Econometrica Supplementary Material

SUPPLEMENT TO "SCREENING IN VERTICAL OLIGOPOLIES"

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S1. OMITTED PROOFS FOR SECTION 4.1

S1.1. *Proofs for Section 4.1.1*

PROPOSITION 5: *Fix n, s^{-n}, and $s^n = (\alpha, v)$. Let $P \equiv \{\theta | \pi(\theta, \alpha, v) \geq 0\}$. Then there is $(\hat{\alpha}, \hat{v})$ with $\pi(\cdot, \hat{\alpha}, \hat{v}) \geq 0$ that agrees on P with (α, v). If (α, v) is a best response to s^{-n}, then $\pi(\theta, \alpha, v) \geq 0$ for almost all θ where $\varphi > 0$.*

PROOF: The idea is simply to remove all menu items for which θ is not in P. Let us first show that P can be taken to be closed. Fix n and let $G(\theta, v)$ be the subdifferential to v at θ. Since v is convex, G is singleton-valued almost everywhere, and every selection from G is increasing. Thus, since G is compact-valued, it is wlog to assume that $\alpha(\theta) \in \arg\max_{a \in G(\theta, v)} \pi(\theta, a, v)$ for all θ. But then, since G is upper hemicontinuous in θ, $\pi(\cdot, \alpha, v)$ is upper semicontinuous (Aliprantis and Border (2006, Lemma 17.30, p. 569)), and so P is a closed subset of $[0, 1]$ and, hence, compact.

Now let us build the menu that results when menu items with θ not in P are removed. For each $\theta' \in [0, 1]$, let $v_L(\cdot, \theta')$ be the line given by $v_L(\theta, \theta') = v(\theta') + (\theta - \theta')\alpha(\theta')$ for all $\theta \in [0, 1]$. Note that $v_L(\theta, \theta) = v(\theta)$, that since v is convex with $\alpha(\theta') \in G(\theta', v)$, $v_L(\cdot, \theta')$ lies below v for each θ', and that along $v_L(\cdot, \theta')$, the profits to the firm are constant using private values and since the action is constant. If P is empty, set $(\alpha, v) = (\alpha_*^n, v_*^n)$ and we are done. If P is nonempty, define $\hat{v}(\theta) = \max_{\theta' \in P} v_L(\theta, \theta')$. Then $\hat{v}$, which is the maximum of a set of lines, is convex, with $\hat{v} \leq v$ (since each $v_L(\cdot, \theta')$ lies below v) and $\hat{v} = v$ on P (using that $v_L(\theta, \theta) = v(\theta)$). Let $\hat{\alpha}$ be a selection from $G(\cdot, \hat{v})$, where we can take $\hat{\alpha} = \alpha$ on P and where at any $\theta \notin P$, we can take $\hat{\alpha}(\theta) = \alpha(\theta')$ for some $\theta' \in \arg\max_{\theta' \in P} v_L(\theta, \theta')$. Then by using $(\hat{\alpha}, \hat{v})$, the firm implements the same action on P at the same profit as before (the types in P have no new deviations available), and the firm earns positive profits on any other type, since that type either leaves or, if served, is now imitating a type in P.

Note finally that if (α, v) is a best response to s^{-n} and $\pi(\theta, \alpha, v) < 0$ for some positive measure set of θ where $\varphi > 0$, then $(\hat{\alpha}, \hat{v})$ gives strictly higher profits than (α, v), a contradiction. *Q.E.D.*

S1.2. *Proofs for Section 4.1.4*

DETAILS FOR THE PROOF OF LEMMA 4: Let $\hat{s}(q, \varepsilon) = (\alpha(\cdot, q, \varepsilon), v(\cdot, q, \varepsilon))$ be the menu described in Appendix A, and note that for $\theta \in [\theta_J - \varepsilon, \theta_J)$, $\alpha_q(\theta, q, \varepsilon) = 1$ and

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$v_q(\theta, q, \varepsilon) \le \varepsilon$. Hence,

$$\frac{\partial}{\partial q}\pi\big(\theta, s(q, \varepsilon)\big) \ge \pi_a\big(\theta, s(q, \varepsilon)\big) - \varepsilon \ge \pi_a\big(\theta, \underline{a} + q, v(q)\big) - \varepsilon,$$

since π is concave in a. Similarly, for $[\theta_J, \theta_J + \varepsilon]$,

$$\frac{\partial}{\partial q}\pi\big(\theta, s(q, \varepsilon)\big) \ge -\pi_a\big(\theta, s(q, \varepsilon)\big) - \varepsilon \ge -\pi_a\big(\theta, \bar{a} - q, v(q)\big) - \varepsilon.$$

Hence, recalling that $=_s$ means "has strictly the same sign as,"

$$\begin{aligned}
&\frac{\partial}{\partial q}\Pi\big(\hat{s}(q, \varepsilon), s^{-n}\big)\\
&\quad\ge \int_{\theta_J-\varepsilon}^{\theta_J}\big(\pi_a\big(\theta, \underline{a} + q, v(q)\big) - \varepsilon\big)h(\theta)\,d\theta + \int_{\theta_J}^{\theta_J+\varepsilon}\big(-\pi_a\big(\theta, \bar{a} - q, v(q)\big) - \varepsilon\big)h(\theta)\,d\theta\\
&\quad= \varepsilon\big[\big(\pi_a\big(\theta', \underline{a} + q, v(q)\big) - \varepsilon\big)h\big(\theta'\big) - \big(\pi_a\big(\theta'', \bar{a} - q, v(q)\big) - \varepsilon\big)h\big(\theta''\big)\big]\\
&\quad\underset{s}{=} \big(\pi_a\big(\theta', \underline{a} + q, v(q)\big) - \varepsilon\big)h\big(\theta'\big) - \big(\pi_a\big(\theta'', \bar{a} - q, v(q)\big) - \varepsilon\big)h\big(\theta''\big)\\
&\quad\cong \big(\pi_a\big(\theta_J, \underline{a} + q, v(q)\big) - \pi_a\big(\theta_J, \bar{a} - q, v(q)\big)\big)h(\theta_J)\\
&\quad> 0,
\end{aligned}$$

where the first equality uses the mean value theorem for some $\theta' \in [\theta_J - \varepsilon, \theta_J]$ and $\theta'' \in [\theta_J, \theta_J + \varepsilon]$, where the approximation is arbitrarily good when ε is small, and where the last inequality holds for $q < (\bar{a} - \underline{a})/2$. But then, for ε and q small, $\partial\Pi(\hat{s}(q, \varepsilon), s^{-n})/\partial q > 0$, and we are done. *Q.E.D.*

S1.3. *Proofs for Section 4.1.5*

Let us first reexpress the profits of the firm in a useful and standard way.

LEMMA 22: *Fix n, and for any feasible α and v, define*

$$M(\theta, \alpha, v) = V\big(\alpha(\theta)\big) + \alpha(\theta)\theta - v(\theta_l) - \alpha(\theta)\frac{H(\theta_h) - H(\theta)}{h(\theta)}. \tag{18}$$

Then $\int_{\theta_l}^{\theta_h} \pi(\theta, \alpha, v)h(\theta)\,d\theta = \int_{\theta_l}^{\theta_h} M(\theta, \alpha, v)h(\theta)\,d\theta$.

PROOF: For any α and v, $\int_{\theta_l}^{\theta_h} \pi(\theta, \alpha, v)h(\theta)\,d\theta = \int_{\theta_l}^{\theta_h}(V(\alpha(\theta)) + \alpha(\theta)\theta - v(\theta_l) - \int_{\theta_l}^{\theta} \alpha(\tau)\,d\tau)h(\theta)\,d\theta$ and, integrating by parts, $\int_{\theta_l}^{\theta_h}(\int_{\theta_l}^{\theta} \alpha(\tau)\,d\tau)h(\theta)\,d\theta = \int_{\theta_l}^{\theta_h} \alpha(\theta)(H(\theta_h) - H(\theta))\,d\theta$. Substituting and rearranging yields the result. *Q.E.D.*

PROOF OF LEMMA 1: Existence is standard and uniqueness follows since the set of feasible strategies is convex, and the objective function is strictly concave (since $\pi(\theta, a, v)$ is strictly concave in a and linear in v for each θ). Fix (θ_l, θ_h) and fix the optimum $\tilde{s} = (\tilde{\alpha}, \tilde{v})$.

Step 1. Let us show that there is η such that for all $\theta \in (\theta_l, \theta_h)$, $\pi_a(\theta, \tilde{s}) = (-\eta + H(\theta_h) - H(\theta))/h(\theta)$, where we can then take $\kappa = -\eta + H(\theta_h)$. To see the idea, choose any point θ in (θ_l, θ_h). We will consider perturbations which subtract a small amount from the action schedule near θ and replace it just to the left of θ_h. We can do this without worrying about monotonicity since this is the relaxed problem. The perturbation has cost $\pi_a(\theta, \tilde{s})h(\theta)$ near θ, benefit $\pi_a(\theta_h, \tilde{s})h(\theta_h)$ near θ_h, and benefit $H(\theta_h) - H(\theta)$ because v is lowered between θ and θ_h. Setting the cost equal to the benefit, we have $-\pi_a(\theta, \tilde{s})h(\theta) + H(\theta_h) - H(\theta) + \pi_a(\theta_h, \tilde{s})h(\theta_h) = 0$, and so

$$\pi_a(\theta, \tilde{s}) = \frac{H(\theta_h) + \pi_a(\theta_h, \tilde{s})h(\theta_h) - H(\theta)}{h(\theta)} = \frac{-\eta + H(\theta_h) - H(\theta)}{h(\theta)},$$

where $\eta = -\pi_a(\theta_h, \tilde{s})h(\theta_h)$.

To formalize this, fix $\theta' \in (\theta_l, \theta_h)$ and $0 < \varepsilon < \min\{\theta_h - \theta', \theta' - \theta_l\}/2$. Define $\hat{\alpha}(\cdot, y, \varepsilon)$ to be $\tilde{\alpha} - y/\varepsilon$ on $[\theta' - \varepsilon, \theta']$, $\tilde{\alpha} + y/\varepsilon$ on $[\theta_h - \varepsilon, \theta_h]$, and $\tilde{\alpha}$ elsewhere, and define $\hat{v}(\theta, y, \varepsilon) = \tilde{v}(\theta_l) + \int_{\theta_l}^{\theta} \hat{\alpha}(\tau, y, \varepsilon)\, d\tau$, noting that $\hat{v}(\theta_h, y, \varepsilon) = \tilde{v}(\theta_h)$, and so for each y, $\hat{s}(y, \varepsilon) = (\hat{\alpha}(\cdot, y, \varepsilon), \hat{v}(\cdot, y, \varepsilon))$ is feasible in $\mathcal{P}(\theta_l, \theta_h)$. Note that $\hat{v}_y(\theta, y, \varepsilon) = -1$ on $[\theta', \theta_h - \varepsilon]$, and $\hat{v}_y(\theta, y, \varepsilon) \in [-1, 0]$ on $[\theta' - \varepsilon, \theta']$ and $[\theta_h - \varepsilon, \theta_h]$.

Let the profit of this perturbation be $j(y, \varepsilon) = \int_{\theta_l}^{\theta_h} \pi(\theta, \hat{s}(y, \varepsilon))h(\theta)\, d\theta$. Then, since $\pi_v = -1$,

$$\begin{aligned} j_y(y, \varepsilon) = {} & \int_{\theta' - \varepsilon,}^{\theta'} \left(-\pi_a\big(\theta, \hat{s}(y, \varepsilon)\big)\frac{1}{\varepsilon} - \hat{v}_y(\theta, y, \varepsilon) \right) h(\theta)\, d\theta + H(\theta_h - \varepsilon) - H\big(\theta'\big) \\ & + \int_{\theta_h - \varepsilon,}^{\theta_h} \left(\pi_a\big(\theta, \hat{s}(y, \varepsilon)\big)\frac{1}{\varepsilon} - \hat{v}_y(\theta, y, \varepsilon) \right) h(\theta)\, d\theta, \end{aligned}$$

where between θ' and $\theta_h - \varepsilon$, we use $\hat{\alpha}_y = 0$ and $\hat{v}_y = -1$. Note that $\hat{s}(0, \varepsilon) = (\tilde{\alpha}, \tilde{v})$. Hence, evaluating $j_y(y, \varepsilon)$ at $y = 0$ and using the mean value theorem, there is $\tau' \in [\theta' - \varepsilon, \theta']$ and $\tau_h \in [\theta_h - \varepsilon, \theta_h]$ such that

$$\begin{aligned} j_y(0, \varepsilon) = {} & \varepsilon\left(-\pi_a\big(\tau', \tilde{\alpha}, \tilde{v}\big)\frac{1}{\varepsilon} - \hat{v}_y\big(\tau', 0, \varepsilon\big) \right) h\big(\tau'\big) + \big(H(\theta_h - \varepsilon) - H\big(\theta'\big)\big) \\ & + \varepsilon\left(\pi_a(\tau_h, \tilde{\alpha}, \tilde{v})\frac{1}{\varepsilon} - \hat{v}_y(\tau_h, 0, \varepsilon) \right) h(\tau_h). \end{aligned}$$

But then, since $\hat{v}_y(\tau', 0, \varepsilon)$ and $\hat{v}_y(\tau_h, 0, \varepsilon)$ are bounded,

$$\lim_{\varepsilon \to 0} j_y(0, \varepsilon) = -\pi_a\big(\theta', \tilde{\alpha}, \tilde{v}\big)h\big(\theta'\big) + H(\theta_h) - H\big(\theta'\big) + \pi_a(\theta_h, \tilde{\alpha}, \tilde{v})h(\theta_h) = 0,$$

since the perturbation is feasible for y in a neighborhood of zero. Rearranging and taking $\eta = -\pi_a(\theta_h, \tilde{\alpha}, \tilde{v})h(\theta_h)$, we are done.

Step 2. Let us next show that if one fixes surplus to equal $\tilde{v}(\theta_l)$ at θ_l and then varies κ, ignoring (3), then profits are single-peaked at $\kappa = H(\theta_h)$. Similarly, if one fixes surplus to equal $\tilde{v}(\theta_h)$ at θ_h and then varies κ, ignoring (2), then profits are single-peaked at $\kappa = H(\theta_l)$.

To formalize this, let $v_l(\theta,\kappa)=\tilde{v}(\theta_l)+\int_{\theta_l}^{\theta}\gamma(\tau,\kappa)\,d\tau$ and let $s_l(\kappa)=(\gamma(\cdot,\kappa),v_l(\cdot,\kappa))$. Since $v_l(\theta_l,\kappa)=\tilde{v}(\theta_l)$ and so is independent of κ, it follows from (18) that on (θ_l,θ_h),

$$\begin{aligned}\frac{d}{d\kappa}M\big(\theta,s_l(\kappa)\big)&=\left(\pi_a\big(\theta,s_l(\kappa)\big)-\frac{H(\theta_h)-H(\theta)}{h(\theta)}\right)\gamma_\kappa(\theta,\kappa)\\&=\left(\frac{\kappa-H(\theta_h)}{h(\theta)}\right)\gamma_\kappa(\theta,\kappa)=_s-\big(\kappa-H(\theta_h)\big),\end{aligned}$$

where the equality follows by Step 1 and the equality in sign since $\gamma_\kappa<0$. But then letting $Y_l(\kappa)\equiv\int_{\theta_l}^{\theta_h}\pi(\theta,s_l(\kappa))h(\theta)\,d\theta$, by Lemma 22, $dY_l(\kappa)/d\kappa=_s-(\kappa-H(\theta_h))$, and so $Y_l(\kappa)$ is strictly single-peaked at $\kappa=H(\theta_h)$.

Similarly, if we define $v_h(\theta,\kappa)=\tilde{v}(\theta_h)-\int_{\theta}^{\theta_h}\gamma(\tau,\kappa)\,d\tau$, then $Y_h(\kappa)\equiv\int_{\theta_l}^{\theta_h}\pi(\theta,\gamma(\cdot,\kappa),v_h(\cdot,\kappa))h(\theta)\,d\theta$ is strictly single-peaked in κ with maximum at $\kappa=H(\theta_l)$, where to show this, one integrates

$$\int_{\theta_l}^{\theta_h}\pi(\theta,\alpha,v)h(\theta)\,d\theta=\int_{\theta_l}^{\theta_h}\left(V\big(\alpha(\theta)\big)+\alpha(\theta)\theta-v(\theta_h)+\int_{\theta}^{\theta_h}\alpha(\tau)\,d\tau\right)h(\theta)\,d\theta$$

by parts to arrive at an analogue to M.

Step 3. Finally, let us show that the optimal κ is $\kappa_o=\tilde{\kappa}(\theta_l,\theta_h)$. Note that one of (2) or (3) must bind; otherwise, reducing $\tilde{v}$ by a small positive constant, holding fixed $\tilde{\alpha}$, is profitable. Assume that $\tilde{v}(\theta_h)>v^{-n}(\theta_h)$. Then $s_l(\kappa)$ is feasible for κ on a neighborhood of κ_o, and so since by Step 2, Y_l is strictly single-peaked with maximum at $H(\theta_h)$, we have $\kappa_o=H(\theta_h)$. Let us see that $\tilde{\kappa}(\theta_l,\theta_h)=H(\theta_h)$ as well, so that $\kappa_o=\tilde{\kappa}(\theta_l,\theta_h)$. Since $\tilde{v}(\theta_l)=v^{-n}(\theta_l)$ and since $(\tilde{\alpha},\tilde{v})=(\gamma(\cdot,\kappa_o),\tilde{v})$ is feasible, we have $\tilde{v}(\theta_h)=v^{-n}(\theta_l)+\int_{\theta_l}^{\theta_h}\gamma(\tau,H(\theta_h))\,d\tau>v^{-n}(\theta_h)$ and so $\iota(\theta_l,\theta_h,H(\theta_h))<0$, and thus by definition of $\tilde{\kappa}$, we have $\tilde{\kappa}(\theta_l,\theta_h)=H(\theta_h)$ as well. Similarly, if $\tilde{v}(\theta_l)>v^{-n}(\theta_l)$, then using Y_h, we must have $\kappa_o=H(\theta_l)=\tilde{\kappa}(\theta_l,\theta_h)$.

Assume finally that (2) and (3) both bind. Then, by definition, $\iota(\theta_l,\theta_h,\kappa_o)=0$. Assume $\kappa_o>H(\theta_h)$. Then

$$\begin{aligned}v_l\big(\theta_h,H(\theta_h)\big)&=\tilde{v}(\theta_l)+\int_{\theta_l}^{\theta_h}\gamma\big(\tau,H(\theta_h)\big)\,d\tau>\tilde{v}(\theta_l)+\int_{\theta_l}^{\theta_h}\gamma(\tau,\kappa_o)\,d\tau\\&=\tilde{v}(\theta_h)=v^{-n}(\theta_h),\end{aligned}$$

so that $s_l(H(\theta_h))$ is feasible, which contradicts the optimality of $(\tilde{\alpha},\tilde{v})$ since Y_l is uniquely maximized at $H(\theta_h)$, and Y_l ignores (3). Hence $\kappa_o\le H(\theta_h)$. Similarly, $\kappa_o\ge H(\theta_l)$ and, thus, $\kappa_o\in[H(\theta_l),H(\theta_h)]$, from which $\kappa_o=\tilde{\kappa}(\theta_l,\theta_h)$, again by definition of $\tilde{\kappa}$. *Q.E.D.*

We now prove that any optimum of the original problem has the form given by Lemma 1.

PROPOSITION 6: *Let s be Nash with NEO. Then, for each n, there is $\kappa^n\in[H(\theta_l^n),H(\theta_h^n)]$ such that $\alpha^n=\gamma^n(\cdot,\kappa^n)$ on (θ_l^n,θ_h^n), where $\kappa^1=0$, and $\kappa^N=1$.*

PROOF: We will show that if on (θ_l,θ_h), (α,v) is not equal to $(\tilde{\alpha},\tilde{v})$—the optimal solution to the relaxed problem—then we can profitably perturb (α,v) in the direction of

$(\tilde{\alpha}, \tilde{v})$.[54] We need this perturbation to respect monotonicity, and the fact that workers both within and outside of (θ_l, θ_h) may be affected. This proof would be substantially simpler if all crossings where transversal, but we know this will fail when firms are not very differentiated.

Let $\check{s}(\delta)$ be given by $\check{\alpha}(\cdot, \delta) = (1-\delta)\alpha + \delta\tilde{\alpha}$ and $\check{v}(\cdot, \delta) = (1-\delta)v + \delta\tilde{v}$, so that $\check{s}(0) = (\alpha, v)$ and $\check{s}(1) = (\tilde{\alpha}, \tilde{v})$. The problem with $\check{s}$ is that when crossings are not transversal, $\check{s}(\delta)$ need not serve all of (θ_l, θ_h) even for small δ. So let $\bar{v} = v^{-n}/2 + v/2$, so that $\bar{v} > v^{-n}$ on (θ_l, θ_h). Now let $\grave{v}(\cdot, \delta) = \max(\bar{v}, \check{v}(\cdot, \delta))$, let $\grave{\alpha}(\cdot, \delta)$ be a subgradient to $\grave{v}(\cdot, \delta)$, and let $\grave{s}(\delta) = (\grave{\alpha}(\cdot, \delta), \grave{v}(\cdot, \delta))$. By construction, $\grave{s}$ always wins on (θ_l, θ_h) and may serve other types as well. Also, since on (θ_l, θ_h), $v > v^{-n}$, then $\grave{s}(0) = (\alpha, v)$. Finally, let $P(\delta)$ be the set upon which $\grave{s}(\delta)$ is profitable and construct $\hat{s}(\delta) = (\hat{\alpha}(\cdot, \delta), \hat{v}(\cdot, \delta))$ from $\grave{s}(\delta)$ as in Proposition 5. We then have

$$\begin{aligned}
\Pi\big(\hat{s}(\delta), s^{-n}\big) &= \int \pi\big(\theta, \hat{s}(\delta)\big)\varphi\big(\theta, \hat{s}(\delta), s^{-n}\big)h(\theta)\, d\theta \\
&\geq \int_{P(\delta)\cap(\theta_l,\theta_h)} \pi\big(\theta, \hat{s}(\delta)\big)\varphi\big(\theta, \hat{s}(\delta), s^{-n}\big)h(\theta)\, d\theta \\
&= \int_{P(\delta)\cap(\theta_l,\theta_h)} \pi\big(\theta, \grave{s}(\delta)\big)h(\theta)\, d\theta \\
&\geq \int_{\theta_l}^{\theta_h} \pi\big(\theta, \grave{s}(\delta)\big)h(\theta)\, d\theta.
\end{aligned}$$

The first inequality follows since $\pi(\cdot, \hat{s}(\delta)) \geq 0$, the second equality since $\hat{s}(\delta)$ and $\grave{s}(\delta)$ agree on $P(\delta)$ and $\varphi(\cdot, \grave{s}(\delta)) = 1$ on (θ_l, θ_h), and the second inequality since $\pi(\theta, \grave{s}(\delta)) \leq 0$ outside of $P(\delta)$.

It is thus enough to show that for δ sufficiently small, $\int_{\theta_l}^{\theta_h} \pi(\theta, \grave{s}(\delta))h(\theta)\, d\theta > \int_{\theta_l}^{\theta_h} \pi(\theta, \alpha, v)h(\theta)\, d\theta$, since by PS, $\varphi(\theta, s) = 0$ outside of $[\theta_l, \theta_h]$. Because $\grave{s}(0) = (\alpha, v)$, it is sufficient that $\frac{d}{d\delta}\int_{\theta_l}^{\theta_h} \pi(\theta, \grave{s}(\delta))h(\theta)\, d\theta|_{\delta=0} > 0$. But $\frac{d}{d\delta}\int_{\theta_l}^{\theta_h} \pi(\theta, \grave{s}(\delta))h(\theta)\, d\theta|_{\delta=0} = \frac{d}{d\delta}\int_{\theta_l}^{\theta_h} \pi(\theta, \check{s}(\delta))h(\theta)\, d\theta|_{\delta=0}$, since for each $\theta \in (\theta_l, \theta_h)$, $v(\theta) > \bar{v}(\theta)$, and so at $\delta = 0$, $(\check{\alpha}(\theta, \delta))_\delta = (\grave{\alpha}(\theta, \delta))_\delta$ and $(\check{v}(\theta, \delta))_\delta = (\grave{v}(\theta, \delta))_\delta$. And since $(\tilde{\alpha}, \tilde{v})$ is the unique solution on (θ_l, θ_h) to the relaxed problem $\mathcal{P}(\theta_l, \theta_h)$, and since $\check{s}(0) = (\alpha, v)$ and so is feasible in $\mathcal{P}(\theta_l, \theta_h)$,

$$\begin{aligned}
\int_{\theta_l}^{\theta_h} \pi(\theta, \tilde{\alpha}, \tilde{v})h(\theta)\, d\theta &= \int_{\theta_l}^{\theta_h} \pi\big(\theta, \check{s}(1)\big)h(\theta)\, d\theta \\
&> \int_{\theta_l}^{\theta_h} \pi\big(\theta, \check{s}(0)\big)h(\theta)\, d\theta \\
&= \int_{\theta_l}^{\theta_h} \pi(\theta, \alpha, v)h(\theta)\, d\theta.
\end{aligned}$$

[54] It is not important how $(\tilde{\alpha}, \tilde{v})$ is defined outside of (θ_l, θ_h) as long as monotonicity, continuity of actions, and the integral condition hold.

Now $\check{s}$ is linear in δ, and $\pi(\theta, \cdot, \cdot)$ is concave in the action and utility, and thus $\int_{\theta_l}^{\theta_h} \pi(\theta, \check{s}(\delta))h(\theta)\, d\theta$ is concave in δ. But then, by the previous strict inequality, it must be that $\frac{d}{d\delta}\int_{\theta_l}^{\theta_h} \pi(\theta, \check{s}(\delta))h(\theta)\, d\theta|_{\delta=0} > 0$.

Finally, let us see that $\kappa^N = 1$ (where the proof that $\kappa^1 = 0$ is similar). Note first that for $\theta \geq \theta_l^N$, $v^{-N} = v^{N-1}$. Thus, by definition $v^N(\theta_l^N) = v^{N-1}(\theta_l^N)$. But by NEO, for all $\theta > \theta_l^N$, we have $\alpha^N(\theta) > \alpha^N(\theta_l^N) \geq a_e^{N-1} \geq \alpha^{N-1}(\theta)$ and, thus, $v_\theta^N > v_\theta^{N-1}$. Thus, $v^N(1) > v^{N-1}(1)$ and, hence, $\iota(\theta_l^N, 1, \kappa^N) < 0$, which by definition of $\tilde{\kappa}$ can only hold if $\kappa^N = H(1) = 1$. *Q.E.D.*

S1.4. *Proofs for Section 4.1.6*

PROPOSITION 7: *Let s be Nash with NEO. Then* (5) *holds*.

PROOF: Fix n. We will prove (5) for θ_h, with the case at θ_l analogous. We will consider perturbations that add or subtract workers in a continuous fashion immediately to the right or left of θ_h. We need to respect monotonicity and the integral condition, and ensure that our perturbed menus continue to serve an interval of workers (as opposed to a disconnected set thereof).[55]

If $\theta_l^{n+1} > \theta_h^n = \theta_h$, then (5) is automatic, since by Corollary 1 and the definition of PS, $\pi(\theta_h, \alpha, v) = 0$ and $\alpha(\theta_h) = \alpha^{n+1}(\theta_h)$. So assume $\theta_l^{n+1} = \theta_h$, and note that by Proposition 6, α is strictly increasing to the left of θ_h and $a^{-n} = \alpha^{n+1}$ is strictly increasing to the right of θ_h.

Step 1. Let us first define a basic perturbation $(\hat{\alpha}(\cdot, y), \hat{v}(\cdot, y))$ indexed by y. Fix n and $0 < \varepsilon < \theta_h - \theta_l$. For y positive or negative, define $\hat{\alpha}(\theta, y)$ as $\alpha(\theta)$ if $\theta < \theta_h - \varepsilon$ and $\max\{\alpha(\theta_h - \varepsilon), \min\{\alpha(\theta) + y, \alpha(\theta_h)\}\}$ if $\theta \geq \theta_h - \varepsilon$. That is, above $\theta_h - \varepsilon$, change actions by y, but censor them to be above $\alpha(\theta_h - \varepsilon)$ and below $\alpha(\theta_h)$. Note that $\hat{\alpha}$ is continuous and $\hat{\alpha}(\cdot, y)$ is increasing. Define $\hat{v}(\theta, y) = v(\theta_l) + \int_{\theta_l}^{\theta} \hat{\alpha}(\tau, y)\, d\tau$. Because $\hat{\alpha}(\tau, y)$ is bounded and for each y, differentiable in y for almost all τ, with $\hat{\alpha}_y(\tau, y) \in \{0, 1\}$ wherever it is defined, $\hat{v}$ is continuously differentiable in (θ, y) wherever $\theta > \theta_h - \varepsilon$, with $\hat{v}_y(\theta_h, 0) = \varepsilon > 0$.

Step 2. Let us now use the basic perturbation to add or subtract types near θ_h. Define $\hat{y}(\theta')$ implicitly by $\hat{v}(\theta', \hat{y}(\theta')) - v^{-n}(\theta') = 0$. Then $\hat{y}$ is well defined on an interval around θ_h, with

$$\hat{y}_{\theta'}(\theta') = \frac{a^{-n}(\theta') - \hat{\alpha}(\theta', \hat{y}(\theta'))}{\hat{v}_y(\theta', \hat{y}(\theta'))} \geq 0. \tag{19}$$

Further, when $\hat{y}(\theta') > 0$, then $\hat{v}(\theta, \hat{y}(\theta')) - v^{-n}(\theta) > 0$ for all $\theta \in (\theta_l, \theta_h]$ and, hence, any crossing of zero by $\hat{v}(\cdot, \hat{y}(\theta')) - v^{-n}(\cdot)$ above θ_l occurs where $\theta > \theta_h$ and, thus, where

$$\big(\hat{v}(\theta, \hat{y}(\theta')) - v^{-n}(\theta)\big)_\theta = \hat{\alpha}(\theta, \hat{y}(\theta')) - a^{-n}(\theta) = \alpha(\theta_h) - a^{-n}(\theta) < 0,$$

since $a^{-n}(\theta) > a^{-n}(\theta_h) \geq \alpha(\theta_h)$. Thus, indeed θ' is the unique crossing, and so $\varphi = 1$ for all $\theta \in (\theta_l, \theta')$ and $\varphi = 0$ outside of $[\theta_l, \theta']$. Similarly, if $\hat{y}(\theta') < 0$, then any crossing of zero by $\hat{v}(\cdot, \hat{y}(\theta')) - v^{-n}(\cdot)$ above θ_l occurs where $\theta \in (\theta_h - \varepsilon, \theta_h)$ and, thus, where $\hat{\alpha}(\theta, \hat{y}(\theta')) < \alpha(\theta)$, and, hence,

$$\big(\hat{v}(\theta, \hat{y}(\theta')) - v^{-n}(\theta)\big)_\theta = \hat{\alpha}(\theta, \hat{y}(\theta')) - a^{-n}(\theta) < \alpha(\theta) - a^{-n}(\theta) < 0,$$

[55]This proof would be much easier if all crossing were strictly transversal. Then we could use $\gamma(\cdot, \kappa)$ and vary κ, holding fixed $v(\theta_l)$.

since $\alpha(\theta) < \alpha(\theta_h) \leq a^n_e \leq a^{-n}(\theta)$ by NEO, and so again $\varphi = 1$ for all $\theta \in (\theta_l, \theta')$ and $\varphi = 0$ outside of $[\theta_l, \theta']$.

Step 3. Since this perturbation is feasible, it must be unprofitable. Let us show that this implies (5). To do so, let $j(\theta')$ be the profit from the perturbation. Then

$$j(\theta') = \int_{\theta_l}^{\theta_h-\varepsilon} \pi(\theta, \alpha, v)h(\theta)\, d\theta + \int_{\theta_h-\varepsilon}^{\theta'} \pi\big(\theta, \hat{\alpha}\big(\cdot, \hat{y}(\theta')\big), \hat{v}\big(\cdot, \hat{y}(\theta')\big)\big)h(\theta)\, d\theta,$$

since for $\theta < \theta_h - \varepsilon$, $\hat{\alpha} = \alpha$ and $\hat{v} = v$. Thus,

$$\begin{aligned} j_{\theta'}(\theta') &= \pi\big(\theta', \hat{\alpha}\big(\cdot, \hat{y}(\theta')\big), \hat{v}\big(\cdot, \hat{y}(\theta')\big)\big)h(\theta') \\ &\quad + \hat{y}_\theta(\theta') \int_{\theta_h-\varepsilon}^{\theta'} \big(\pi_a\big(\theta, \hat{\alpha}\big(\cdot, \hat{y}(\theta')\big), \hat{v}\big(\cdot, \hat{y}(\theta')\big)\big)\hat{\alpha}_y\big(\theta, \hat{y}(\theta')\big) - \hat{v}_y\big(\theta, \hat{y}(\theta')\big)\big)h(\theta)\, d\theta. \end{aligned}$$

To evaluate this at $\theta' = \theta_h$, note that $\hat{y}(\theta_h) = 0$, $\hat{\alpha}(\theta, 0) = \alpha(\theta)$, and $\hat{\alpha}^n_y(\theta, 0) = 1$ for $\theta \in (\theta_h - \varepsilon, \theta_h)$ and $= 0$ outside of $[\theta_h - \varepsilon, \theta_h]$, and that $\hat{v}(\cdot, 0) = v$, so using (19) and $\hat{v}_y(\theta_h, 0) = \varepsilon$,

$$\begin{aligned} j_{\theta'}(\theta_h) &= \pi(\theta_h, \alpha, v)h(\theta_h) + \frac{a^{-n}(\theta_h) - \alpha(\theta_h)}{\varepsilon} \int_{\theta_h-\varepsilon}^{\theta_h} \big(\pi_a(\theta, \alpha, v) - \hat{v}_y(\theta, 0)\big)h(\theta)\, d\theta \\ &= \pi(\theta_h, \alpha, v)h(\theta_h) + \big(a^{-n}(\theta_h) - \alpha(\theta_h)\big)\big(\pi_a(\tau, \alpha, v) - \hat{v}_y(\tau, 0)\big)h(\tau) \end{aligned}$$

for some $\tau \in [\theta_h - \varepsilon, \theta_h]$ by the mean value theorem and where we note that $\hat{v}_y(\tau, 0) = \tau - (\theta_h - \varepsilon) \in [0, \varepsilon]$. Since (α, v) is optimal, we have $j_{\theta'}(\theta_h) = 0$. Taking $\varepsilon \to 0$, we have $\tau \to \theta_h$ and, hence, cancelling $h(\theta_h)$, we arrive at $0 = \pi(\theta_h, \alpha, v) + (a^{-n}(\theta_h) - \alpha(\theta_h))\pi_a(\theta_h, \alpha, v)$. Thus, (5) holds, and we are done. *Q.E.D.*

S2. NUMERICAL ANALYSIS AND FIGURE 1

We take four firms, with $\mathcal{V}^n(a) = \zeta^n + \beta^n \log(\rho + a)$, where with some mild abuse we set $\rho = 0$. Assume that $\mathcal{V}(a, \theta) = -(3 - \theta)a$ and that h is uniform on $[0, 1]$. From IO, $\gamma^n(\theta, \kappa^n) = \beta^n/(3 + \kappa^n - 2\theta)$ for $n = 1, \dots, 4$, so stacking holds if $\beta^{n+1}/\beta^n > 2$. So assume $\beta^1 = 1$, $\beta^2 = 4$, $\beta^3 = 9$, $\beta^4 = 20$, $\zeta^1 = 2.5$, $\zeta^2 = 3$, $\zeta^3 = -2$, and $\zeta^4 = -23$, where the values of ζ^n are chosen so that each firm is relevant (the fact that $\zeta^1 < \zeta^2$ reflects that in this parameterization, ζ^n is the value of $\mathcal{V}^n(1)$ and that the point where $\mathcal{V}^1 = \mathcal{V}^2$ is to the left of 1). Integrating γ^n yields $v^n(\theta) = v^n(0) + (\beta^n/2)\log((3 + \kappa^n)/(3 + \kappa^n - 2\theta))$, so the nine equations that characterize equilibria are

$$\begin{aligned} v^n(\theta^n) - v^{n+1}(\theta^n) &= 0, \quad n = 1, 2, 3, \\ \pi^n\big(\theta^n, \gamma^n(\cdot, \kappa^n), v^n\big) + \big(\kappa^n - \theta^n\big)\big(\gamma^{n+1}(\theta^n) - \gamma^n(\theta^n)\big) &= 0, \quad n = 1, 2, 3, \\ \pi^{n+1}\big(\theta^n, \gamma^{n+1}(\cdot, \kappa^{n+1}), v^{n+1}\big) + \big(\kappa^{n+1} - \theta^n\big)\big(\gamma^n(\theta^n) - \gamma^{n+1}(\theta^n)\big) &= 0, \quad n = 1, 2, 3, \end{aligned}$$

with nine unknowns κ^2, κ^3, θ^1, θ^2, θ^3, $v^1(0)$, $v^2(0)$, $v^3(0)$, and $v^4(0)$. Solving numerically, inserting the values for $v^n(0)$ and κ^n into v^n, and graphing gives us Figure 1.[56]

[56]The solution is $\kappa^2 = 0.53318$, $\kappa^3 = 0.84976$, $\theta^1 = 0.26105$, $\theta^2 = 0.68527$, $\theta^3 = 0.91815$, $v^1(0) = 0.1502$, $v^2(0) = -0.074$, $v^3(0) = -1.0726$, and $v^4(0) = -4.3006$.

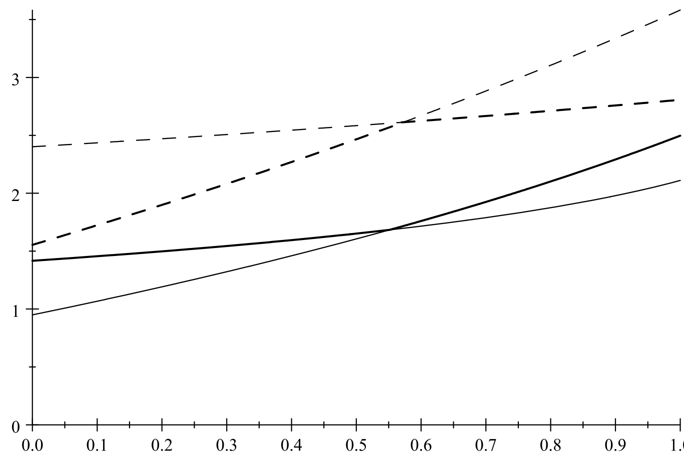

FIGURE 6.—A complete welfare reversal. The thin solid lines are the incomplete information equilibrium surplus of the two firms, with the thicker solid line the equilibrium surplus as a function of the type. The thin dashed lines are the efficient surplus each firm can offer, with the thicker dashed line the equilibrium surplus under incomplete information.

S3. AN EXTREME WELFARE REVERSAL EXAMPLE

Examples where all types prefer complete information are easy to build if one brings h into play. To see this, modify the distribution over types so that there is $\tau < 1/2$ weight at each of $\theta = 0$ and $\theta = 1$, and let the remaining weight be uniform. (The point masses could be dispensed with, but make things simple.) When τ is zero, we are back in the uniform case, while as τ grows, there are more consumers that are essentially captive to one firm or the other, muting competition. Then it is a routine calculation that stacking holds for $\beta^1 = 1$, $\beta^2 = 5$, and $\tau = 0.2$, and it can be calculated numerically that if $k_1 = 4.5$ and $k_2 = 4$, then the solution for $\tau = 0.2$ is given by $\theta^1 = 0.55129$, $v^1(0) = 1.4169$, and $v^2(0) = 0.94957$. As can be seen from Figure 6, all types prefer the complete information case.

S4. OMITTED PROOFS FOR SECTION 5

S4.1. *Proofs for Section 5.2*

PROOF OF THEOREM 4: We proceed in a series of steps.

Step 0. Define $\tilde{a}$ by $V_a(\tilde{a}, \bar{z}) = -1 - (1/h(1))$. Define $z^a(a) = \arg\max V(a, \cdot)$ as the optimal technology to implement action a, and define $a^\theta(\theta) = \arg\max(V(a, z^a(a)) + a\theta)$ as the action that, when implemented with technology $z^a(a)$, maximizes the surplus for type θ. Define $z^\theta(\theta) \equiv z^a(a^\theta(\theta))$, so that type θ is best served by a firm with technology $z^\theta(\theta)$ and action $a^\theta(\theta)$. For any given $\kappa \in [0, 1]$, let $\gamma(\cdot, \kappa, z)$ solve (1) with technology z. Then $\tilde{a}$ is an upper bound for $\gamma(\theta, \kappa, z)$ for all (θ, κ, z) with $\kappa \in [0, 1]$, and z^a, a^θ, and z^θ are positive, well defined, continuously differentiable, and bounded away from zero and ∞ for all $\theta \in [0, 1]$ and $a \in [0, \tilde{a}]$.

That $\tilde{a}$ is a relevant upper bound on $\gamma(\theta, \kappa, z)$ follows from (1). The properties of z^a, a^θ, and z^θ follow from our ambient assumptions and the implicit function theorem. Formally, for any $\theta \in [0, 1]$ and $z \in [0, \bar{z}]$, $\gamma(\theta, \kappa, z)$ uniquely solves $V_a = -\theta + (\kappa - H)/h$. Thus, since $V_{az} > 0$,

$$V_a\big(\gamma(\theta,\kappa,z),\bar{z}\big) \geq V_a\big(\gamma(\theta,\kappa,z),z\big) = -\theta + \frac{\kappa - H(\theta)}{h(\theta)} \geq -1 - \frac{1}{h(1)} = V_a(\tilde{a},\bar{z}),$$

and since $V_{aa} < 0$, we obtain that $\gamma(\theta,\kappa,z) \leq \tilde{a}$ for all (θ,z) and $\kappa \in [0,1]$. Note that z^a is implicitly defined by $V_z(a,z^a(a)) = 0$ and, hence, $z^a_a(a) = -V_{az}(a,z^a = (a))/V_{zz}(a,z^a(a)) > 0$. Consider $j(a,\theta) = V(a,z^a(a)) + a\theta$. Then $j_a(0,\theta) = V_a(0,z^a(0)) + \theta \geq V_a(0,0) > 0$, and for all $a \geq \tilde{a}$, $j_a(a,\theta) = V_a(a,z^a(a)) + \theta \leq V_a(a,\bar{z}) + 1 \leq V_a(\tilde{a},\bar{z}) + 1 < 0$. Thus, j has a maximum on $(0,\tilde{a})$. Further,

$$\begin{aligned} j_{aa}(a,\theta) &= \big(V_a\big(a,z^a(a)\big)\big)_a \\ &= V_{aa}\big(a,z^a(a)\big) + V_{az}\big(a,z^a(a)\big)z^a_a(a) \\ &= V_{aa}\big(a,z^a(a)\big) - V_{az}\big(a,z^a(a)\big)\frac{V_{az}\big(a,z^a(a)\big)}{V_{zz}\big(a,z^a(a)\big)} \\ &=_s -\big(V_{aa}\big(a,z^a(a)\big)V_{zz}\big(a,z^a(a)\big) - V^2_{az}\big(a,z^a(a)\big)\big) < 0, \end{aligned}$$

since we assumed that the Hessian of V was strictly positive. Hence, $j(\cdot,\theta)$ has a unique maximum $a^\theta(\theta) \in (0,\tilde{a})$, and the pair $a^\theta(\theta)$ and $z^\theta(\theta) \equiv z^a(a^\theta(\theta))$ jointly maximize $V(a,z) + a\theta$ for each θ, and, thus, $v_*(\theta) = j(a^\theta(\theta),\theta)$. Differentiating the identity $j_a(a^\theta(\theta),\theta) = 0$ and using the expression for j_{aa}, we have $a^\theta_\theta(\theta) = -1/j_{aa}(a^\theta(\theta),\theta) > 0$ and, hence, $z^\theta_\theta(\theta) = z^a_a(a^\theta(\theta))a^\theta_\theta(\theta) > 0$.

Since the type space $[0,1]$ is compact, since all relevant actions will come from the compact interval $[0,\tilde{a}]$ and technologies from $[0,z]$, and since V is $\mathcal{C}^2$, we have that $\ell_{a^\theta_\theta} > 0$, and $\ell_{z^\theta_\theta} > 0$, where for any function g, we will use ℓ_g as shorthand for the infimum of g over its domain.

Let the maximum surplus a firm with technology z can offer to type θ be $\bar{v}(\theta,z) = \max_a(V(a,z) + \theta a) = V(\bar{a}(\theta,z),z) + \theta\bar{a}(\theta,z)$, where $\bar{a}$ is defined by $V_a(\bar{a}(\theta,z),z) + \theta = 0$, and, hence,

$$\bar{a}_\theta(\theta,z) = \frac{-1}{V_{aa}\big(\bar{a}(\theta,z),z\big)} > 0, \quad \text{and} \quad \bar{a}_z(\theta,z) = \frac{-V_{az}\big(\bar{a}(\theta,z),z\big)}{V_{aa}\big(\bar{a}(\theta,z),z\big)} > 0.$$

Each of these is finite on the compact set $[0,1] \times [0,\bar{z}]$, and, hence, has a strictly positive uniform lower bound, $\ell_{\bar{a}_\theta}$ and $\ell_{\bar{a}_z}$, and finite upper bound, $\upsilon_{\bar{a}_\theta}$ and $\upsilon_{\bar{a}_z}$.

In what follows fix N, $\{z^n\}_{n=1}^N$, and an equilibrium s.

Step 1. Let us show first a lower bound on how much an entering firm can earn as a function of how far apart its competitors are. Partition the interval $[z^\theta(0), z^\theta(1)]$ by those elements of $\{z^n\}_{n=1}^N$ that lie in $[z^\theta(0), z^\theta(1)]$. Let d_z be the length of the longest element of the partition. We claim there is $\rho_1 \in (0,\infty)$ such that an entrant can earn at least $\rho_1 d_z^3$.

Let $[z_l, z_h]$ be a largest element of the partition, so that $z_h - z_l = d_z$. Associated with $[z_l, z_h]$ is an interval of types $[\theta_l,\theta_h] = [(z^\theta)^{-1}(z_l), (z^\theta)^{-1}(z_h)]$, where, by the mean value theorem,

$$\theta_h - \theta_l \geq \frac{d_z}{\upsilon_{z^\theta_\theta}}. \tag{20}$$

Let $\tilde{\theta} = (\theta_l + \theta_h)/2$. Since V is concave and there are no firms in (z_l, z_h), $\max\{\bar{v}(\tilde{\theta}, z_l), \bar{v}(\tilde{\theta}, z_h)\} \geq v^O(\tilde{\theta})$, since z_l does a better job of serving $\tilde{\theta}$ than any $z < z_l$, and similarly for

$z > z_h$, and where, if, for example, $z_l = z^\theta(0)$, so that z_l need not be an existing firm, then the inequality holds a fortiori.

Note that $v_*(\tilde{\theta}) = \bar{v}(\tilde{\theta}, z^\theta(\tilde{\theta}))$ and $\bar{v}(\theta_l, z^\theta(\theta_l)) - \bar{v}(\theta_l, z_l) = 0$, since $z^\theta(\theta_l) = z_l$. Hence,

$$\begin{aligned} v_*(\tilde{\theta}) - \bar{v}(\tilde{\theta}, z_l) &= \bar{v}\big(\tilde{\theta}, z^\theta(\tilde{\theta})\big) - \bar{v}(\tilde{\theta}, z_l) \\ &= \bar{v}\big(\theta_l, z^\theta(\theta_l)\big) - \bar{v}(\theta_l, z_l) + \int_{\theta_l}^{\tilde{\theta}} \big(\big(\bar{v}\big(\theta, z^\theta(\theta)\big)\big)_\theta - \bar{v}_\theta(\theta, z_l)\big)\, d\theta \\ &= \int_{\theta_l}^{\tilde{\theta}} \big(\big(\bar{v}\big(\theta, z^\theta(\theta)\big)\big)_\theta - \bar{v}_\theta(\theta, z_l)\big)\, d\theta \\ &= \int_{\theta_l}^{\tilde{\theta}} \big(\bar{v}_\theta\big(\theta, z^\theta(\theta)\big) - \bar{v}_\theta(\theta, z_l)\big)\, d\theta \\ &= \int_{\theta_l}^{\tilde{\theta}} \big(\bar{a}\big(\theta, z^\theta(\theta)\big) - \bar{a}(\theta, z_l)\big)\, d\theta, \end{aligned}$$

where the fourth and fifth equalities use the envelope theorem. But again since $z^\theta(\theta_l) = z_l$, $\bar{a}(\theta, z^\theta(\theta_l)) - \bar{a}(\theta, z_l) = 0$, so

$$\begin{aligned} \bar{a}\big(\theta, z^\theta(\theta)\big) - \bar{a}(\theta, z_l) &= \int_{\theta_l}^{\theta} \frac{\partial}{\partial \tau} \bar{a}\big(\theta, z^\theta(\tau)\big)\, d\tau \\ &= \int_{\theta_l}^{\theta} \bar{a}_z\big(\theta, z^\theta(\tau)\big) z^\theta_\theta(\tau)\, d\tau \geq (\theta - \theta_l)\ell_{\bar{a}_z}\ell_{z^\theta_\theta} \end{aligned}$$

and, hence, substituting,

$$v_*(\tilde{\theta}) - \bar{v}(\tilde{\theta}, z_l) \geq \ell_{\bar{a}_z}\ell_{z^\theta_\theta} \int_{\theta_l}^{\tilde{\theta}} (\theta - \theta_l)\, d\theta = \ell_{\bar{a}_z}\ell_{z^\theta_\theta} \frac{(\tilde{\theta} - \theta_l)^2}{2} = \frac{\ell_{\bar{a}_z}\ell_{z^\theta_\theta}}{8}(\theta_h - \theta_l)^2$$

and similarly for z_h. Hence, using (20),

$$v_*(\tilde{\theta}) - \max\big\{\bar{v}(\tilde{\theta}, z_l), \bar{v}(\tilde{\theta}, z_h)\big\} \geq \frac{\ell_{\bar{a}_z}\ell_{z^\theta_\theta}}{8}(\theta_h - \theta_l)^2 \geq \frac{\ell_{\bar{a}_z}\ell_{z^\theta_\theta}}{8 \upsilon^2_{z^\theta_\theta}} d_z^2 \equiv \delta. \tag{21}$$

Let $\tilde{z} = z^\theta(\tilde{\theta})$ enter, offer $\bar{a}(\cdot, \tilde{z})$, and offer surplus $\bar{v}(\cdot, \tilde{z}) - \delta/2$. This earns $\delta/2$ on each type served. Let $\hat{\theta}_l$ be the lowest type with $\hat{\theta}_l \geq \theta_l$ who accepts versus $\bar{v}(\hat{\theta}_l, z_l)$, and let $\hat{\theta}_h$ be the highest type with $\hat{\theta}_h \leq \theta_h$ who accepts versus $\bar{v}(\hat{\theta}_h, z_h)$. Since $\bar{v}(\cdot, z_l)$ is an upper bound on the surplus that z_l (if such a firm even exists) will offer in any post-entry equilibrium, and similarly for $\bar{v}(\cdot, z_h)$, any type between $\hat{\theta}_l$ and $\hat{\theta}_h$ is certainly attracted to the entrant. Note in particular that since V is strictly supermodular, $\bar{v}(\cdot, \tilde{z})$ is steeper than $\bar{v}(\cdot, z_l)$, so no type above $\tilde{\theta}$ prefers any firm at or below z_l to $\tilde{z}$, and, similarly, no type below $\tilde{\theta}$ prefers any firm at or above z_h to $\tilde{z}$.

If $\hat{\theta}_l = \theta_l$ or $\hat{\theta}_h = \theta_h$, then $\hat{\theta}_h - \hat{\theta}_l \geq (\theta_h - \theta_l)/2$, and the firm earns at least $((\theta_h - \theta_l)/2)\ell_h(\delta/2) \geq (\ell_h \ell_{\bar{a}_z}\ell_{z^\theta_\theta}/32\upsilon^3_{z^\theta_\theta})d_z^3$, using (20) and the definition of δ. Otherwise, $\hat{\theta}_l$ is defined by $\bar{v}(\hat{\theta}_l, \tilde{z}) - \bar{v}(\hat{\theta}_l, z_l) = \delta/2$. But $\bar{v}(\tilde{\theta}, \tilde{z}) - \bar{v}(\tilde{\theta}, z_l) = v_*(\tilde{\theta}) - \bar{v}(\tilde{\theta}, z_l) \geq \delta$ by ((21)), so

$(\bar{v}(\tilde{\theta}, \tilde{z}) - \bar{v}(\tilde{\theta}, z_l)) - (\bar{v}(\hat{\theta}_l, \tilde{z}) - \bar{v}(\hat{\theta}_l, z_l)) \geq \delta/2$, from which, since $(\bar{v}(\theta, \tilde{z}) - \bar{v}(\theta, z_l))_\theta = \bar{a}(\theta, \tilde{z}) - \bar{a}(\theta, z_l) \leq (\tilde{z} - z_l)\upsilon_{\bar{a}_z}$, we have $\tilde{\theta} - \hat{\theta}_l \geq \delta/(2(\tilde{z} - z_l)\upsilon_{\bar{a}_z})$, and similarly for $\hat{\theta}_h - \tilde{\theta}$, and, thus,

$$\hat{\theta}_h - \hat{\theta}_l \geq \frac{\delta}{2\upsilon_{\bar{a}_z}}\left(\frac{1}{\tilde{z} - z_l} + \frac{1}{z_h - \tilde{z}}\right) = \frac{\delta}{2\upsilon_{\bar{a}_z}} \frac{z_h - z_l}{(\tilde{z} - z_l)(z_h - \tilde{z})} \geq \frac{\delta}{2\upsilon_{\bar{a}_z}} \frac{4}{d_z},$$

where the second inequality follows since $(\tilde{z} - z_l)(z_h - \tilde{z}) \leq (z_h - z_l)^2/4$, and so using (21), the firm earns at least $(\hat{\theta}_h - \hat{\theta}_l)\ell_h\delta/2 \geq \delta^2\ell_h/(\upsilon_{\bar{a}_z} d_z) = (\ell_{\bar{a}_z}^2 \ell_{z_\theta^\theta}^2 \ell_h / 64\upsilon_{z_\theta^\theta}^4 \upsilon_{\bar{a}_z})d_z^3$. Thus, we have established that an entrant can earn at least $\rho_1 d_z^3$, where $\rho_1 = \min\{\ell_{\bar{a}_z}\ell_{z_\theta^\theta}\ell_h/32\upsilon_{z_\theta^\theta}^3, (\ell_{\bar{a}_z}^2 \ell_{z_\theta^\theta}^2 \ell_h/64\upsilon_{z_\theta^\theta}^4 \upsilon_{\bar{a}_z})\}$.

Step 2. Let us next show that for each firm $1 < n < N$, there is an upper bound on how much firm n can earn as function of how far apart its competitors are. In particular, we claim there is $\rho_2 \in (0, \infty)$ such that each such firm earns at most $\rho_2(z^{n+1} - z^{n-1})^3$.

Fix $1 < n < N$, let n serve $[\theta_l, \theta_h]$, choose $\hat{\theta} \in [\theta_l, \theta_h]$, and let $\hat{a} = \alpha^n(\hat{\theta})$. Our first task is to show that

$$V(\hat{a}, z^n) - \max_{n'} V(\hat{a}, z^{n'}) \leq \frac{-\ell_{V_{zz}}}{2}(z^{n+1} - z^{n-1})^2. \tag{22}$$

If $z^n = z^{n+1}$, then this holds trivially, since the lhs is zero. So assume $z^{n+1} > z^n$. Recall by NP that $V(\hat{a}, z^n) \geq V(\hat{a}, z^{n+1})$. Let $\hat{z} = z^a(\hat{a}) = \arg\max_z V(\hat{a}, z)$ be the technology that is most efficient at $\hat{a}$. We claim that $\hat{z} \in [z^{n-1}, z^{n+1}]$. To see this, assume $\hat{z} > z^{n+1}$ (the case $\hat{z} < z^{n-1}$ is similar). Then, since $z^{n+1} > z^n$, there is $p \in (0, 1)$ such that $z^{n+1} = pz^n + (1 - p)\hat{z}$. But then, since V is strictly concave in z, $V(\hat{a}, z^{n+1}) > pV(\hat{a}, z^n) + (1 - p)V(\hat{a}, \hat{z}) > V(\hat{a}, z^{n+1})$, a contradiction. But then, since $V_z(\hat{a}, \hat{z}) = 0$, we have by Taylor's theorem that

$$\begin{aligned} V(\hat{a}, z^n) - \max_{n'} V(\hat{a}, z^{n'}) &\leq V(\hat{a}, \hat{z}) - V(\hat{a}, z^{n-1}) \\ &\leq \frac{-\ell_{V_{zz}}}{2}(\hat{z} - z^{n-1})^2 \\ &\leq \frac{-\ell_{V_{zz}}}{2}(z^{n+1} - z^{n-1})^2, \end{aligned}$$

where the last inequality follows since $\hat{z} \in [z^{n-1}, z^{n+1}]$, and we have established (22).

Recall from Section 4.1.2 that $\pi^n(\hat{\theta}, \hat{a}, v^n) \leq V^n(\hat{a}) - \max_{n' \neq n} V^{n'}(\hat{a})$. Thus, using (22), the total profit of firm n is at most $-\ell_{V_{zz}}\upsilon_h(z^{n+1} - z^{n-1})^2(\theta_h - \theta_l)/2$, and it is enough to show that $(\theta_h - \theta_l)$ is bounded by a multiple of $(z^{n+1} - z^{n-1})$. Let $a^z = (z^a)^{-1}$, so that at any z, $a^z(z)$ is the action that z is uniquely best at providing. But since $[\alpha^n(\theta_l), \alpha^n(\theta_h)] \subseteq [a_e^{n-1}, a_e^n] \subseteq (a^z(z^{n-1}), a^z(z^{n+1}))$ and since $\alpha^n = \gamma^n(\cdot, \kappa^n)$, we have $\gamma^n(\theta_h, \kappa^n) - \gamma^n(\theta_l, \kappa^n) \leq a^z(z^{n+1}) - a^z(z^{n-1})$, so $\theta_h - \theta_l \leq (\upsilon_{a_{\tilde{z}}^z}/\ell_{\gamma_\theta^n})(z^{n+1} - z^{n-1})$, where $\ell_{\gamma_\theta^n} > 0$ is taken over $\theta \in [0, 1]$, $\kappa^n \in [0, 1]$, and $\upsilon_{a_{\tilde{z}}^z} < \infty$ is taken over the compact subset $z^a([0, \tilde{a}])$. Thus, the total profit of firm n is at most $\rho_2(z^{n+1} - z^{n-1})^3$, where $\rho_2 = (-\ell_{V_{zz}}\upsilon_h\upsilon_{a_{\tilde{z}}^z})/2\ell_{\gamma_\theta^n}$, and we are done.

Step 3. There is ρ such that $1/(\rho F^{1/3}) \leq N \leq (\rho/F^{1/3}) + 2$ and d_z is $O(1/N)$.

This follows from the last two steps since no firms wish to enter or exit. For entry not to be profitable, we must have $\rho_1 d_z^3 \leq F$ and, thus, $d_z \leq (F/\rho_1)^{1/3}$; hence, since $N \geq (z^\theta(1) - z^\theta(0))/d_z$, we have $N \geq (z^\theta(1) - z^\theta(0))(\rho_1)^{1/3}/F^{1/3}$. Similarly, for firm $1 < n < N$ not to want to exit, we must have $F \leq \rho_2(z^{n+1}-z^{n-1})^3$ and so $(F/\rho_2)^{1/3} \leq (z^{n+1}-z^{n-1})$. But

$$(N-2)\left(\frac{F}{\rho_2}\right)^{\frac{1}{3}} \leq \sum_{n=2}^{N-1}\left(z^{n+1}-z^{n-1}\right) = \sum_{n=2}^{N-1}\left(z^{n+1}-z^{n}\right) + \sum_{n=2}^{N-1}\left(z^{n}-z^{n-1}\right) \leq 2\bar{z},$$

so rearranging the end terms, $N \leq (2\bar{z}\rho_2^{1/3}/F^{1/3})+2$, and taking ρ large enough, we have established the first claim. That d_z is $O(1/N)$ follows immediately, since $d_z \leq (F/\rho_1)^{1/3}$.

Step 4. Let $d_1 = \max_{a\in[a^\theta(0),a^\theta(1)],n}(V^n(a) - \max_{n'\neq n} V^{n'}(a)) \geq 0$ be the largest difference over the efficient range of actions between the first and second highest V. We claim that there is ρ_3 such that $d_1 \leq \rho_3 d_z^2$ and, hence, by Step 3, d_1 is of order $1/N^2$.

This is much as in Step 2. Fix $\hat{a} \in [0, \tilde{a}]$ and let $\hat{z} = z^a(\hat{a})$. Let z be any other technology. Then, much as in (22),

$$V(\hat{a},\hat{z}) - V(\hat{a},z) \leq -\frac{1}{2}(\hat{z}-z)^2\ell_{V_{zz}}. \tag{23}$$

Now assume that $\hat{a} \in [a^\theta(0), a^\theta(1)]$, so that $\hat{z} \in [z^\theta(0), z^\theta(1)]$. Pick the two operating firms closest to $\hat{z}$. Each is at most $2d_z$ away from $\hat{z}$. Then the first most efficient such firm has V at most $V(\hat{a},\hat{z})$, and the second has V at least $V(\hat{a},\hat{z}) + \frac{1}{2}(2d_z)^2\ell_{V_{zz}}$, so $d_1 \leq -2\ell_{V_{zz}}d_z^2$, and we can take $\rho_3 = -2\ell_{V_{zz}}$.

Step 5. Each type θ must in equilibrium receive an amount close to $v_*(\theta) = \max_{a,z}(V(a,z)+\theta a)$. In particular, we claim that when $d_z < (z^\theta(1)-z^\theta(0))/4$, then there is ρ_4 such that θ's equilibrium payoff is at least $v_*(\theta) - \rho_4 d_z^2$; hence, by Step 3, the difference between θ's equilibrium payoff and $v_*(\theta)$ is of order $1/N^2$.

The idea here is that since firms are densely packed on $[z^\theta(0), z^\theta(1)]$, there are at least two firms that are very well positioned to meet the needs of any given type. Competition and incentive compatibility then force the equilibrium payoff to be near v_*. Formally, consider any type $\tilde{\theta} \in [0,1]$. By definition of d_z, there is a firm n' for whom $z^{n'}$ is within d_z of $z^\theta(\tilde{\theta})$ and, hence, a firm n with $z^\theta(0) \leq z^{n-1} \leq z^{n+1} \leq z^\theta(1)$ for whom z^n is within $2d_z$ of $z^\theta(\tilde{\theta})$. Let $\hat{\theta}$ be any type n serves and let $\hat{a}$ be the associated action. Let us show that $\tilde{\theta}$ can attain at least $v_*(\theta) - \rho_4 d_z^2$ for a suitable ρ_4 by imitating $\hat{\theta}$.

Note first that since $\hat{a} \in [a_e^{n-1}, a_e^n] \subseteq [a^z(z^{n-1}), a^z(z^{n+1})]$,

$$\hat{a} - a^\theta(\tilde{\theta}) = \hat{a} - a^z\big(z^\theta(\tilde{\theta})\big) \leq a^z\big(z^{n+1}\big) - a^z\big(z^\theta(\tilde{\theta})\big) \leq |z^{n+1} - z^\theta(\tilde{\theta})|\upsilon_{a_z^z} \leq 3d_z\upsilon_{a_z^z}.$$

Take a first-order Taylor expansion (Mardsen and Tromba (2012, Theorem 2, pp. 160–162)) of $V(a,z)+a\tilde{\theta}$ as a function of a and z around $(a^\theta(\tilde{\theta}), z^\theta(\tilde{\theta}))$, noting that the first-order terms disappear since $(a^\theta(\tilde{\theta}), z^\theta(\tilde{\theta}))$ is a maximum of $V(a,z)+a\tilde{\theta}$. Then using that $v_*(\tilde{\theta}) = V(a^\theta(\tilde{\theta}), z^\theta(\tilde{\theta})) + a^\theta(\tilde{\theta})\tilde{\theta}$, write

$$\begin{aligned} V\big(\hat{a}, z^n\big) + \hat{a}\tilde{\theta} = v_*(\tilde{\theta}) &+ \frac{V_{aa}\big(a', z'\big)}{2}\big(\hat{a} - a^\theta(\tilde{\theta})\big)^2 \\ &+ \frac{V_{zz}\big(a', z'\big)}{2}\big(z^n - z^\theta(\tilde{\theta})\big)^2 + V_{az}\big(a', z'\big)\big(\hat{a} - a^\theta(\tilde{\theta})\big)\big(z^n - z^\theta(\tilde{\theta})\big), \end{aligned}$$

where (a', z') is some point on the line segment between $(\hat{a}, z^n)$ and $(a^\theta(\tilde{\theta}), z^\theta(\tilde{\theta}))$. Hence,

$$V\big(\hat{a}, z^n\big) + \hat{a}\tilde{\theta} \geq v_*(\tilde{\theta}) + \frac{\ell_{V_{aa}}}{2}(3d_z\upsilon_{a_z^z}))^2 + \frac{\ell_{V_{zz}}}{2}(2d_z)^2 - \upsilon_{V_{az}}6\upsilon_{a_z^z}d_z^2,$$

which, since $d_1 \leq \rho_3 d_z^2$, implies that the value to $\tilde{\theta}$ of imitating $\hat{\theta}$ is at least $V(\hat{a}, z^n) + \hat{a}\tilde{\theta} - d_1 \geq v_*(\tilde{\theta}) - \rho_4 d_z^2$, where $\rho_4 = (9/2)\ell_{V_{aa}} \upsilon_{a_z^z}^2 + 2\ell_{V_{zz}} - \upsilon_{V_{az}} 6\upsilon_{a_z^z} - \rho_3$.

Step 6. The firm serving $\theta = 1$ is not very far above $z^\theta(1)$, and the firm serving $\theta = 0$ is not far below $z^\theta(0)$. Indeed, there is ρ_5 such that each difference is at most $\rho_5 d_z$.

The idea is that when z is much above $z^\theta(1)$, then a firm with technology z—even if it offers all available surplus to the agent—is unable to beat the offerings of firms near $z^\theta(1)$. To begin, for given $\hat{z} > z^\theta(1)$, let us bound $\bar{v}(1, \hat{z})$, the most surplus $\hat{z}$ can offer type 1. A similar argument will apply for $\hat{z} < z^\theta(0)$. Since $\bar{v}(1, z^\theta(1)) = v^*(1)$ and $\bar{v}_z(1, z^\theta(1)) = 0$, we can take a first-order Taylor expansion of $\bar{v}(1, \cdot)$ around $z^\theta(1)$, for some $\tilde{z} \in [z^\theta(1), \hat{z}]$, to obtain

$$\bar{v}(1, \hat{z}) = v^*(1) + \big(\hat{z} - z^\theta(1)\big)^2 \bar{v}_{zz}(1, \tilde{z})/2 \leq v^*(1) + \big(\hat{z} - z^\theta(1)\big)^2 \upsilon_{\bar{v}_{zz}(1,\cdot)}/2.$$

Let us show next that $\upsilon_{\bar{v}_{zz}(1,\cdot)} < 0$. By the envelope theorem, $\bar{v}_z(1, z) = (V(\bar{a}(1, z), z) + \bar{a}(1, z))_z = V_z(\bar{a}(1, z), z)$ and so

$$\begin{aligned} \bar{v}_{zz}(1, z) &= V_{za}\big(\bar{a}(1, z), z\big)\bar{a}_z(1, z) + V_{zz}\big(\bar{a}(1, z), z\big) \\ &= -V_{za}\big(\bar{a}(1, z), z\big)\frac{V_{za}\big(\bar{a}(1, z), z\big)}{V_{aa}\big(\bar{a}(1, z), z\big)} + V_{zz}\big(\bar{a}(1, z), z\big) \\ &\leq \max_{[0,\tilde{a}]\times[0,z]}\left(\frac{1}{V_{aa}(a, z)}\big(V_{aa}(a, z)V_{zz}(a, z) - V_{za}^2(a, z)\big)\right) = \upsilon_{\bar{v}_{zz}(1,\cdot)} < 0, \end{aligned}$$

where the second inequality follows by the concavity assumptions on V and because $[0, \tilde{a}] \times [0, z]$ is compact.

To complete this step, assume that $\hat{z} > z^\theta(1)$ serves $\theta = 1$. Then, since type 1 earns at least $v_*(1) - \rho_4 d_z^2$ by dealing with a firm with $z \cong z^\theta(1)$, we have by PP that $v_*(1) + \upsilon_{\bar{v}_{zz}(1,\cdot)}(\hat{z} - z^\theta(1))^2/2 \geq v_*(1) - \rho_4 d_z^2$, and so $\hat{z} - z^\theta(1) \leq \sqrt{(-2\rho_4)/\upsilon_{\bar{v}_{zz}(1,\cdot)}}\, d_z \equiv \rho_5 d_z$.

Step 7. The profits on each type are bounded by $\rho_6 d_z^2$ and so are of order $1/N^2$.

This follows since, by the preceding steps, for each equilibrium firm, type, and action, there is a nearby firm of similar capabilities, since by Step 6, there is no firm very far outside of $[z^\theta(0), z^\theta(1)]$, and by the preceding steps, firms are closely packed in $[z^\theta(0), z^\theta(1)]$. Formally, fix $\hat{\theta}$ and assume that $\hat{\theta}$ receives action $\hat{a}$ in equilibrium. If $\hat{a} \in [a^\theta(0), a^\theta(1)]$, then by Step 4, we have $\pi^n(\hat{\theta}, \hat{a}, v^n) \leq d_1 \leq \rho_3 d_z^2$, where the first inequality is as proven in Step 2. If $\hat{a} \notin [a^\theta(0), a^\theta(1)]$, then by Step 6, the firm serving $\hat{\theta}$ has $\hat{z}$ within $\rho_5 d_z$ of $[z^\theta(0), z^\theta(1)]$ and so $\hat{z}$ is within $(\rho_5 + 1)d_z$ of $\tilde{z}$, where $\tilde{z} \in [z^\theta(0), z^\theta(1)]$ is some other operating firm. But then, again as in Step 2, and using (23),

$$\begin{aligned} \pi^n\big(\hat{\theta}, \hat{a}, v^n\big) &\leq V(\hat{a}, \hat{z}) - V(\hat{a}, \tilde{z}) \\ &\leq -\frac{1}{2}(\hat{z} - \tilde{z})^2 \ell_{V_{zz}} \\ &\leq -\frac{1}{2}\ell_{V_{zz}}(\rho_5 + 1)^2 d_z^2, \end{aligned}$$

and so defining $\rho_6 = \max\{\rho_3, -(1/2)\ell_{V_{zz}}(\rho_5 + 1)^2\}$, we are done, noting that from Step 3, d_z is $O(1/N)$ and, hence, π^n is $O(1/N^2)$. *Q.E.D.*

S5. A MERGER OF ALL FIRMS EXCEPT FIRM 1

Consider the situation in which all firms except for firm 1 merge and, for simplicity, assume that the merged firm continues to operate firm 2. Then the conditions defining the boundary θ^1 between firm 1 and firm 2 in the pre-merger setting are

$$V^1(\gamma^1(\theta^1,0))+\theta^1\gamma^1(\theta^1,0)-v^O(\theta^1)-\frac{H(\theta^1)}{h(\theta^1)}(\gamma^2(\theta^1,\kappa^2)-\gamma^1(\theta^1,0))=0,$$

$$V^2(\gamma^2(\theta^1,\kappa^2))+\theta^1\gamma^2(\theta^1,\kappa^2)-v^O(\theta^1)+\frac{\kappa^2-H(\theta^1)}{h(\theta^1)}(\gamma^1(\theta^1,0)-\gamma^2(\theta^1,\kappa^2))=0,$$

while post-merger, the key difference is that instead of κ^2, we have $\kappa^M=1$. Since at the boundary type, θ^1, the equilibrium surplus $v^O(\theta^1)$ equals both the surplus of firm 1 and of firm 2, we can replace it by $v^1(0)+\int_0^{\theta^1}\gamma^1(\tau,0)\,d\tau$. The system of equations then becomes

$$\begin{aligned}&V^1(\gamma^1(\theta^1,0))+\theta^1\gamma^1(\theta^1,0)-v^1(0)-\int_0^{\theta^1}\gamma^1(\tau,0)\,d\tau\\&\quad-\frac{H(\theta^1)}{h(\theta^1)}(\gamma^2(\theta^1,\kappa^2)-\gamma^1(\theta^1,0))=0,\\&V^2(\gamma^2(\theta^1,\kappa^2))+\theta^1\gamma^2(\theta^1,\kappa^2)-v^1(0)\\&\quad-\int_0^{\theta^1}\gamma^1(\tau,0)\,d\tau+\frac{\kappa^2-H(\theta^1)}{h(\theta^1)}(\gamma^1(\theta^1,0)-\gamma^2(\theta^1,\kappa^2))=0.\end{aligned}$$

This system can be solved for $(\theta^1,v^1(0))$ as a function of κ^2. Since post-merger, κ increases (to 1), it is thus enough to understand how $(\theta^1,v^1(0))$ changes when κ^2 increases. Totally differentiating these equations with respect to κ^2 and solving for $\partial\theta^1/\partial\kappa^2$ and $\partial v^1(0)/\partial\kappa^2$ yields, after some algebra,

$$\begin{aligned}\frac{\partial\theta^1}{\partial\kappa^2}&=_s-\frac{H(\theta^1)}{h(\theta^1)}\gamma^2_\kappa(\theta^1,\kappa^2)+\frac{1}{h(\theta^1)}(\gamma^2(\theta^1,\kappa^2)-\gamma^1(\theta^1,0))\\\frac{\partial v^1(0)}{\partial\kappa^2}&=_s-\frac{\kappa^2-H(\theta^1)}{h(\theta^1)}\gamma^1_\theta(\theta^1,0)\frac{H(\theta^1)}{h(\theta^1)}\gamma^2_\kappa(\theta^1,\kappa^2)\\&\qquad-\left(\frac{H(\theta^1)}{h(\theta^1)}\right)_\theta(\gamma^2(\theta^1,\kappa^2)-\gamma^1(\theta^1,0))^2\frac{1}{h(\theta^1)},\end{aligned}$$

where in the last line we have used the expressions for the partial derivatives of γ^2 (which follow from differentiating IO) to cancel two terms.

Since $\gamma^2_\kappa<0$ and since by stacking, $\gamma^2(\theta^1,\kappa^2)>\gamma^1(\theta^1,0)$, it follows immediately that θ^1 strictly increases in κ^2. That is, *the market share of the nonmerged firm unambiguously increases.*

Regarding $\partial v^1(0)/\partial\kappa^2$, the effect is ambiguous, since it is the difference of two strictly positive terms. For some intuition, rewrite

$$V^1\big(\gamma^1(\theta^1,0)\big)+\theta^1\gamma^1(\theta^1,0)-v^1(0)-\int_0^{\theta^1}\gamma^1(\tau,0)\,d\tau$$
$$-\frac{H(\theta^1)}{h(\theta^1)}\big(\gamma^2(\theta^1,\kappa^2)-\gamma^1(\theta^1,0)\big)=0$$

as $v^1(0)=z(\theta^1,\kappa^2)$, where z is strictly decreasing in θ^1 and strictly increasing in κ^2. Then

$$\frac{\partial v^1(0)}{\partial\kappa^2}=\frac{\partial z}{\partial\theta^1}\frac{\partial\theta^1}{\partial\kappa^2}+\frac{\partial z}{\partial\kappa^2}.$$

The first term is strictly negative, since we have shown that θ^1 strictly increases in κ^2. This is the market-share effect described in the text: having more types to serve provides firm 1 with an incentive to *lower* the surplus it offers to each of them. The second term is the differentiation effect described in the text: once merged, κ increases, so at the boundary the two firms are less differentiated, which provides incentives to firm 1 to *increase* the surplus offered so as to grab more types from the merged firm (which is easier to do given the increase in κ). Which effect prevails depends on the strength of these forces.

One case in which we can tame these forces is when $V^n_{aa}=-\tau^n$ for some $\tau^n>0$ and, thus, $V^n_a=\iota^n-\tau^n a$, $\iota^n>0$. Then from IO we obtain $\gamma^n(\theta,\kappa)=(\iota^n+\theta-((\kappa-H)/h)))/\tau^n$ and, thus,

$$\begin{aligned}&\gamma^2(\theta,\kappa^2)-\gamma^1(\theta,0)\\&=\frac{1}{\tau^1\tau^2}\left(\tau^1\left(\iota^2+\theta-\frac{\kappa^2-H(\theta)}{h(\theta)}\right)-\tau^2\left(\iota^1+\theta-\frac{-H(\theta)}{h(\theta)}\right)\right)\\&=\frac{1}{h(\theta)\tau^1\tau^2}\big(\tau^1\big(h(\theta)(\iota^2+\theta)-(\kappa^2-H(\theta))\big)-\tau^2\big(h(\theta)(\iota^1+\theta)+H(\theta)\big)\big).\end{aligned}$$

Note next that stacking holds if and only if

$$\begin{aligned}&\gamma^2(\theta,1)-\gamma^1(\theta,0)\\&=\frac{1}{\tau^1\tau^2}\left(\tau^1\left(\iota^2+\theta-\frac{1-H(\theta)}{h(\theta)}\right)-\tau^2\left(\iota^1+\theta+\frac{H(\theta)}{h(\theta)}\right)\right)\\&=\frac{1}{\tau^1\tau^2h(\theta)}\big(\tau^1\big(h(\theta)(\iota^2+\theta)-(1-H(\theta))\big)-\tau^2\big(h(\theta)(\iota^1+\theta)+H(\theta)\big)\big)\geq 0\quad\forall\theta.\end{aligned}$$

Inserting the expressions for the γs into $\partial v^1(0)/\partial\kappa^2$, we obtain that it is equal in sign to

$$\begin{aligned}&\frac{1}{2}\big(\kappa^2-H(\theta^1)\big)H(\theta^1)\\&\quad-\left(\frac{H(\theta^1)}{h(\theta^1)}\right)_\theta\frac{1}{\tau^1\tau^2}\big(\big(\tau^1\big(h(\theta^1)(\iota^2+\theta)-(\kappa^2-H(\theta^1))\big)\big)\\&\quad-\tau^2\big(h(\theta^1)(\iota^1+\theta)+H(\theta^1)\big)\big)\big)^2.\end{aligned}$$

It is clear that the expression will be negative if $\iota^2 - \iota^1$ is sufficiently large, that is, if firms are sufficiently differentiated. For a sharp illustration, assume that $h = 1$. Then $\partial v^1(0)/\partial \kappa^2$ has strictly the same sign as

$$\frac{1}{2}(\kappa^2 - \theta^1)\theta^1 - \frac{1}{\tau^1\tau^2}\left(\tau^1(\iota^2 + 2\theta^1 - \kappa^2) - \tau^2(\iota^1 + 2\theta^1)\right)^2,$$

which is negative if and only if

$$\tau^1(\iota^2 + 2\theta^1) - \tau^2(\iota^1 + 2\theta^1) \geq \tau^1\kappa^2 + \sqrt{\tau^1\tau^2\frac{1}{2}(\kappa^2 - \theta^1)\theta^1}.$$

Since this must hold for all values of κ^2, the most difficult case is $\kappa^2 = 1$. For example, if $\tau^1 = \tau^2 = \tau > 0$, then this expression reduces to

$$\iota^2 - \iota^1 \geq 1 + \frac{1}{2\sqrt{2}}.$$

Since stacking in this case is simply $\iota^2 - \iota^1 \geq 1$, it follows that a stronger level of differentiation ensures that firm 1 will reduce the surplus it offers to each of its types after the merger.

S6. OMITTED PROOFS FOR APPENDIX C

LEMMA 23: *Consider γ and $\tilde{\kappa}$ as functions on $\tilde{R} \cap \Theta$. Then $(\gamma(\theta_l, \tilde{\kappa}))_{\theta_l} > \gamma_\kappa(\theta_l, \tilde{\kappa})\tilde{\kappa}_{\theta_l} > 0$ and $(\gamma(\theta_h, \tilde{\kappa}))_{\theta_h} > \gamma_\kappa(\theta_h, \tilde{\kappa})\tilde{\kappa}_{\theta_h} > 0$, with*

$$\begin{vmatrix} (\gamma(\theta_l, \tilde{\kappa}))_{\theta_l} & (\gamma(\theta_l, \tilde{\kappa}))_{\theta_h} \\ (\gamma(\theta_h, \tilde{\kappa}))_{\theta_l} & (\gamma(\theta_h, \tilde{\kappa}))_{\theta_h} \end{vmatrix} > 0.$$

PROOF: Note that $(\gamma(\theta_l, \tilde{\kappa}))_{\theta_l} = \gamma_\theta(\theta_l, \tilde{\kappa}) + \gamma_\kappa(\theta_l, \tilde{\kappa})\tilde{\kappa}_{\theta_l} > \gamma_\kappa(\theta_l, \tilde{\kappa})\tilde{\kappa}_{\theta_l}$, since $\gamma_\theta > 0$ using that $\tilde{\kappa} \in [0, 1]$. But since $\iota(\theta_l, \theta_h, \tilde{\kappa}) = 0$ on Θ, we have $\tilde{\kappa}_{\theta_l} = -\iota_{\theta_l}/\iota_\kappa < 0$, since $\iota_\theta > 0$ and $\iota_\kappa > 0$ using the definition of ι. Thus, $\gamma_\kappa(\theta_l, \tilde{\kappa})\tilde{\kappa}_{\theta_l} > 0$, since $\gamma_\kappa < 0$. Similarly, $\tilde{\kappa}_{\theta_h} < 0$ and so $(\gamma(\theta_h, \tilde{\kappa}))_{\theta_h} > \gamma_\kappa(\theta_h, \tilde{\kappa})\tilde{\kappa}_{\theta_h} > 0$. But then

$$\begin{vmatrix} (\gamma(\theta_l, \tilde{\kappa}))_{\theta_l} & (\gamma(\theta_l, \tilde{\kappa}))_{\theta_h} \\ (\gamma(\theta_h, \tilde{\kappa}))_{\theta_l} & (\gamma(\theta_h, \tilde{\kappa}))_{\theta_h} \end{vmatrix} > \begin{vmatrix} \gamma_\kappa(\theta_l, \tilde{\kappa})\tilde{\kappa}_{\theta_l} & \gamma_\kappa(\theta_l, \tilde{\kappa})\tilde{\kappa}_{\theta_h} \\ \gamma_\kappa(\theta_h, \tilde{\kappa})\tilde{\kappa}_{\theta_l} & \gamma_\kappa(\theta_h, \tilde{\kappa})\tilde{\kappa}_{\theta_h} \end{vmatrix} = 0. \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA 13: From (12), and recalling that π_a does not depend on $\tilde{v}$,

$$\begin{aligned} \frac{r_{\theta_h\theta_l}(\theta_l, \theta_h)}{h(\theta_h)} &= \pi_{aa}\big(\theta_h, \gamma(\theta_h, \tilde{\kappa}), \tilde{v}\big)\big(\gamma(\theta_h, \tilde{\kappa})\big)_{\theta_l}\big(a^{-n}(\theta_h) - \gamma(\theta_h, \tilde{\kappa})\big) \\ &\quad + \pi_a\big(\theta_h, \gamma(\theta_h, \tilde{\kappa}), \tilde{v}\big)\big(-\big(\gamma(\theta_h, \tilde{\kappa})\big)_{\theta_l}\big) \\ &\quad + \pi_a\big(\theta_h, \gamma(\theta_h, \tilde{\kappa}), \tilde{v}\big)\big(\gamma(\theta_h, \tilde{\kappa})\big)_{\theta_l}, \\ &= \pi_{aa}\big(\theta_h, \gamma(\theta_h, \tilde{\kappa}), \tilde{v}\big)\big(\gamma(\theta_h, \tilde{\kappa})\big)_{\theta_l}\big(a^{-n}(\theta_h) - \gamma(\theta_h, \tilde{\kappa})\big), \end{aligned} \qquad (24)$$

and similarly, from (13),

$$\frac{r_{\theta_l\theta_h}(\theta_l,\theta_h)}{h(\theta_l)} = \pi_{aa}\big(\theta_l,\gamma(\theta_l,\tilde{\kappa}),\tilde{v}\big)\big(\gamma^n(\theta_l,\tilde{\kappa})\big)_{\theta_h}\big(\gamma(\theta_l,\tilde{\kappa}) - a^{-n}(\theta_l)\big).^{57} \tag{25}$$

To see that $r_{\theta_l\theta_h} < 0$, start from (25) (or analogously from (24)), and note that $\pi_{aa} < 0$, that $(\gamma^n(\theta_l,\tilde{\kappa}))_{\theta_h} = \gamma_\kappa(\theta_l,\tilde{\kappa})\tilde{\kappa}_{\theta_h} > 0$, and that by stacking, $\gamma(\theta_l,\tilde{\kappa}) - a^{-n}(\theta_l) > 0$.

Note next that since $\iota(\theta_l,\theta_h,\tilde{\kappa}) = 0$, $\tilde{v}(\theta_l) = v^{-n}(\theta_l)$ and $\tilde{v}(\theta_h) = v^{-n}(\theta_h)$. Note that $\pi_a(\theta_h,\gamma(\cdot,\tilde{\kappa}),\tilde{v}) =_s \tilde{\kappa} - H(\theta_h) \le 0$, since $\tilde{\kappa} \in [H(\theta_l), H(\theta_h)]$. Similarly, $\pi_a(\theta_l,\gamma(\cdot,\tilde{\kappa}),\tilde{v}) \ge 0$. Assume that $r_{\theta_h}(\theta_l,\theta_h) = 0$. Then using (12),

$$\begin{aligned}\frac{r_{\theta_h\theta_h}(\theta_l,\theta_h)}{h(\theta_h)} &= \big(1+\pi_{aa}\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_h,\tilde{\kappa})\big)_{\theta_h}\big)\big(a^{-n}(\theta_h) - \gamma(\theta_h,\tilde{\kappa})\big)\\ &\quad + \pi_a\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)\big(a_\theta^{-n}(\theta_h) - \big(\gamma(\theta_h,\tilde{\kappa})\big)_{\theta_h}\big)\\ &\quad + \gamma(\theta_h,\tilde{\kappa}) + \pi_a\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_h,\tilde{\kappa})\big)_{\theta_h} - a^{-n}(\theta_h),\end{aligned}$$

where the term involving h_θ disappears since $r_{\theta_h} = 0$ and where we use that $\tilde{v}(\theta_h) = v^{-n}(\theta_h)$, and, hence, $(\tilde{v}(\theta_h))_{\theta_h} = (v^{-n}(\theta_h))_{\theta_h} = a^{-n}(\theta_h)$. Cancelling yields

$$\begin{aligned}&\frac{r_{\theta_h\theta_h}(\theta_l,\theta_h)}{h(\theta_h)}\\ &\quad = \pi_{aa}\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_h,\tilde{\kappa})\big)_{\theta_h}\big(a^{-n}(\theta_h) - \gamma(\theta_h,\tilde{\kappa})\big) + \pi_a\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)a_\theta^{-n}(\theta_h)\\ &\quad \le \pi_{aa}\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_h,\tilde{\kappa})\big)_{\theta_h}\big(a^{-n}(\theta_h) - \gamma(\theta_h,\tilde{\kappa})\big)\\ &\quad < 0,\end{aligned} \tag{26}$$

where the first inequality uses that $\pi_a(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}) \le 0$, and the second uses that $\pi_{aa} < 0$, that by Lemma 23, $(\gamma(\theta_h,\tilde{\kappa}))_{\theta_h} > 0$, and that by stacking, Cn1, and $\tilde{\kappa} \in [0,1]$, $a^{-n}(\theta_h) - \gamma(\theta_h,\tilde{\kappa}) > 0$.[58]

Similarly, taking cancellations as before, if $r_{\theta_l} = 0$, then

$$\frac{r_{\theta_l\theta_l}(\theta_l,\theta_h)}{h(\theta_l)} \le \pi_{aa}\big(\theta_l,\gamma(\theta_l,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_l,\tilde{\kappa})\big)_{\theta_l}\big(\gamma(\theta_l,\tilde{\kappa}) - a^{-n}(\theta_l)\big) < 0. \tag{27}$$

For strict local concavity, it remains to show that if $r_{\theta_l} = r_{\theta_h} = 0$, then $r_{\theta_l\theta_l}r_{\theta_h\theta_h} - r^2_{\theta_l\theta_h} > 0$. From (26) and (27),

$$\begin{aligned}\frac{r_{\theta_l\theta_l}r_{\theta_h\theta_h}}{h(\theta_l)h(\theta_h)} &\ge \pi_{aa}\big(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_h,\tilde{\kappa})\big)_{\theta_h}\big(a^{-n}(\theta_h) - \gamma(\theta_h,\tilde{\kappa})\big)\\ &\quad \times \pi_{aa}\big(\theta_l,\gamma(\theta_l,\tilde{\kappa}),\tilde{v}\big)\big(\gamma(\theta_l,\tilde{\kappa})\big)_{\theta_l}\big(\gamma(\theta_l,\tilde{\kappa}) - a^{-n}(\theta_l)\big),\end{aligned}$$

[57]These two expressions must of course be equal, but it is convenient to express them in these two different ways.

[58]To be careful, a_θ^{-n} and, hence, $r_{\theta_h\theta_h}$, may not be everywhere defined. But since a^{-n} is increasing, $\liminf_{\varepsilon\downarrow 0} a_\theta^{-n}(\theta_h+\varepsilon) \ge 0$ and $\liminf_{\varepsilon\downarrow 0} a_\theta^{-n}(\theta_h-\varepsilon) \ge 0$, and so since $\pi_a(\theta_h,\gamma(\theta_h,\tilde{\kappa}),\tilde{v}) \le 0$, we have $\limsup_{\varepsilon\downarrow 0} r_{\theta_h\theta_h}(\theta_l,\theta_h+\varepsilon) < 0$ and $\limsup_{\varepsilon\downarrow 0} r_{\theta_h\theta_h}(\theta_l,\theta_h-\varepsilon) < 0$. We henceforth ignore this technical detail.

while from (24) and (25),

$$\frac{r_{\theta_l\theta_h}r_{\theta_h\theta_l}}{h(\theta_l)h(\theta_h)} = \pi_{aa}\big(\theta_l, \gamma(\theta_l, \tilde{\kappa}), \tilde{v}\big)\big(\gamma(\theta_l, \tilde{\kappa})\big)_{\theta_h}\big(\gamma(\theta_l, \tilde{\kappa}) - a^{-n}(\theta_l)\big)$$
$$\times \pi_{aa}\big(\theta_h, \gamma(\theta_h, \tilde{\kappa}), \tilde{v}\big)\big(\gamma(\theta_h, \tilde{\kappa})\big)_{\theta_l}\big(a^{-n}(\theta_h) - \gamma(\theta_h, \tilde{\kappa})\big).$$

Collecting common positive terms, it suffices that $(\gamma(\theta_h, \tilde{\kappa}))_{\theta_h}(\gamma(\theta_l, \tilde{\kappa}))_{\theta_l} - (\gamma(\theta_l, \tilde{\kappa}))_{\theta_h} \times (\gamma(\theta_h, \tilde{\kappa}))_{\theta_l} > 0$, which follows from Lemma 23. *Q.E.D.*

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