s(\theta)}) - \hat{t}(\theta) = u(\hat{\theta}; f^{s(\theta)}) - \hat{t}(\hat{\theta})$ and $u(\hat{\theta}; f^{s(\hat{\theta})}) - \hat{t}(\hat{\theta}) = u(\theta; f^{s(\hat{\theta})}) - \hat{t}(\theta)$. By strict implementability, indifference implies that θ and $\hat{\theta}$ must receive the same outcome: $s(\theta) = s(\hat{\theta})$. The rest of the proof of the second part of Lemma 6 is unchanged.

APPENDIX C: CONTINUOUS DISTRIBUTIONS OF TYPES

In this appendix, I extend the definition of cutoff mechanisms to continuous type spaces, re-prove all the results from Sections 3–4, and provide a proof of Proposition 3.

I assume that the product distribution of types is continuous, that is, it admits a density f on some compact convex $\Theta \subset \mathbb{R}_+^N$. I let f_i denote a density of the marginal distribution of types of agent i with respect to the Lebesgue measure on Θ_i .

A mechanism $(\mathbf{x}, \boldsymbol{\pi}, \mathbf{t})$ is defined as before, except that it is assumed that all functions are measurable, and the signal spaces \mathcal{S}_i are allowed to be arbitrary (possibly infinite) measurable spaces. Because I do not distinguish between two mechanisms that induce the same distribution of posterior beliefs for every prior, it is without loss of generality to assume that the cardinality of the message space is at most a continuum—I will thus assume throughout that, for all $i \in \mathcal{N}$, $\mathcal{S}_i \subset \mathbb{R}^+$ and that \mathcal{S}_i is endowed with a Borel σ -field. I will equate mechanisms that differ on a measure-zero set of type profiles: $(\mathbf{x}, \boldsymbol{\pi}, \mathbf{t})$ and $(\mathbf{x}', \boldsymbol{\pi}', \mathbf{t}')$ are treated as the same mechanism if $\mathbf{x}(\boldsymbol{\theta}) = \mathbf{x}'(\boldsymbol{\theta})$, $\boldsymbol{\pi}(\cdot|\boldsymbol{\theta}) = \boldsymbol{\pi}'(\cdot|\boldsymbol{\theta})$, and $\mathbf{t}(\boldsymbol{\theta}) = \mathbf{t}'(\boldsymbol{\theta})$, for almost all $\boldsymbol{\theta}$. Consequently, all statements of the form “for all types”

should be interpreted as “for almost all types,” and profitable deviations are allowed for a measure-zero set of types of any agent.

The payoffs $u_i(\theta_i; \bar{f})$ and $V_i(\theta_i; \bar{f})$ are assumed bounded and measurable. $V_i(\theta_i; \bar{f})$ is additionally upper semi-continuous in \bar{f} (in the weak* topology), for any i .²

The definition of implementability remains identical, except that the sum operator $\sum_{s \in \mathcal{S}_i}$ is replaced by an integral $\int_{\mathcal{S}_i}$ with respect to the measure induced by $\pi_i(s|\cdot)$.

A cutoff mechanism is defined as follows. Suppose that the interim allocation rule $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in θ_i for any $\boldsymbol{\theta}_{-i}$. A non-decreasing function is continuous almost everywhere, and thus there exists a non-decreasing, right-continuous $x'_i(\theta_i, \boldsymbol{\theta}_{-i})$ which differs from $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ on a measure-zero set of types θ_i . Because I equate mechanisms that differ on measure-zero set of types, I can without loss of generality assume that $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ is right-continuous. Thus, $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ can be extended to a cdf on $C_i = \Theta_i \cup \{\bar{\theta}_i\}$ by defining $x_i(\bar{\theta}_i, \boldsymbol{\theta}_{-i}) = 1$. The random variable defined by this cdf is the random-cutoff representation of the allocation rule $x_i(\theta_i, \boldsymbol{\theta}_{-i})$. For any measurable function g on C_i , $\int g(c) dx_i(c, \boldsymbol{\theta}_{-i})$ denotes the Lebesgue integral of g with respect to the distribution of the cutoff induced by the allocation rule $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ on C_i .

DEFINITION 9—Cutoff Rules: A mechanism frame $(\mathbf{x}, \boldsymbol{\pi})$ is a *cutoff rule* if $x_i(\theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in θ_i for all $\boldsymbol{\theta}_{-i}$, and π_i can be represented as

$$\pi_i(S|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \gamma_i(S|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}), \quad (\text{C.1})$$

for some measurable signal function $\gamma_i : C_i \times \Theta_{-i} \rightarrow \Delta(\mathcal{S}_i)$, for all $i \in \mathcal{N}$, $\theta_i \in \Theta_i$, $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$, and measurable $S \subset \mathcal{S}_i$.

The only difference in the definition is that (C.1) must be expressed for all measurable subsets of \mathcal{S}_i rather than for all elements $s \in \mathcal{S}_i$. Similarly, condition (M) becomes

$$\begin{aligned} &\pi_i(S|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) \\ &\text{is non-decreasing in } \theta_i \text{ for all measurable } S \in \mathcal{S}_i \text{ and all } \boldsymbol{\theta}_{-i} \in \Theta_{-i}. \end{aligned} \quad (\text{M})$$

I first prove Proposition 1 showing that condition (M) characterizes cutoff rules.

PROOF OF PROPOSITION 1: The proof is analogous to the proof for the discrete type case, but the infinite type and signal spaces require additional care. I only have to prove the “if” direction. I fix i and $\boldsymbol{\theta}_{-i}$, and drop them from the notation to simplify exposition. It is without loss of generality to normalize $\mathcal{S} = [0, 1]$ —this will simplify notation in the proof.

Denote $\beta_S(\tau) \equiv \pi(S|\tau)x(\tau)$, for any measurable $S \subseteq \mathcal{S}$. Unlike in the discrete type case, β_S corresponds to the probability that the signal lies in the set S to account for the fact that \mathcal{S} is an infinite space.³ Because $\beta_S(\tau)$ is non-decreasing, it has one-sided limits everywhere and is continuous almost everywhere. According to the convention that I

²The representation of posterior beliefs by a density \bar{f} is only justified within the class of cutoff mechanisms; outside of the class, posterior beliefs might not be represented by a continuous distribution.

³Each $\beta_S(\tau)$ is a measurable function of τ because both $x_i(\tau, \boldsymbol{\theta}_{-i})$ and $\pi_i(S|\tau, \boldsymbol{\theta}_{-i})$ were assumed to be measurable in τ .

identify mechanisms that differ on a measure-zero set of types, it is without loss of generality to assume that $\beta_S(\tau)$ is right-continuous in τ . It follows that β_S induces a positive σ -additive measure μ_S on C defined by $\mu_S((a, b]) = \beta_S(b) - \beta_S(a)$. Because a σ -additive measure on the Borel σ -field is uniquely defined by the values it takes on intervals, the above definition uniquely characterizes μ_S .

I will show that the measure μ_S , for any measurable $S \subseteq \mathcal{S}$, is absolutely continuous with respect to the cutoff distribution dx induced by the allocation rule x . For any $a, b \in C$, $a < b$, we have

$$\beta_S(b) - \beta_S(a) \leq \beta_S(b) - \beta_S(a) = x(b) - x(a).$$

It follows that if $x(b) = x(a)$, then $\beta_S(b) - \beta_S(a) = 0$. Because a and b were arbitrary, μ_S is absolutely continuous with respect to dx .

By the Radon–Nikodym theorem, for any measurable $S \subseteq \mathcal{S}$, there exists a measurable nonnegative function g_S supported on C that is a density of μ_S with respect to the measure dx . In particular,

$$\beta_S(\theta) = \pi(S|\theta)x(\theta) \equiv \mu_S([0, \theta]) = \int_0^\theta g_S(c) dx(c), \quad (\text{C.2})$$

for all θ and measurable $S \subseteq \mathcal{S}$.

Recall that I normalized $\mathcal{S} = [0, 1]$. With $S = [0, s]$, I define $y_c(s) \equiv g_{[0, s]}(c)$, for any $s \in \mathcal{S}$. It can be directly verified that $y_c(s)$, treated as a function of s , satisfies, for dx -almost all c : (i) $y_c(0) = 0$, $y_c(1) = 1$, (ii) $y_c(s)$ is non-decreasing in s , and (iii) $y_c(s)$ is right-continuous in s . Thus, $y_c(s)$ is a cdf for dx -almost all c . We can thus define γ , for dx -almost all $c \in C$, by $\gamma([0, s]|c) = y_c(s)$, for any $s \in \mathcal{S}$. (It is irrelevant how we define γ on the remaining dx -measure-zero set of points c .) Because a σ -additive distribution γ is uniquely determined by the value it assigns to sets of the form $[0, s]$, for all $s \in \mathcal{S}$, by equation (C.2) we get

$$\pi(S|\theta)x(\theta) = \int_0^\theta \gamma(S|c) dx(c),$$

for all measurable $S \subseteq \mathcal{S}$. Therefore, (x, π) satisfies (C.1). Because i and θ_{-i} were arbitrary, (x, π) is a cutoff rule. Q.E.D.

The definition of DS implementability remains the same, except that sums are replaced by integrals: The payoff to agent i from reporting $\hat{\theta}_i$ to a direct mechanism (x, π, t) , when her true type is θ_i and other agents report truthfully, is

$$\int_{\mathcal{S}_i} u_i(\theta_i; f_i^s) d\pi(s|\hat{\theta}_i, \theta_{-i})x_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i}).$$

Next, I prove that cutoff rules (and only cutoff rules) are implementable for any prior distribution of types and any monotone aftermarket (Theorem 1).

PROOF OF THEOREM 1: If (x, π) is a cutoff rule, then condition (M) follows directly from Definition 9 of cutoff rules. I will show that condition (M) implies implementability for any prior distribution and any (monotone) aftermarket. Fix i and θ_{-i} . Using the definition of cutoff mechanisms, for any $\tau \in \Theta_i$,

$$\int_{\mathcal{S}_i} u_i(\tau; f_i^s) d\pi_i(s|\tau, \theta_{-i})x_i(\tau, \theta_{-i}) = \int_0^\tau \int_{\mathcal{S}_i} u_i(\tau; f_i^s) d\gamma_i(s|c, \theta_{-i}) dx_i(c, \theta_{-i}).$$

For any $\theta_i \geq \hat{\theta}_i$, we have

$$\int_{\hat{\theta}_i}^{\theta_i} \int_{\mathcal{S}_i} [u_i(\theta_i; f_i^s) - u_i(\hat{\theta}_i; f_i^s)] d\gamma_i(s|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}) \geq 0,$$

where the inequality follows from monotonicity of the aftermarket. Therefore,

$$\int_{\mathcal{S}_i} [u_i(\theta_i; f_i^s) - u_i(\hat{\theta}_i; f_i^s)] [d\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) - d\pi_i(s|\hat{\theta}_i, \boldsymbol{\theta}_{-i})x_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] \geq 0. \quad (\text{C.3})$$

To show that condition (C.3) implies implementability, I use a condition for implementability in arbitrary type and allocation spaces from Dworzak and Zhang (2017) which is a version of Rochet (1987)'s classic cyclic monotonicity condition: Given a set of types and their final allocations, the assignment is implementable if and only if, for any finite subset of the type space, the matching between types and final allocations is efficient (see Dworzak and Zhang for a formal definition⁴). A monotone aftermarket guarantees that, for any $s \in \mathcal{S}_i$, the payoff of each agent is non-decreasing in her type. Thus, matching efficiency is implied by pairwise stability—it is enough to show that joint surplus cannot be increased by swapping the allocations of any pair of types. This is exactly what condition (C.3) states.

The proof of the converse part is virtually identical to the proof for the discrete type space, and hence skipped. *Q.E.D.*

The results on optimal cutoff mechanisms with a single agent (Section 4.1) go through with virtually no change to the argument (with sum operators replaced with integrals).⁵ The Matthews–Border condition (M-B) has a direct analog for a continuous type space, so the only difficulty in extending the results to the multi-agent model lies in proving Lemma 2: Because the signal space is now potentially infinite, I cannot directly apply Lemma 3 from Gershkov et al. (2013) because Gershkov et al. only allowed for a finite set of social alternatives. I circumvent this difficulty by proving an approximation result.

I say that a sequence of mechanism frames $\{(\mathbf{x}, \boldsymbol{\pi}^n)\}_{n=1}^\infty$ on the same signal space $\times_{i \in \mathcal{N}} \mathcal{S}_i$ converges to $(\mathbf{x}, \boldsymbol{\pi})$, if, for all i , $\pi_i^n(\cdot|\boldsymbol{\theta})x_i(\boldsymbol{\theta})$ converges to $\pi_i(\cdot|\boldsymbol{\theta})x_i(\boldsymbol{\theta})$ in the weak* topology of measures on \mathcal{S}_i , for almost all $\boldsymbol{\theta}$. Call a mechanism frame $(\mathbf{x}, \boldsymbol{\pi})$ \mathcal{S} -finite if there are finitely many signal realizations in the support of $\boldsymbol{\pi}$.

LEMMA 7: *A mechanism frame $(\mathbf{x}, \boldsymbol{\pi})$ is a cutoff rule if and only if it is the limit of \mathcal{S} -finite cutoff rules with the same allocation rule \mathbf{x} .*

PROOF OF LEMMA 7: First, suppose that a sequence of \mathcal{S} -finite cutoff rules $\{(\mathbf{x}, \boldsymbol{\pi}^n)\}_{n=1}^\infty$ converges to some mechanism frame $(\mathbf{x}, \boldsymbol{\pi})$. I show that $(\mathbf{x}, \boldsymbol{\pi})$ is a cutoff rule.

Fix $\boldsymbol{\theta}$ and $i \in \mathcal{N}$. Convergence in the weak* topology means that, for any continuous bounded function g on \mathcal{S}_i , we have

$$\lim_n \int_{\mathcal{S}_i} g(s) d\pi_i^n(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_{\mathcal{S}_i} g(s) d\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i})x_i(\theta_i, \boldsymbol{\theta}_{-i}).$$

⁴Although Dworzak and Zhang considered single-agent mechanisms, checking DS implementability in a model with multiple agents reduces to checking conditions **IR** and **IC** for any fixed $\boldsymbol{\theta}_{-i}$.

⁵The results on concavification now follow from the Online Appendix of Kamenica and Gentzkow (2011) where they extend their methods to continuous state spaces.

Because, for each n , $(\mathbf{x}, \boldsymbol{\pi}^n)$ is a (\mathcal{S} -finite) cutoff rule, we have

$$\int_{\mathcal{S}_i} g(s) d\pi_i^n(s|\theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \int_{\mathcal{S}_i} g(s) d\gamma_i^n(s|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}),$$

for some probability measure $\gamma_i^n(\cdot|c, \boldsymbol{\theta}_{-i})$ on \mathcal{S}_i . By the Banach–Alaoglu theorem, the set of probability measures is compact in the weak* topology, so (after passing to a subsequence if necessary) we can assume that γ_i^n converges to some γ_i . Thus,

$$\lim_n \int_{\mathcal{S}_i} g(s) d\gamma_i^n(s|c, \boldsymbol{\theta}_{-i}) = \int_{\mathcal{S}_i} g(s) d\gamma_i(s|c, \boldsymbol{\theta}_{-i}).$$

By the Lebesgue dominated convergence theorem,

$$\lim_n \int_0^{\theta_i} \int_{\mathcal{S}_i} g(s) d\gamma_i^n(s|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \left(\int_{\mathcal{S}_i} g(s) d\gamma_i(s|c, \boldsymbol{\theta}_{-i}) \right) dx_i(c, \boldsymbol{\theta}_{-i}).$$

Combining the above equations,

$$\int_{\mathcal{S}_i} g(s) d\pi_i(s|\theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \int_{\mathcal{S}_i} g(s) d\gamma_i(s|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}).$$

Because the above equality is true for all continuous bounded functions g , the two measures must be equal, that is,

$$\pi_i(\mathcal{S}|\theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \gamma_i(\mathcal{S}|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}),$$

for all measurable $\mathcal{S} \subseteq \mathcal{S}_i$. Thus, $(\mathbf{x}, \boldsymbol{\pi})$ is a cutoff rule.

Conversely, suppose that $(\mathbf{x}, \boldsymbol{\pi})$ is a cutoff rule. I have to find a sequence $\{(\mathbf{x}, \boldsymbol{\pi}^n)\}_{n=1}^\infty$ of \mathcal{S} -finite cutoff rules that converges to $(\mathbf{x}, \boldsymbol{\pi})$.

Fix $\boldsymbol{\theta}$ and $i \in \mathcal{N}$, and consider the measure $\gamma_i(\cdot|\theta_i, \boldsymbol{\theta}_{-i})$ satisfying equation (C.1), defined on \mathcal{S}_i . Take an arbitrary discrete approximation of the probability measure $\gamma_i(\cdot|\theta_i, \boldsymbol{\theta}_{-i})$, that is, a sequence $\{\gamma_i^n(\cdot|\theta_i, \boldsymbol{\theta}_{-i})\}_{n=1}^\infty$ of finite-support measures on \mathcal{S}_i that converges in weak* topology to γ_i .⁶ For each n , define a mechanism frame $(\mathbf{x}, \boldsymbol{\pi}^n)$ by

$$\pi_i^n(\mathcal{S}|\theta_i, \boldsymbol{\theta}_{-i}) x_i(\theta_i, \boldsymbol{\theta}_{-i}) = \int_0^{\theta_i} \gamma_i^n(\mathcal{S}|c, \boldsymbol{\theta}_{-i}) dx_i(c, \boldsymbol{\theta}_{-i}),$$

for all $\boldsymbol{\theta}$, $i \in \mathcal{N}$, and measurable $\mathcal{S} \subseteq \mathcal{S}_i$. Because γ_i^n has finite support, $(\mathbf{x}, \boldsymbol{\pi}^n)$ is an \mathcal{S} -finite cutoff rule. By the same argument as in the first part of the proof, $(\mathbf{x}, \boldsymbol{\pi})$ is a limit of $\{(\mathbf{x}, \boldsymbol{\pi}^n)\}_{n=1}^\infty$. *Q.E.D.*

I am now ready to extend the proof of Lemma 2.

PROOF OF LEMMA 2: The “only if” direction requires no changes in the continuous type case (except that I now use the continuous type version of Proposition 1 proven above). I focus on the “if” direction. First, because the results of [Gershkov et al. \(2013\)](#)

⁶Such an approximation can be constructed by discretizing the compact domain of γ_i .

allow for a continuous type space, the proof technique extends with no changes in the argument to the case of \mathcal{S} -finite mechanism frames. Consider a general mechanism frame $(\bar{x}, \bar{\pi})$ (not necessarily \mathcal{S} -finite). By Lemma 7, $(\bar{x}, \bar{\pi})$ can be represented as a limit of a sequence of \mathcal{S} -finite reduced-form cutoff rules $\{(\bar{x}_n, \bar{\pi}_n)\}_{n=1}^\infty$.⁷ By the result for \mathcal{S} -finite mechanism frames, we know that, for each n , there exists a (\mathcal{S} -finite) cutoff rule (x, π_n) such that $(x^f, \pi_n^f) = (\bar{x}_n, \bar{\pi}_n)$, where (x^f, π_n^f) denotes the reduced form of (x, π_n) under f . Passing to a subsequence if necessary, we can assume that (x, π_n) converges to some (x, π_*) . Then, (x, π_*) is also a cutoff rule. Moreover, $(x^f, \pi_*^f) = (\bar{x}, \bar{\pi})$ (because this equality holds along the sequence). *Q.E.D.*

The remaining part of the proof of Theorem 3 is fully analogous to the discrete type case.

Finally, I extend the results from Section 4.3. With continuous distributions, I say that a distribution with density g *likelihood-ratio dominates* a distribution with full-support density f (denoted $g \succ^{\text{LR}} f$) if $g(\theta)/f(\theta)$ is bounded and non-decreasing. The proof of Lemma 5 and Proposition 2 is then virtually identical with a continuous type space—in fact, except for using the sum operator instead of the integral operator, the proof in the main text did not make use of finiteness of the type space.

C.1. Proof of Proposition 3

I will apply the results from Sections 3 and 4 for a continuous type space—the previous section showed that these results extend to this case. Because I have assumed that agents are symmetric, I drop all the subscripts.

Case 1. Suppose that W is concave and non-decreasing. Because W is concave, and $M(\bar{f})$ is linear in \bar{f} , the functional \mathcal{W} is concave. Thus, for any interim allocation function \bar{x} , it is optimal to disclose no information, by Corollary 2. Using Proposition 2, we can write the problem as

$$\max_{\bar{x}} \left(\int_0^1 \bar{x}(\theta) f(\theta) d\theta \right) W(M(f^{\bar{x}})) \quad (\text{C.4})$$

$$\text{s.t. } \bar{x}(\theta) \text{ is non-decreasing in } \theta, \quad (\text{C.5})$$

$$\text{and } \int_\tau^1 \bar{x}(\theta) f(\theta) d\theta \leq \frac{1 - F^N(\tau)}{N}, \quad \forall \tau \in [0, 1], \quad (\text{C.6})$$

where (C.6) is a version of the Matthews–Border condition (4.7) for continuous type spaces, and $f^{\bar{x}}$, defined analogously as (4.1),

$$f^{\bar{x}}(\theta) = \frac{\bar{x}(\theta) f(\theta)}{\int_\theta \bar{x}(\tau) f(\tau) d\tau},$$

⁷Lemma 7 was stated for cutoff rules, but it also applies to reduced-form cutoff rules which, for any fixed i , are equivalent to one-agent cutoff rules.

is the belief over the winner's type conditional on no disclosure. We can also write the objective function explicitly as

$$\left(\int_0^1 \bar{x}(\theta) f(\theta) d\theta \right) W \left(\frac{\int_0^1 \theta \bar{x}(\theta) f(\theta) d\theta}{\int_0^1 \bar{x}(\theta) f(\theta) d\theta} \right).$$

Consider an auxiliary problem in which we fix $\int_0^1 \bar{x}(\theta) f(\theta) d\theta = \beta$ for some $\beta \leq 1/N$. Since W is non-decreasing, the problem becomes

$$\max_{\bar{x}} \int_0^1 \theta \bar{x}(\theta) f(\theta) d\theta, \quad (\text{C.7})$$

subject to (C.5), (C.6), and

$$\int_0^1 \bar{x}(\theta) f(\theta) d\theta = \beta. \quad (\text{C.8})$$

In the above problem, we can think of constraint (C.8) as an equal mass condition. Intuitively, it is optimal to shift as much mass as possible to the right, subject to constraint (C.6), which will thus hold with equality for large enough τ . Formally, I will show optimality of $\bar{x}(\theta) = F^{N-1}(\theta) \mathbf{1}_{\{\theta \geq r\}}$, where r is chosen so that condition (C.8) holds. Using integration by parts,

$$\int_0^1 \theta \bar{x}(\theta) f(\theta) d\theta = \int_0^1 \underbrace{\left(\int_{\theta}^1 \bar{x}(\tau) f(\tau) d\tau \right)}_{\Gamma(\theta)} d\theta.$$

Ignoring constraint (C.5) for now, the problem is to maximize the above expression over Γ subject to $\Gamma(0) = \beta$, Γ is non-increasing, and $\Gamma(\theta) \leq (1 - F^N(\theta))/N$, for all θ . Clearly, this problem is solved by $\Gamma(\theta) = \min\{\beta, (1 - F^N(\theta))/N\}$. But then $\Gamma(\theta) = \int_{\theta}^1 F^{N-1}(\tau) \mathbf{1}_{\{\theta \geq r\}} f(\tau) d\tau$, by the definition of r . Moreover, $F^{N-1}(\theta) \mathbf{1}_{\{\theta \geq r\}}$ satisfies constraint (C.5), so it is a solution to problem (C.7) subject to (C.5), (C.6), and (C.8).

In the second step, I optimize over $\beta \in [0, 1/N]$ in condition (C.8), which corresponds to optimizing over $r \in [0, 1]$ in the optimal solution to the auxiliary problem. By plugging in the optimal solution from the auxiliary problem to (C.4), we obtain

$$\max_{r \in [0,1]} \left(\int_r^1 F^{N-1}(\theta) f(\theta) d\theta \right) W \left(\frac{\int_r^1 \theta F^{N-1}(\theta) f(\theta) d\theta}{\int_r^1 F^{N-1}(\theta) f(\theta) d\theta} \right).$$

This corresponds to equation (4.11) in Proposition 3, and thus the first case is proven.

Case 2. Consider the case when W is concave and non-increasing. Following the same steps as previously, I consider the auxiliary problem with constraint (C.8). Because W is non-increasing, the objective is

$$\min_{\bar{x}} \int_0^1 \theta \bar{x}(\theta) f(\theta) d\theta,$$

subject to (C.5), (C.6), and (C.8). This time, all the mass under \bar{x} should be shifted to the left, subject to the monotonicity constraint (C.5). Thus, the optimal \bar{x} will be constant, equal to β . Because $\beta \leq 1/N$, such \bar{x} satisfies the Matthews–Border condition (C.6), and corresponds to allocating the object uniformly at random.

In the second step, because W was assumed nonnegative, optimization over β yields $\beta = 1/N$, that is, β should be set to the maximal feasible level. Such a mechanism always allocates the good (to a randomly selected agent).

Case 3. Finally, assume that W is convex. Then, the functional \mathcal{W} is convex, so it is optimal to fully disclose the cutoff representing the interim allocation rule \bar{x} , by Corollary 2. Full disclosure means that any posterior belief $\bar{f} \in M_f$ is decomposed into a distribution over truncations of the prior distribution f . Recall that \bar{x} can be treated as a cdf of the cutoff. Therefore,

$$\text{co}^{M_f} \mathcal{W}(\bar{f}) = \int_0^1 W(m(c)) \frac{1 - F(c)}{\int_0^1 \bar{x}(\theta) f(\theta) d\theta} d\bar{x}(c).$$

The additional term $(1 - F(c))/(\int_0^1 \bar{x}(\theta) f(\theta) d\theta)$ appears because, by definition, the payoff W is a conditional expected payoff conditional on allocating the good. The distribution with cdf \bar{x} is the ex ante distribution of the cutoff for agent i . Conditional on agent i being the winner, the posterior distribution of the cutoff for agent i must be adjusted (intuitively, lower cutoffs are more likely). The ex ante probability of cutoff c is transformed into a conditional probability by conditioning on the event $\tilde{\theta} \geq c$. The objective function (4.10) can be written as

$$\max_{\bar{x}} \int_0^1 W(m(c))(1 - F(c)) d\bar{x}(c).$$

Using integration by parts (by assumption, W is differentiable), we obtain

$$\int_0^1 W(m(c))(1 - F(c)) d\bar{x}(c) = -W(m(0))\bar{x}(0^-) - \int_0^1 \frac{d}{dc} [W(m(c))(1 - F(c))] x(c) dc.$$

Because \bar{x} represents a cdf in the above equation, $\bar{x}(0^-)$, the left limit of \bar{x} at 0, is equal to 0. By letting $w(c) \equiv W(m(c))$, the objective function can be written as

$$\begin{aligned} & \max_{\bar{x}} \int_0^1 \frac{-\frac{d}{dc} [W(m(c))(1 - F(c))]}{f(c)} \bar{x}(c) f(c) dc \\ & = \max_{\bar{x}} \int_0^1 \underbrace{\left[w(c) - w'(c) \frac{1 - F(c)}{f(c)} \right]}_{J_w(c)} \bar{x}(c) f(c) dc. \end{aligned}$$

The conclusion of Proposition 3 now follows from an argument analogous to the one used in previous cases. If $J_w(c)$ is non-positive for $c \leq \underline{c}$, and positive non-decreasing for $c \geq \underline{c}$, then it is optimal to set $\bar{x}(\theta) = 0$ for $\theta \in [0, \underline{c}]$, and push all the mass under \bar{x} on $[\underline{c}, 1]$ to the right, subject to constraint (C.6). This gives us $\bar{x}(\theta) = F^{N-1}(\theta) \mathbf{1}_{\{\theta \geq \underline{c}\}}$. Under this \bar{x} , the distribution of the cutoff has a continuous part which is the distribution of a second

highest type conditional on that type exceeding \underline{r} , and an atom at \underline{r} , with mass equal to the probability that the second highest type is below \underline{r} .

To finish the proof of Proposition 3, I have to show that when $W(c)$ is non-decreasing and log-concave, then there exists \underline{r} such that $J_w(c)$ is non-positive for $c \leq \underline{r}$, and positive non-decreasing for $c \geq \underline{r}$. It is enough to prove that $J_w(c) \geq 0$ implies $J'_w(c) \geq 0$. We have $m'(c) = (m(c) - c)f(c)/(1 - F(c))$. The inequality $J_w(c) \geq 0$ implies that $m(c) - c \leq W(m(c))/W'(m(c))$. Using the assumption that $W'' \geq 0$, and the above inequality,

$$J'_w(c) = W'(m(c)) - W''(m(c))(m(c) - c) \geq W'(m(c)) - W''(m(c)) \frac{W(m(c))}{W'(m(c))}.$$

Using the fact that $W' \geq 0$, the above expression is greater than zero if and only if $(W')^2 \geq W''W$ which is equivalent to log-concavity of W .

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