Econometrica Supplementary Material

SUPPLEMENT TO "A QUANTITATIVE THEORY OF THE CREDIT SCORE"

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APPENDIX A: MODEL APPENDIX

A.1. *Construction of $Q^s(s'|\psi)$ and Proof of Lemma 1*

LET $G \equiv \{0, 1/K, 2/K, \dots, 1\}$ BE a uniform discrete approximation of $[0, 1]$. Let $D = 1/K$ denote the distance between adjacent (grid) points of G. Let $\mathcal{S} = \{(s_1, s_2, \dots, s_B) | s_i \in G$ and $\sum_{i=1}^{B} s_i = 1\}$ be the associated probability simplex.

LEMMA A.1: *Let $s_i \in G$ for $i = 1, 2, \dots, B-1$. If $\sum_{i=1}^{B-1} s_i < 1$, then $1 - \sum_{i=1}^{B-1} s_i \in G$.*

PROOF: $\sum_{i=1}^{B-1} s_i < 1 \Rightarrow \sum_{i=1}^{B-1} (\ell_i/K) < 1 \Rightarrow \sum_{i=1}^{B-1} \ell_i < K$, where the ℓ_i's are integers between 0 and K. Since a sum of integers is an integer and a difference of two integers is also an integer, $K - \sum_{i=1}^{B-1} \ell_i$ is a positive integer and it is less than K. Therefore, by the definition of G, $1 - \sum_{i=1}^{B-1} \ell_i/K \in G$. *Q.E.D.*

DEFINITION A.1: All elements of the matrix Q^β are strictly positive.

LEMMA A.2: *Let $\psi = (\psi_1, \psi_2, \dots, \psi_B)$ be any vector of type scores resulting from the Bayesian update. Then, $\psi_i \geq \underline{Q} > 0$.*

PROOF: Let $\underline{Q}$ be the smallest element of Q^β. By Assumption 1, $\underline{Q} > 0$.

$$\begin{aligned}\psi_i &= \sum_j Q^\beta(i|j) \times \text{ posterior probability of } j| \text{ actions} \\ &\geq \sum_j \underline{Q} \times \text{ posterior probability of } j| \text{ actions} = \underline{Q}.\end{aligned}$$

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The first equality follows from the definition of ψ_i, the inequality follows from Assumption 1, and the last line follows from the fact that the sum of posterior probabilities is 1. *Q.E.D.*

We now identify the elements of $\mathcal{S}$ that approximate any given type-score vector ψ resulting from the Bayesian update. Let $s_{i,L} = \max_{s\in G} s \le \psi_i$ and $s_{i,H} = s_{i,L} + D$. Consider the collection of 2^{B-1} vectors:

$$S_\psi = \left\{\left(s_{1,l(1)}, s_{2,l(2)}, \dots, 1 - \sum_{i=1}^{B-1} s_{i,l(i)}\right)\right\} \quad \text{where for each } i, l(i) \in \{L, H\}.$$

LEMMA A.3: *If $D < \underline{Q}/(B-1)$, then $S_\psi \subset \mathcal{S}$.*

PROOF: By construction, $s_{i,L} \in G$. Next, observe that $s_{i,L}$ cannot be 1 since that would imply that $\psi_i = 1$ and, therefore, $\psi_{j\neq i} = 0$ in contradiction to Lemma A.2. Therefore, $s_{i,H} = s_{i,L} + D$ must belong in G for all i. To show that $(s_{1,l(1)}, s_{2,l(2)}, \dots, 1 - \sum_{i=1}^{B-1} s_{i,l(i)})$ belongs in $\mathcal{S}$, it is sufficient to show, by virtue of Lemma A.1, that $\sum_{i=1}^{B-1} s_{i,l(i)} < 1$:

$$\begin{aligned}\sum_{i=1}^{B-1} s_{i,l(i)} &\le \sum_{i=1}^{B-1} s_{i,H}\\ &\le \sum_{i=1}^{B-1}(\psi_i + D)\\ &= (1-\psi_B) + (B-1)D\\ &\le 1 - \underline{Q} + (B-1)D < 1.\end{aligned}$$

The first inequality follows because $s_{i,l(i)} \le s_{i,H}$. The second inequality follows because $s_{i,L} = s_{i,H} + D$ and $\psi_i \ge s_{i,L}$. The third equality follows because $\sum_{i=1}^{B} \psi_i = 1$. The fourth inequality follows from Lemma A.2, and the final inequality follows from the hypothesis of the lemma. *Q.E.D.*

By Lemma A.3, we can take S_ψ to be the collection of approximating vectors. Note that for each member of this set, the first $B-1$ components are within $\psi_i \pm D$ so, in this sense, the vectors are close to ψ.

We now determine the probability assigned to each of these vectors. To this end, let

$$p(s_{i,L}) = \frac{s_{i,H} - \psi_i}{D} \quad \text{and} \quad p(s_{i,H}) = \frac{\psi_i - s_{i,L}}{D} \quad \text{for } i = 1, 2, 3, \dots, B-1. \tag{27}$$

Since $s_{i,L} \le \psi_i < s_{i,H}$ and $s_{i,H} - s_{i,L} = D$, $p(s_{i,L})$ and $p(s_{i,H})$ are nonnegative and sum to 1. We set

$$\Pr\left[\left(s_{1,l(1)}, s_{2,l(2)}, s_{3,l(3)}, \dots, 1 - \sum_{i=1}^{B-1} s_{i,l(i)}\right)\right] = \prod_{i=1}^{B-1} p(s_{i,l(i)}),$$
$$l(i) \in \{L, H\}, i = 1, 2, \dots, B-1.$$

Then our assignment rule $Q^s(s'|\psi):\mathcal{S}\to[0,1]$ is given by

$$Q^s(s'|\psi)=\begin{cases}\prod_{i=1}^{B-1} p(s'_{i,l(i)}) & \text{if } s'\in S_\psi,\\ 0 & \text{otherwise.}\end{cases} \tag{28}$$

For this assignment rule, we can prove the following:

LEMMA 1: *(i)* $\sum_{s'\in\mathcal{S}} s'_i Q^s(s'|\psi)=\psi_i$, $\forall i$ *(consistency)*, *(ii)* $\sum_{s'\in\mathcal{S}}(s'_i-\psi_i)^2 Q^s(s'|\psi)\le 2(B-1)D^2$, $\forall i$ *(variance of the approximation error can be made arbitrarily small)*, *and (iii)* $Q^s(s'|\psi)$ *is continuous in* ψ *(continuity)*.

PROOF: (i) First, note that $\sum_{s'\in\mathcal{S}} s'_i Q^s(s'|\psi)=\sum_{s'\in S_\psi} s'_i Q^s(s'|\psi)$ since (28) assigns positive probability only to vectors that are in S_ψ. Let $i\in\{1,2,\dots,B-1\}$. Now, group the collection of vectors in S_ψ into two: In the first group are all vectors for which $s'_i=s_{i,L}$ and in the second group are all vectors for which $s'_i=s_{i,H}$. Denote these groups as S^L_ψ and S^H_ψ. Then,

$$\begin{aligned}\sum_{s'\in S_\psi} s'_i Q^s(s'|\psi)&=\sum_{s'\in S^L_\psi} s'_i Q^s(s'|\psi)+\sum_{s'\in S^H_\psi} s'_i Q^s(s'|\psi)\\ &=s_{i,L}\sum_{s'\in S^L_\psi} Q^s(s'|\psi)+s_{i,H}\sum_{s'\in S^H_\psi} Q^s(s'|\psi)\\ &=s_{i,L}p(s_{i,L})+s_{i,H}p(s_{i,H})=\psi_i.\end{aligned}$$

The third equality follows from the fact that the first and second sums in the second line are the probabilities of selecting a vector from group L and group H, respectively. Since the assignment of $s_{i,L}$ or $s_{i,H}$ for s'_i is done independently of the assignments to the other $B-2$ components, the probability of selecting a vector in group L is $p(s_{i,L})$ and in group H is $p(s_{i,H})$. The last equality follows from (27).

Next, let $i=B$. Then,

$$\begin{aligned}\sum_{s'\in S_\psi} s'_B Q^s(s'|\psi)&=\sum_{s'\in S_\psi}[1-s'_1-s'_2-\cdots-s'_{B-1}]Q^s(s'|\psi)\\ &=\sum_{s'\in S_\psi} Q^s(s'|\psi)-\sum_{i=1}^{B-1}\sum_{s'\in S_\psi} s'_i Q^s(s'|\psi)\\ &=1-\sum_{i=1}^{B-1}\psi_i=\psi_B.\end{aligned}$$

(ii) Let $i\in\{1,2,\dots,B-1\}$:

$$\begin{aligned}\sum_{s'\in S_\psi}(s'_i-\psi_i)^2 Q^s(s'|\psi)&=\sum_{s'\in S^L_\psi}(s'_i-\psi_i)^2 Q^s(s'|\psi)+\sum_{s'\in S^H_\psi}(s'_i-\psi_i)^2 Q^s(s'|\psi)\\ &=\sum_{s'\in S^L_\psi}(s_{i,L}-\psi_i)^2 Q^s(s'|\psi)+\sum_{s'\in S^H_\psi}(s_{i,H}-\psi_i)^2 Q^s(s'|\psi)\end{aligned}$$

$$
\begin{aligned}
&\leq D^2 \sum_{s' \in S^L_\psi} Q^s(s'|\psi) + D^2 \sum_{s' \in S^H_\psi} Q^s(s'|\psi) \\
&= D^2\big(p(s_{i,L}) + p(s_{i,H})\big) = D^2.
\end{aligned}
$$

Let $i = B$. Then,

$$
\begin{aligned}
\sum_{s' \in S_\psi} (s'_B - \psi_B)^2 Q^s(s'|\psi) &= \sum_{s' \in S_\psi} \left(1 - \sum_{i=1}^{B-1} s'_i - 1 + \sum_{i=1}^{B-1} \psi_i\right)^2 Q^s(s'|\psi) \\
&= \sum_{s' \in S_\psi} \left(\sum_{i=1}^{B-1} (s'_i - \psi_i)\right)^2 Q^s(s'|\psi) \\
&= \sum_{i=1}^{B-1} \sum_{s' \in S_\psi} (s'_i - \psi_i)^2 Q^s(s'|\psi) + \text{expectations of cross-product terms} \\
&\leq (B-1)D^2.
\end{aligned}
$$

The inequality in the final line follows from the bound on each of the variances and from the fact that the assignments of s'_i for $i \in \{1, 2, \dots, B-1\}$ are independent of each other so that the expectation of all the cross-product terms is zero.

(iii) Let ψ_n be a sequence converging to ψ^*. Consider first the case where $\psi^*_i \notin G$. Then, for $n > N$, N sufficiently large, $\psi^n_i \in (s^*_{i,L}, s^*_{i,H})$ and, so,

$$
p^n(s_{i,L}) = \frac{s^*_{i,H} - \psi^n_i}{D} \quad \text{and} \quad p^n(s_{i,H}) = \frac{\psi^n_i - s^*_{i,L}}{D}.
$$

It follows that $\lim_{n\to\infty} p^n(s_{i,L}) = p^*(s_{i,L})$ and $\lim_{n\to\infty} p^n(s_{i,H}) = p^*(s_{i,H})$. Next consider the case where $\psi^*_i \in G$. Then, by construction,

$$
s^*_{i,L} = \psi^*_i, \qquad s^*_{i,H} = s^*_{i,L} + D \quad \text{and} \quad p^*(s^*_{i,L}) = 1.
$$

Then, for $n > N$, N sufficiently large, either $\psi^n_i \in (s^*_{i,L} - D, s^*_{i,L})$ or $\psi^n_i \in (s^*_{i,L}, s^*_{i,L} + D)$. Therefore, $p^n(s^*_{i,L})$ converges to $1 = p^*(s^*_{i,L})$ as ψ^n_i converges to ψ^*_i. *Q.E.D.*

Note that by reducing the distance D between adjacent points of G, or, equivalently, increasing the number of (uniformly-placed) grid points K approximating the unit interval, the dispersion of s' around ψ can be made arbitrarily small.

A.2. *Proof of Theorem 1 (Existence of the Value Function)*

THEOREM 1: *Given f, there exists a unique solution $W(\beta, z, \omega|f)$ to the decision problem in* (3)–(8).

PROOF: The proof relies on the contraction mapping theorem. However, since the extreme value shocks ν and ϵ can take any value on the real line, it is mathematically more convenient to seek a solution to (3), (4), (12), and (13) since the extreme value shocks do not appear in these. Define the operator $(T_f)(W) : \mathbb{R}^{B+Z+|\Omega|} \to \mathbb{R}^{B+Z+|\Omega|}$ as the map that takes a vector W in $\mathbb{R}^{B+Z+|\Omega|}$ and returns a vector $(T_f)(W)$ via (4), (12), and (13) using (3). We may easily verify that T_f satisfies Blackwell's sufficiency condition for a contraction map (with modulus $\beta\rho$). Since $\mathbb{R}^{B+Z+|\Omega|}$ is a complete metric space (with, say, the uniform metric $\rho(W, W') = \max_{1\leq i\leq B+Z+|\Omega|} \|W_i - W_i'\|$), by Theorem 3.2 of Stokey and Lucas (1989), there exists a unique $W(\beta, z, \omega|f)$ satisfying $(T_f)(W) = W$. *Q.E.D.*

A.3. *Proof of Lemma 2 (Existence of the Invariant Distribution)*

LEMMA 2: *There exists a unique invariant distribution* $\overline{\mu}(\cdot|f)$ *and* $\{\mu_0 T^n\}$ *converges to* $\bar{\mu}(\cdot|f)$ *at a geometric rate for any initial distribution* μ_0.

PROOF: We will use Theorem 11.4 in Stokey and Lucas (1989) to establish this result. To connect to that theorem, let i be a typical element of the finite state space $\mathcal{B}\times\mathcal{Z}\times\Omega$. Let the transition matrix Π in their theorem correspond to T in (19) and let π_{ij} denote the probability of transitioning to j from i. Further, let $\epsilon_j = \min_i \pi_{ij}$ and $\epsilon = \sum_j \epsilon_j$. Then it is sufficient to establish that $\epsilon > 0$. To this end, consider the state $\hat{j} = (\hat{\beta}, \hat{z}, \hat{e}, 0, F_\beta)$ with the property that $F_\beta(\hat{\beta})H(\hat{z})F_e(\hat{e}) > 0$. Then, (19) implies $\pi_{i\hat{j}} \geq (1-\rho)F_\beta(\hat{\beta})H(\hat{z})F_e(\hat{e}) > 0$ for all i. Hence $\epsilon_{\hat{j}} \geq (1-\rho)F_\beta(\hat{\beta})H(\hat{z})F_e(\hat{e}) > 0$. Since $\epsilon_j \geq 0$ for all other j, it follows that $\epsilon > 0$. *Q.E.D.*

A.4. *Proof of Lemma 3 (Value Continuity) and Theorem 2 (Equilibrium Existence)*

The fact that there are zero profits in equilibrium implies $q^{(0,a')}(\omega|f) = \frac{\rho}{1+r}$ for $a' \geq 0$ (i.e., the price on savings is a function only of parameters). In what follows, we take $F^* \subset F$ to contain only those f_1 for which $f_1(a', \omega) = \frac{\rho}{1+r}$ for $a' \geq 0$.

LEMMA 3: $W(\beta, z, \omega|f)$ *is continuous in* f, *and for any* $(d, a') \in \mathcal{F}(z, \omega|f)$, $\sigma^{(d,a')}(\beta, z, \omega|f)$ *is continuous in* f.

PROOF: We first show that the operator T_f defined in Theorem 1 is continuous in f (meaning that for any given W, small changes in f lead to small changes in $T_f(W)$). Inspection of (6) and (8) shows that this will be true if the conditional value functions $v^{(d,a')}(\beta, z, e, a, s|f)$ in (4) are continuous in f. Let $\bar{f} \in F^*$ and let $(\hat{d}, \hat{a}') \in \mathcal{F}(z, \omega|\bar{f})$. Let $f^n \in F^*$ be a sequence converging to $\bar{f}$. By Assumption 1, (0, 0) and (1, 0) are feasible choices regardless of the value of any inherited debt (i.e., $a < 0$), so all debt choices ($a' < 0$) and the default choice belong in $\mathcal{F}(z, \omega|f^n)$. Furthermore, if an asset choice (i.e., $a' \geq 0$) is feasible for $\bar{f}$, that asset choice remains feasible for f^n since the price of any asset is the same in $\bar{f}$ and f^n (namely, $\rho/(1+r)$). Therefore, $(\hat{d}, \hat{a}') \in \mathcal{F}(z, \omega|f^n)$ and so $v^{(\hat{d},\hat{a}')}(\beta, z, e, a, s|f^n)$ is well-defined for all n. Observe that f^n affects $v^{(d,a')}(\beta, z, e, a, s|f^n)$ in (4) via how q^n affects the feasible set given in (3) and how ψ^n affects $Q^s(s'|\psi^n)$ in (4). Since $\lim_{n\to\infty} c^{(\hat{d},\hat{a}')}(z, \omega|f^n) = c^{(\hat{d},\hat{a}')}(z, \omega|\bar{f})$, the continuity of u gives $\lim_{n\to\infty} u(c^{(\hat{d},\hat{a}')}(z, \omega|f^n)) = u(\lim_{n\to\infty} c^{(\hat{d},\hat{a}')}(z, \omega|f^n)) = u(c^{(\hat{d},\hat{a}')}(z, \omega|\bar{f}))$. From Lemma 1, $\lim_{n\to\infty} Q^s(s'|\psi_{\beta'}^{(d,a')}(\omega|f^n)) = Q^s(s'|\psi_{\beta'}^{(d,a')}(\omega|\bar{f}))$. It follows that $v^{(d,a')}(\beta, z, e, a, s|f)$ is continuous in f and hence $\lim_{n\to\infty} T_{f^n} = T_{\bar{f}}$. Since F is a Banach space and T_f is a contraction map, we may apply Theorem 4.3.6 in Hutson and Pym (1980) to conclude that W is continuous in f. The continuity of $\sigma^{(d,a')}(\beta, z, \omega|f)$ in f follows directly by continuity of σ in W. *Q.E.D.*

THEOREM 2: *There exists a stationary recursive competitive equilibrium.*

PROOF: The proof of existence uses Brouwer's fixed point theorem (Theorem 17.3 in Stokey and Lucas (1989)). To connect to that theorem, we reinterpret the function f as a point in a unit (hyper)cube in high-dimensional Euclidean space. To this end, let $\mathcal{G} = \{((d, a'), \beta, z, \omega) : (d, a') \in \mathcal{Y}, \beta \in \mathcal{B}, z \in \mathcal{Z}, \omega \in \Omega\} \subset \mathcal{Y} \times \mathcal{B} \times \mathcal{Z} \times \Omega$, where $\mathcal{Y} = \{(d, a') : (d, a') \in \{0\} \times \mathcal{A}$ or $(d, a') = (1, 0)\}$. Let M and K be the cardinalities of $\mathcal{G}$ and $\mathcal{Y} \setminus \{(1, 0)\}$. Then, $f \in F^*$ can be thought of as a vector composed by stacking $q \in [0, 1]^K$ and $\psi \in [0, 1]^{B \cdot M}$. Then $f \in [0, 1]^{K+B\cdot M}$ and $F^* \subset [0, 1]^{K+B\cdot M}$. Next, use (15) (with equality) to construct the vector $q^{a'}_{\text{new}}(\omega|f)$ and use (16) to construct the vector $\psi^{(d,a')}_{\text{new}}(\omega|f)$. Then, let J be the mapping

$$f_{\text{new}} \equiv \left(q^{a'}_{\text{new}}, \psi^{(0,a')}_{\text{new}}, \psi^{(1,0)}_{\text{new}}\right) = J(f) : F^* \to F^*.$$

Since $\sigma^{(d,a')}(\beta, z, \omega|f)$ is a continuous function of f (Lemma 3), J is a continuous self-map as (15) and (16) are continuous functions of $\sigma^{(d,a')}(\beta, z, \omega|f)$. And since F^* is a nonempty, closed, bounded, and convex subset of a finite-dimensional normed vector space, by Brouwer's FPT there exists $f^* \in F^*$ such that $f^* = J(f^*)$. *Q.E.D.*

A.5. *Equivalence*

Given an RCE, let $\mathcal{P}(e, a) = \bigcup_{s\in\mathcal{S}}\{m : m = p^{\bar{a}*}(e, a, s)\}$ and $\hat{\Omega} = \{(e, a, m) : (e, a) \in \mathcal{E} \times \mathcal{A}$ and $m \in \mathcal{P}(e, a)\}$ with typical element $\hat{\omega} \in \hat{\Omega}$. An individual in state $(\beta, z, \hat{\omega})$ chooses whether to default d and, conditional on not defaulting, chooses asset a' taking as given

- a price function $q^{a'}(\hat{\omega}) : \mathcal{A} \times \hat{\Omega} \to [0, 1]$,
- credit-score transition functions $Q^{(0,a')}_m(m'|e', \hat{\omega}) : \mathcal{P}(e', a') \times \mathcal{D} \times \mathcal{A} \times \mathcal{E} \times \hat{\Omega} \to [0, 1]$ and $Q^{(1,0)}_m(m'|e', \hat{\omega}) : \mathcal{P}(e', a') \times \mathcal{D} \times \mathcal{A} \times \mathcal{E} \times \hat{\Omega} \to [0, 1]$.

As in (3), this implies that an individual of type β in state $(z, \hat{\omega})$ chooses $(d, a') \in \mathcal{F}(z, \hat{\omega})$ inducing consumption $c^{(d,a')}(z, \hat{\omega})$ satisfying

$$c^{(d,a')}(z, \hat{\omega}) = \begin{cases} y\big(e(\hat{\omega}), z\big) + a(\hat{\omega}) - q^{a'}(\hat{\omega}) \cdot a' & \text{if } \big(d, a'\big) = \big(0, a'\big), \\ y\big(e(\hat{\omega}), z\big)(1 - \kappa_1) - \kappa & \text{if } a < 0 \text{ and } \big(d, a'\big) = (1, 0). \end{cases} \tag{29}$$

For all $(d, a') \in \mathcal{F}(z, \hat{\omega})$, the value functions given by equations (5), (7), (12), and (13) and choice probabilities given by equations (9), (10), and (11) associated with the individual's problem are unchanged in form after substituting $\hat{\omega}$ for ω except for equation (4) now given by

$$\begin{aligned} &v^{(d,a')}(\beta, z, \hat{\omega}) \\ &\quad = u\big(c^{(d,a')}(z, \hat{\omega})\big) \\ &\qquad + \beta\rho \cdot \sum_{\beta', z', e', m'} Q^{\beta}\big(\beta'|\beta\big) Q^{e}\big(e'|e\big) H\big(z'\big) Q^{(d,a')}_m\big(m'|e', \hat{\omega}\big) W\big(\beta', z', \hat{\omega}\big). \end{aligned} \tag{30}$$

Intermediaries issue a positive measure of contracts taking the price function $q^{a'}(\hat{\omega})$ and probability of repayment function $p^{a'}(\hat{\omega})$ as given to maximize profits:

$$\pi^{a'}(\hat{\omega}) = \begin{cases} \rho \cdot \dfrac{p^{a'}(\hat{\omega}) \cdot (-a')}{1+r} - q^{a'}(\hat{\omega}) \cdot (-a') & \text{if } a' < 0, \\ q^{a'} \cdot a' - \rho \cdot \dfrac{a'}{1+r} & \text{if } a' \geq 0. \end{cases} \tag{31}$$

If the intermediary issues a strictly positive measure of credit contracts, then zero profits require

$$q^{a'}(\hat{\omega}) = \begin{cases} \dfrac{\rho \cdot p^{a'}(\hat{\omega})}{1+r} & \text{if } a' < 0, \\ \dfrac{\rho}{1+r} & \text{if } a' \geq 0, \end{cases} \tag{32}$$

which is the analogue of (15).

Consistency requires that the probability of repayment satisfy the analog of (17), namely,

$$p^{a'}(\hat{\omega}) = \sum_{\beta', z', e', m'} H(z') \cdot Q^e(e'|e) \cdot Q_m^{(d,a')}(m'|e', \hat{\omega}) \cdot M_{\beta'}(\hat{\omega}') \cdot \left(1 - \sigma^{(1,0)}(\beta', z', \hat{\omega}')\right). \tag{33}$$

Here, $M(\hat{\omega}) : \hat{\Omega} \to \mathcal{S}$, where $M(\hat{\omega}) = (M_{\beta_1}(\hat{\omega}), \ldots, M_{\beta_B}(\hat{\omega}))$ with the function $M_\beta(\hat{\omega})$ mapping m to the probability an individual is of a given type β.

The transition function in equation (19) which tracks the probability that an individual in state $(\beta, z, \hat{\omega})$ transitions to state $(\beta', z', \hat{\omega}')$ is now given by

$$\begin{aligned} &T(\beta', z', \hat{\omega}'; \beta, z, \hat{\omega}) \\ &\quad = \rho \cdot Q^\beta(\beta'|\beta) \cdot H(z') \cdot Q^e(e'|e) \cdot \sigma^{(d,a')}(\beta, z, m) \cdot Q_m^{(d,a')}(m'|e', \hat{\omega}) \\ &\qquad + (1-\rho) \cdot F_\beta(\beta') \cdot H(z') \cdot F_e(e') \cdot \mathbf{1}_{\{a'=0\}} \cdot \mathbf{1}_{\{m'=p^{\bar{a}*}(e_1,0,F_\beta)\}}. \end{aligned} \tag{34}$$

We can now give the definition of a stationary recursive competitive equilibrium with credit scores.

DEFINITION 6—Stationary Recursive Competitive Equilibrium with Credit Scores: A stationary Recursive Competitive Equilibrium with Credit Scores (RCECS) is a pricing function $q^{a'*}(\hat{\omega})$, a credit-scoring function $Q_m^{(d,a')*}(m'|e', \hat{\omega})$, a choice probability function $\sigma^{(d,a')*}(\beta, z, \hat{\omega})$, a repayment probability function $p^{a'*}(\hat{\omega})$, a credit-score-to-type-probability function $M^*(\hat{\omega})$, and a distribution $\overline{\mu}^*(\hat{\omega})$ such that:

(i) Optimality: Given $q^{a'*}(\hat{\omega})$ and $Q_m^{(d,a')*}(m'|e', \hat{\omega})$, $\sigma^{(d,a')*}(\beta, z, \hat{\omega})$ satisfies (10) and (11) for all $(\beta, z, \hat{\omega}) \in \mathcal{B} \times \mathcal{Z} \times \hat{\Omega}$ and $(d, a') \in \mathcal{F}(z, \hat{\omega})$,

(ii) Zero Profits: Given $Q_m^{(d,a')*}(m'|e', \hat{\omega})$, $M^*(\hat{\omega})$, and $\sigma^{(1,0)*}(\beta, z, \hat{\omega})$, $p^{a'*}(\hat{\omega})$ satisfies (33) for all $\hat{\omega} \in \hat{\Omega}$, and given $p^{a'*}(\hat{\omega})$, $q^{a'*}(\hat{\omega})$ satisfies (32) with equality for all $\hat{\omega} \in \hat{\Omega}$,

(iii) Stationary Distribution: Given $Q_m^{(d,a')*}(m'|e', \hat{\omega})$ and $\sigma^{(d,a')*}(\beta, z, \hat{\omega})$, $\overline{\mu}^*(\beta, z, \hat{\omega})$ is a fixed point of $\mu'(\beta', z', \hat{\omega}') = \sum_{\beta,z,\hat{\omega}} T^*(\beta', z', \hat{\omega}'|\beta, z, \hat{\omega}) \cdot \mu(\beta, z, \hat{\omega})$ for T^* in (34).

Note the difference between the RCE Definition 3 and the RCECS Definition 6: an RCE requires the updating function to be consistent with Bayes's Law (in (iii) of Definition 3), while Definition 6 simply postulates the existence of $Q_m^{(d,a')*}$ and M^* and requires that these be consistent with zero profits.

THEOREM 3: *Given an RCE, let* $m = p^{\bar{a}*}(e, a, s)$. *Suppose that the inverse function* $s = (p^{\bar{a}*})^{-1}(e, a, m)$ *exists. Then an RCECS exists in which the choice probabilities* $\sigma^{(d,a')*}(\beta, z, e, a, m) = \sigma^{(d,a')*}(\beta, z, e, a, s)$ *for* $s = (p^{\bar{a}*})^{-1}(e, a, m)$.

PROOF: Given an RCE and the existence of the inverse function $s = (p^{\bar{a}*})^{-1}(e, a, m)$, set

(a) $M^*(e, a, m) = (p^{\bar{a}*})^{-1}(e, a, m)$,
(b) $q^{a'}(e, a, m) = q^{a'*}(e, a, M^*(e, a, m))$,
(c) $Q_m^{(d,a')*}(m' = \tilde{m}|e', e, a, m) = Q^s(M^*(e', a', \tilde{m}))|\psi^{(d,a')*}(e, a, M^*(e, a, m))$, if $\tilde{m} \in \mathcal{P}(e', a')$ and 0 otherwise,
(d) $W(\beta, z, e, a, m) = W^*(\beta, z, e, a, M^*(e, a, m))$.

By (b), $\mathcal{F}(z, \hat{\omega}) = \mathcal{F}(z, \omega)$ in (29) and (3), and by (c) and (d), $v^{(d,a')}(\beta, z, \hat{\omega})$ in (30) is identical to $v^{(d,a')*}(\beta, z, \omega)$ in (4). Hence, $\sigma^{(d,a')*}(\beta, z, \hat{\omega}) = \sigma^{(d,a')*}(\beta, z, \omega)$, satisfying condition (i) in Definition 6. If the choice probabilities are the same, then repayment probabilities in (33) and (17) are the same since $s'(\beta') = M^*_{\beta'}(\hat{\omega}')$ and $Q^*_m = Q^s$, thereby satisfying the requirement on $p^{a^**}(\hat{\omega})$ in (ii) in Definition 6. If the repayment probabilities in (33) and (17) are the same, then prices in (32) and (15) are the same, thus satisfying the requirement on $q^{a'*}(\hat{\omega})$ in (ii) in Definition 6. Since $\sigma^{(d,a')*}(\beta, z, \hat{\omega}) = \sigma^{(d,a')*}(\beta, z, \omega)$ and $Q^*_m = Q^s$, then (34) is the same as (19) so that (iii) in Definition 6 holds. *Q.E.D.*

APPENDIX B: COMPUTATIONAL APPENDIX

B.1. *Computational Algorithm for the Baseline Model*

In this subsection, we describe the algorithm used to compute the RCE stated in Definition 3. The model is calibrated by using the procedure below to solve the model for a given set of parameters, and then updating parameters to minimize the distance between the model moments and the data moments. This outer minimization is performed using the Nelder–Mead simplex method over hundreds of (randomly chosen) initial conditions.

1. Specify all grids and parameters. Relevant details:
   (a) Asset grid is log-spaced in both directions from 0 with 50 points between $[-0.15, -0.00001]$ and 130 points between $[0, 15]$.
   (b) Type score grid is linearly spaced with 40 points between $\min\{G_{\beta_H}, Q^\beta(\beta'_H|\beta_L)\}$ and $Q^\beta(\beta'_H|\beta_H)$.
   (c) Equilibrium convergence is on p and ψ functions with gradual updating; since ψ is more sensitive, we use a relaxation parameter of $\theta \in (0, 1)$ on p and $\eta\theta$ on ψ for $\eta \in (0, 1)$.
   (d) Persistent and transitory earnings grids are 5- and 3-point discretizations of the processes in Table I, respectively, yielding $\mathcal{E} = \{-0.71, -0.27, 0, 0.27, 0.71\}$ and $\mathcal{Z} = \{-0.18, 0, 0.18\}$.
   (e) All newborns have no assets, lowest e, and $s = F_{\beta_H}$. They are distributed across β and z according to F_{β_H} and $H(z)$, respectively.
   (f) Given α, we set the mean of the ν shocks to be

$$\bar{\nu} = -\alpha(\gamma_E + \ln 2) \quad \Longrightarrow \quad \mathbb{E}\big[\max\{\nu_D, \nu_{\text{ND}}\}\big] = 0. \tag{35}$$

(g) Compute consumption associated with all non-borrowing actions (since r is exogenous, these do not change iteration to iteration).

i. *Savings*: for each $\omega = (a, s, e)$ and z, compute the consumption associated with each feasible action $a' \geq 0$ such that

$$c^{(0,a')}(z, \omega) = y\big(e(\omega), z\big) + a(\omega) - \frac{\rho}{1+r} a' > 0.$$

Let $\overline{n}(z, \omega)$ denote the index of the largest budget feasible a' for an agent with (z, ω).

ii. *Default*: define the consumption for a defaulter to be

$$c^{(1,0)}(z, \omega) = y\big(e(\omega), z\big)(1 - \kappa_1) - \kappa,$$

where κ is a fixed bankruptcy filing cost and κ_1 is a cost that scales with earnings.

2. **Main equilibrium loop.** Every iteration j starts with a value of: (i) $f_j = (q_j^{a'}(\omega), \psi_j^{(0,a')}(\omega), \psi_j^{(1,0)}(\omega))$; and (ii) the (ex ante) value function $W_j(\beta, z, \omega)$.[47,48]

(a) Compute consumption associated with all $a' < 0$ given current prices:

$$c^{(0,a')}(z, \omega | f_j) = y\big(e(\omega), z\big) + a(\omega) - q_j^{a'}(\omega) a'.$$

Note that our Assumption 1 implies that all debt choices are always feasible, which is critical for keeping our Bayesian updates well-defined.

(b) Compute mean of extreme value shock associated with each $a' \in \mathcal{F}(z, \omega | f_j)$:

i. For $n = 1, \ldots, \overline{n}(z, \omega)$, compute

$$c^{(0,\hat{a}_n)}(z, \omega | f_j) = y\big(e(\omega), z\big) + a(\omega) - q_j^{\hat{a}_n}(\omega)\hat{a}_n$$

$$\text{where } \hat{a}_1 = a_1 \text{ and } \hat{a}_n = a_{n-1} + \frac{a_n - a_{n-1}}{2} \text{ for } n = 2, \ldots, N,$$

and $q^{\hat{a}_n}(\omega)$ is given by the linear interpolation of the q function:

$$q_j^{\hat{a}_1}(\omega) = q_j^{a_1}(\omega) \quad \text{for } a' = a_1,$$

$$q_j^{\hat{a}_n}(\omega) = \frac{q_j^{a_{n-1}}(\omega) + q_j^{a_n}(\omega)}{2} \quad \text{for } n = 2, \ldots, N.$$

ii. Define the measure of consumption associated with choice $a' = a_n$ as

$$\eta^{a_n}(z, \omega | f_j) = \begin{cases} \left| c^{(0,\hat{a}_n)}(z, \omega | f_j) - c^{(0,\hat{a}_{n+1})}(z, \omega | f_j) \right| \\ \quad \text{for } n = 1, \ldots, \overline{n}(z, \omega) - 1, \\ \left| c^{(0,\hat{a}_n)}(z, \omega | f_j) - 0 \right| \\ \quad \text{for } n = \overline{n}(z, \omega). \end{cases} \tag{36}$$

[47]While we index these functions by $f = (q^{a'}(\omega), \psi^{(0,a')}(\omega), \psi^{(1,0)}(\omega)))$ to maintain consistency with notation in the text, the algorithm actually iterates on $p^{a'}(\omega)$ which directly yields $q^{a'}(\omega)$ via (15).

[48]Since the full information version of the model solves very quickly, for the initial $j = 0$ values, the value functions and loan price schedules provide a good initial guess. For type scores, a consistent initial guess is $\psi^{(d,a')}(e, a, s) = sQ^{\beta}(\beta_H | \beta_H) + (1 - s)Q^{\beta}(\beta_H | \beta_L)$.

iii. The mean of ϵ^{a_n} for $n = 1, \dots, \overline{n}(z, \omega)$ is taken to be

$$\overline{\epsilon}^{a_n}(z, \omega | f_j) = -\lambda \gamma_E + \lambda \ln \eta^{a_n}(z, \omega | f_j), \tag{37}$$

where λ is the common scale parameter for all shocks.

(c) Iterate to convergence on the value function. Starting with $W_{j,k=1}(\beta, z, \omega) = W_j(\beta, z, \omega)$

i. Compute the conditional value function in (4):

$$\begin{aligned} v_k^{(d,a')}(\beta, z, \omega | f_j) &= u\big(c^{(d,a')}(z, \omega | f_j)\big) \\ &\quad + \beta\rho \cdot \sum_{(\beta', z', e', s')} Q^\beta(\beta' | \beta) Q^e(e' | e) H(z') Q^s\big(s' | \psi_j^{(d,a')}(\omega)\big) W_{j,k}(\beta', z', \omega'). \end{aligned}$$

ii. As in (12), let

$$\begin{aligned} W_k^{\mathrm{ND}}(\beta, z, \omega | f_j) &= \mathbb{E}\Big[\max_{n=1,\dots,\overline{n}(z,\omega)} v_k^{(0,a'_n)}(\beta, z, \omega | f_j) + \epsilon^{a'_n}\Big] \\ &= \lambda \gamma_E + \lambda \ln\left(\sum_{n=1}^{\overline{n}(z,\omega)} \exp\left(\frac{v_k^{(0,a'_n)}(\beta, z, \omega | f_j) + \overline{\epsilon}^{a'_n}(z, \omega | f_j)}{\lambda}\right)\right) \\ &= \lambda \ln\left(\sum_{n=1}^{\overline{n}(z,\omega)} \eta^{a'_n}(z, \omega | f_j) \exp\left(\frac{v_k^{(0,a'_n)}(\beta, z, \omega | f_j)}{\lambda}\right)\right). \end{aligned} \tag{38}$$

Note that this step applies the definition in (37) from step (2(b)iii).

iii. As in (13), update

$$W_{j,k+1}(\beta, z, \omega) = \begin{cases} W_k^{\mathrm{ND}}(\beta, z, \omega | f_j) \\ \quad \text{if } a(\omega) \geq 0, \\ \mathbb{E}\big[v_k^{(1,0)}(\beta, z, \omega | f_j) + \nu^D, W_k^{\mathrm{ND}}(\beta, z, \omega | f_j) + \nu^{\mathrm{ND}}\big] \\ \quad \text{if } a(\omega) < 0. \end{cases}$$

For the $a(\omega) < 0$ case, using $\overline{\nu}$ from step (1f), we simply have

$$\begin{aligned} W_{j,k+1}(\beta, z, \omega) &= \alpha \gamma_E + \alpha \ln\left(\exp\left(\frac{W_k^{\mathrm{ND}}(\beta, z, \omega | f_j) + \overline{\nu}}{\alpha}\right) + \exp\left(\frac{v_k^{(1,0)}(\beta, z, \omega | f_j) + \overline{\nu}}{\alpha}\right)\right) \\ &= -\alpha \ln 2 + \alpha \ln\left(\exp\left(\frac{W_k^{\mathrm{ND}}(\beta, z, \omega | f_j)}{\alpha}\right) + \exp\left(\frac{v_k^{(1,0)}(\beta, z, \omega | f_j)}{\alpha}\right)\right). \end{aligned}$$

iv. If $\sup |W_{j,k+1}(\beta, z, \omega) - W_{j,k}(\beta, z, \omega)|$ is less than desired tolerance, go to step (2d); otherwise, go to step (2c) starting with $W_{j,k+1}(\beta, z, \omega)$.

(d) Compute decision densities:

i. As in (9) in the text, the probability of choosing $a'_n \in \mathcal{F}(z, \omega|f_j)$ conditional on not defaulting is 0 if $a'_n \notin \mathcal{F}(z, \omega|f_j)$; otherwise,

$$\tilde{\sigma}^{(0,a'_n)}(\beta, z, \omega|f_j) = \frac{\eta^{a'_n}(z, \omega|f_j)\exp\left(\frac{v^{(0,a'_i)}(\beta, z, \omega|f_j)}{\lambda}\right)}{\sum_{n=1}^{\overline{n}(z,\omega|f_j)} \eta^{a'_n}(z, \omega|f_j)\exp\left(\frac{v^{(0,a'_n)}(\beta, z, \omega|f_j)}{\lambda}\right)}. \tag{39}$$

ii. As in (10) in the text, the probability of default for $a(\omega) < 0$ is 0 if $a(\omega) \geq 0$; otherwise,

$$\sigma^{(1,0)}(\beta, z, \omega|f_j) = \frac{\exp\left(\frac{v^{(1,0)}(\beta, z, \omega|f_j)}{\alpha}\right)}{\exp\left(\frac{v^{(1,0)}(\beta, z, \omega|f_j)}{\alpha}\right) + \exp\left(\frac{W^{\mathrm{ND}}(\beta, z, \omega|f_j)}{\alpha}\right)}.$$

iii. Combining these, we obtain the unconditional probability

$$\sigma^{(0,a'_n)}(\beta, z, \omega|f_j) = \left(1 - \sigma^{(1,0)}(\beta, z, \omega|f_j)\right)\tilde{\sigma}^{(0,a'_n)}(\beta, z, \omega|f_j).$$

(e) Given the decision probabilities $\sigma^{(1,0)}(\beta, z, \omega|f_j)$ and $\sigma^{(d,a')}(\beta, z, \omega|f_j)$, compute the new set of equilibrium functions, $f_{j+1} = (q^{a'}_{j+1}(\omega), \psi^{(0,a')}_{j+1}(\omega), \psi^{(1,0)}_{j+1}(\omega))$:

i. Compute $\psi^{(0,a')}_{j+1}(\omega)$ and $\psi^{(1,0)}_{j+1}(\omega)$ according to (16).

ii. Compute $q^{a'}_{j+1}(\omega)$ according to (15) using $p^{a'}_{j+1}(\omega)$ in (17).

(f) Assess equilibrium function convergence in terms of the sup norm metric

$$\max\left\{\sup\left|\psi^{(d.a')}_{j+1}(\omega) - \psi^{(d.a'}_{j}(\omega)\right|, \sup\left|q^{a'}_{j+1}(\omega) - q^{a'}_{j}(\omega)\right|, \sup\left|W_{j+1}(\beta, z, \omega) - W_j(\beta, z, \omega)\right|\right\}.$$

If less than tolerance, proceed to step 3; otherwise, start step 2 with f_{j+1} and $W_{j+1}(\beta, z, \omega)$.

3. Compute the stationary distribution.

(a) Given f_j from step (2), compute $\mu_{k+1}(\beta, z, \omega|f_j)$ using the transition operator T in (19) applied to $\mu_k(\beta, z, \omega|f_j)$.

(b) Assess convergence based on the sup norm metric $\sup|\mu_{k+1}(\beta, z, \omega|f_j) - \mu_k(\beta, z, \omega|f_j)|$. If less than tolerance, stop; otherwise, iterate on $\mu_{k+1}(\beta, z, \omega|f_j)$ using T.

B.2. *Model Moment Definitions*

The bankruptcy rate is computed as the total fraction of the population who files for bankruptcy within a given period. The probability of a given state is given by $\mu(\cdot)$, and the probability of bankruptcy given a state is $\sigma^{(1,0)}(\cdot)$, and so the aggregate bankruptcy rate is $\sum_{\beta,z,\omega}\sigma^{(1,0)}(\beta, z, \omega)\cdot\mu(\beta, z, \omega)$. By type, we have $\sum_{\omega}\sigma^{(1,0)}(\beta, z, \omega)\cdot\mu(\beta, z, \omega)/\sum_{\hat{\omega}}\mu(\beta, z, \hat{\omega})$. (Analogous type conditions hold for all other moments as well; we omit them here for brevity.) The fraction in debt is the share of the population choosing $a' < 0$ in a given period: $\sum_{\beta,z,\omega,a'<0}\mu(\beta,z,\omega)\sigma^{(0,a')}(\beta,z,\omega)$. The debt-to-income ratio is the ratio of average debt to average income: $\frac{\sum_{\beta,z,\omega,a'<0}a'\sigma^{(0,a')}(\beta,z,\omega)\mu(\beta,z,\omega)}{\sum_{\beta,z,\omega,a'<0}y(e(\omega),z)\mu(\beta,z,\omega)}$. The average interest paid in the economy is the weighted average of the interest rates paid, $1/q-1$, over the stationary distribution and decision probabilities: $\sum_{\omega}\overline{\mu}(\omega)\cdot\sum_{\beta,z}\frac{\mu(\beta,z,\omega)}{\sum_{\hat{\beta},\hat{z}}\overline{\mu}(\omega)}\sum_{a'}\frac{\sigma^{(0,a')}(\beta,z,\omega)}{\sum_{\tilde{a}}\sigma^{(0,\tilde{a})}(\beta,z,\omega)}(\frac{1}{q^{a'}(\omega)}-1)$, where $\overline{\mu}(\omega)=\sum_{\beta,z}\mu(\beta,z,\omega)$. The standard deviation is the square root of the second moment of this object.

B.3. *Sensitivity Analysis: Implementation of Andrews, Gentzkow, and Shapiro (2017)*

We begin by computing the 10×8 Jacobian matrix $\hat{G}$ of the 10-vector of model moments with respect to the 8-vector of internally estimated model parameters. We approximate this matrix by taking numerical derivatives. Using a parameter step size of $\delta_p\cdot\hat{\theta}_p$ for $p=1,\dots,8$ (i.e., proportional scaling, where δ_p is the proportional increase and $\hat{\theta}_p$ is the estimated parameter), we solve the model for the baseline calibration $\hat{\theta}=\{\hat{\theta}_p\}_{p=1}^{8}$, obtaining moment vector $\hat{m}=\{\hat{m}_n\}_{n=1}^{10}$, and for a sequence of 8 perturbations in which the pth parameter is increased by the step size. We set $\delta_p=\delta=0.1\%$ for all p. The entry of the estimated Jacobian matrix $\hat{G}$ corresponding to moment n and parameter p is then $\hat{g}_{\mathrm{np}}=(\hat{m}_{\mathrm{np}}-\hat{m}_n)/\delta\hat{\theta}_p$. The transpose of this matrix, $\hat{G}'$, is presented in Table V.

Given our estimate of $\hat{G}'$, we compute an estimate of Andrews, Gentzkow, and Shapiro (2017)'s sensitivity matrix $\hat{\Lambda}$ using equation (24) with the identity weighting matrix $W=I_{10}$. What is presented in Table IV is not $\hat{\Lambda}$ directly, but a more easily interpretable transformation which we now describe. Our goal is to answer the question: "by what percent would the estimated parameter $\hat{\theta}_p$ change if target moment m_n changed by δ_n percent?" We assume a change in moment m_n of $\delta_n\hat{m}_n$; for ease of exposition, we choose $\delta_n=1\%$ for all n. Then, the bias in the $\hat{\theta}_p$ associated with the perturbation to moment m_n is $b_{\mathrm{pn}}=\hat{\lambda}_{\mathrm{pn}}\delta_n\hat{m}_n$, where $\hat{\lambda}_{\mathrm{pn}}$ is the corresponding entry of the $\hat{\Lambda}$ matrix. We then report the implied percentage change relative to the estimated parameter, $\hat{\ell}_{\mathrm{pn}}=\frac{\hat{\theta}_p-b_{\mathrm{pn}}}{\hat{\theta}_p}-1$. Each cell of Table IV is the relevant $\hat{\ell}_{\mathrm{pn}}$ entry.

B.4. *The Role of Extreme Value Preference Shocks*

One of the key modifications in our model relative to standard consumer bankruptcy models in macroeconomics is the inclusion of the additive, action-specific preference shocks.[49] The mean of these shocks is adjusted to ensure that the utility bonus scales with the measure of feasible consumption rather than the density of the grid used for computation (see Briglia, Chatterjee, Corbae, Dempsey, and Rios-Rull (2021) for details). In contrast, we calibrate the scale parameters α and λ which govern the variance of the default and a' shocks, respectively. How does behavior in the model change with respect to these parameters? In this section, we address this question by computing actual decision rules under different parameter combinations and describe the differences graphically.[50]

[49] Dvorkin, Sanchez, Sapriza, and Yurdagul (2021) have employed extreme value shocks to smooth out decision rules in models of sovereign default.

[50] An analytical approach is contained in the Additional Material.

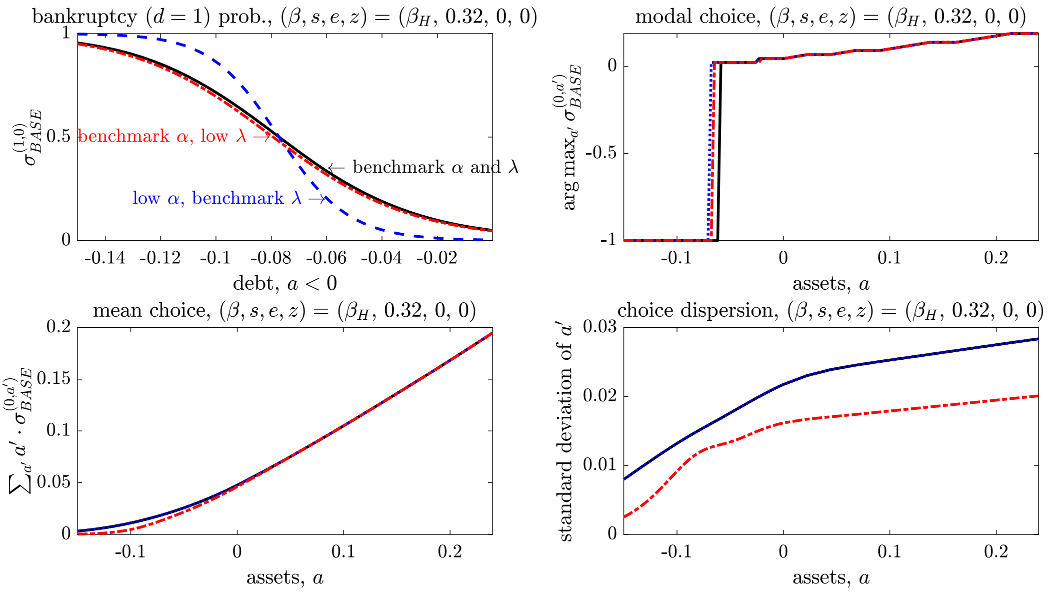

FIGURE 12.—Impact of extreme value preference shocks. *Notes*: "Benchmark" refers to the parameterization of the extreme value shock process from Table III. Low α (λ) is half the baseline value: $\alpha' = \alpha/2$ ($\lambda' = \lambda/2$). All panels fix the state of an agent at $(\beta, s, e, z) = (\beta_H, F_{\beta_H}, 0, 0)$. In the top right panel, a modal choice of -1 corresponds to bankruptcy.

Figure 12 demonstrates the impact of changing α and λ on decisions in our baseline model. Each figure contains three lines, corresponding to: (i) the baseline parameterization of Table III; (ii) a parameterization with low variance α on the bankruptcy decision in which λ is held fixed; and (iii) a parameterization with low variance λ on the a' decision in which α is held fixed. All figures are presented for an agent with $(\beta, s, e, z) = (\beta_H, F_{\beta_H}, 0, 0)$. In each parameterization, the equilibrium pricing function, and therefore the conditional action values, are held fixed, and so the changes in response shown here can be thought of as partial equilibrium in order to highlight the direct effects on decisions.

Consider first the bankruptcy filing decision. The top left panel shows how this decision varies over a range of levels of debt. By lowering α, the slope of increase in filing probability as the level of indebtedness increases is much sharper than in the baseline parameterization. This is because there is less chance for a high value shock to be realized for an action with lower fundamental value, so the decision rule becomes more centered at the mode for each level of a. By lowering λ, the expected value of repaying increases, and so the bankruptcy filing probability shifts down.

The remaining three figures show how a' decisions are affected by changes in the extreme value parameters. The top right panel depicts the modal decision across each case (with bankruptcy depicted as choosing $a' = -1$ for simplicity). Conditional on repaying, there is little change in the modal decision, but lowering either α or λ makes bankruptcy the modal decision only for larger levels of debt. The bottom left and bottom right panels show the mean and standard deviation of the savings decision rule, conditional on repaying, respectively. Changing α has virtually no effect conditional on repaying. Mean decisions are nearly linear in wealth for positive a given the low risk aversion, but there is convexity in the decision rule when in debt since default risk changes the return on

TABLE VIII

MODAL CHOICE METRICS.

Action Type	Share for Whom Action Type Is Modal (%)	Share of Total Action From Modal Agents (%)	Share of Decisions w/in k Grid Pts. of Mode (%)		
			$k=0$	$k=1$	$k=2$
Default	2.72	5.25	-	-	-
Non-Default	99.8	99.8	49.5	83.1	93.3
Borrowing	8.10	86.6	34.7	80.3	93.8
Saving	91.6	99.9	50.8	83.4	93.2

Note: For the right three columns, the share is computed over the population of agents for whom the action type is modal.

borrowing relative to saving. Lowering λ lowers both the mean and standard deviation of savings choices, with the latter effect being more pronounced. Finally, we note that these changes in decision rules are similar (holding price and type-score functions fixed) in the full information and no-tracking economies as well.

B.5. *Modal Choice Metrics*

This section describes a series of metrics which quantify the dispersion in decisions implied by extreme value shocks. These results are summarized in Table VIII, but we first describe the construction of the metrics. Let $x = (\beta, z, \omega)$ be the state variable of an agent, let $\sigma^{(d,a')}(x)$ denote her decision rule, and let $\mu(x)$ be the stationary distribution over individual states in the baseline economy. We want to get a sense of dispersion around the highest value (or modal) choice, which may be defined as

$$y^*(x) \equiv \arg \max_{(d,a')\in\mathcal{F}(x)} \sigma^{(d,a')}(x).$$

Let $\mathcal{Y} \subseteq \{(1,0), \{(0,a')\}|a' \in \mathcal{A}\}$ denote a set of possible actions. The share of agents for whom an action in set $\mathcal{Y}$ is modal is

$$m(\mathcal{Y}) = \sum_x \mu(x)\mathbb{1}\big[y^*(x) \in \mathcal{Y}\big], \tag{40}$$

where $\mathbb{1}[S]$ is an indicator function which takes on the value 1 if S is true. The total mass of agents choosing an action in the set $\mathcal{Y}$ includes those for whom the action is not modal, and so we can compute the share of the actions in this set accounted for by "modal agents," those for whom an action in this set is the mode, via

$$\frac{\displaystyle\sum_{x,(d,a')\in\mathcal{Y}} \mu(x)\sigma^{(d,a')}(x)\mathbb{1}\big[y^*(x) \in \mathcal{Y}\big]}{\displaystyle\sum_{x,(d,a')\in\mathcal{Y}} \mu(x)\sigma^{(d,a')}(x)}. \tag{41}$$

Last, for agents whose modal action is not default, we can compute the share of decisions within k grid points of the mode. For a given individual (whose mode is not default), let $i^*(x)$ denote the grid index of the mode $y^*(x)$. Then let a k-band of actions around the mode be defined by

$$\mathcal{Y}_k(x) = \big\{i^*(x) - k, \dots, i^*(x), \dots, i^*(x) + k\big\}.$$

Finally, define the total weight on decisions in the k-band of the mode for agent x via

$$\zeta_k(x) = \frac{\sum_{(0,a')\in\mathcal{Y}_k(x)} \sigma^{(0,a')}(x)}{1-\sigma^{(1,0)}(x)},$$

where the denominator normalizes to exclude default. We can aggregate over any group of actions $\mathcal{Y}$:

$$\overline{\zeta}(\mathcal{Y}_k) = \frac{\sum_{\{x|y^*(x)\in\mathcal{Y}_k\}} \zeta_k(x)\mu(x)}{m(\mathcal{Y})}. \tag{42}$$

B.6. *Details of Alternative Economies*

B.6.1. *No Tracking (NT)*

The key formal difference in this economy relative to the baseline comes from the separation of the type-score updates (which follow individuals) and the static assessment of types (relevant for pricing). An individual's type score updates based only on exogenous transition probabilities, and so there is no incentive to acquire reputation. As a result, s' evolves from s according to $\psi^1_{\mathrm{NT},\beta'}(s) = \sum_\beta Q^\beta(\beta'|\beta)s(\beta)$. In the two-type case we employ in our quantitative model, we have

$$\psi^1_{\mathrm{NT}}(s) = sQ^\beta(\beta_H|\beta_H) + (1-s)Q^\beta(\beta_H|\beta_L). \tag{43}$$

In this version of the model, lenders perform **intraperiod** updating of type assessments based on the a' chosen by the borrower. That is, the lenders compute

$$\psi^2_{\mathrm{NT},\beta'}(a', s, e) \equiv \Pr(\beta'|a', s, e) = \sum_\beta Q^\beta(\beta'|\beta)\Pr(\beta|a', s, e).$$

All of the action is in the last term of the expression above, and so we analyze it here:

$$\begin{aligned}\Pr(\beta|a', s, e) &= \frac{\Pr(\beta, a', s, e)}{\Pr(a', s, e)} = \frac{\sum_{z,a}\Pr(\beta, a', s, e, z, a)}{\sum_{\tilde\beta,z,a}\Pr(\tilde\beta, a', s, e, z, a)} \\ &= \frac{\sum_{z,a}\sigma^{(0,a')}(\beta, e, z, a, s)\mu(\beta, e, z, a, s)}{\sum_{\tilde\beta,z,a}\sigma^{(0,a')}(\tilde\beta, e, z, a, s)\mu(\tilde\beta, e, z, a, s)},\end{aligned}$$

where the first line uses Bayes's rule, the second sums over unobserved idiosyncratic states, and the third once more applies Bayes's rule via

$$\begin{aligned}\Pr(a', \beta, e, z, a, s) &= \Pr(a'|\beta, e, z, a, s)\Pr(\beta, e, z, a, s) \\ &= \sigma^{(0,a')}(\beta, e, z, a, s)\mu(\beta, e, z, a, s).\end{aligned}$$

Therefore, we obtain

$$\psi^2_{\mathrm{NT},\beta'}(a', s, e) = \sum_{\beta} Q^{\beta}(\beta'|\beta) \frac{\sum_{z,a} \sigma^{(0,a')}(\beta, e, z, a, s)\mu(\beta, e, z, a, s)}{\sum_{\tilde{\beta},z,a} \sigma^{(0,a')}(\tilde{\beta}, e, z, a, s)\mu(\tilde{\beta}, e, z, a, s)}. \tag{44}$$

What the lender must compute is the probability that a' is repaid tomorrow given s, e observed today. For each choice of a', the lender revises the **borrower's assessed type today** via (44). At the same time, though, due to the implicit "anonymity" assumption in this economy, they recognize that the **borrower's type score tomorrow** (which is relevant for tomorrow's default decision) will be determined via (43). Therefore, the $p(\cdot)$ function in this economy is

$$\begin{aligned}
&p(a', s, e) \\
&\quad = \Pr(\text{repay } a'|s, e) \\
&\quad = \frac{\Pr(\text{repay } a', s, e)}{\Pr(s, e)} \\
&\quad = \frac{\sum_{\beta',e',z',s'} \Pr(\text{repay } a'|\beta', e', z', s', a', s, e) \Pr(\beta', e', z', s'|a', s, e)}{\sum_{\beta,a,z} \Pr(\beta, e, z, a, s)} \\
&\quad = \frac{\sum_{\beta',e',z'} [1 - \sigma^{(1,0)}(\beta', e', z', a', \psi^1_{\mathrm{NT}}(s))] \Pr(\beta', e', z'|a', s, e)}{\sum_{\beta,a,z} \mu(\beta, e, z, a, s)} \\
&\quad = \frac{\sum_{\beta,\beta',e',z'} [1 - \sigma^{(1,0)}(\beta', e', z', a', \psi^1_{\mathrm{NT}}(s))] Q^e(e'|e) H(z') Q^{\beta}(\beta'|\beta) \Pr(\beta|a', s, e)}{\sum_{\beta,a,z} \mu(\beta, e, z, a, s)} \\
&\quad = \psi^2_{\mathrm{NT}}(a', s, e) \sum_{e',z'} [1 - \sigma^{(1,0)}(\beta_H, e', z', a', \psi^1_{\mathrm{NT}}(s))] Q^e(e'|e) H(z') \\
&\qquad + (1 - \psi^2_{\mathrm{NT}}(a', s, e)) \sum_{e',z'} [1 - \sigma^{(1,0)}(\beta_L, e', z', a', \psi^1_{\mathrm{NT}}(s))] Q^e(e'|e) H(z'),
\end{aligned} \tag{45}$$

where the last line once more applies the two-type implementation from our quantitative model.

Figure 13 shows the percentage differences between the price menus faced by some agents in the NT economy relative to the BASE economy in comparable states (specifically the lowest persistent earning state $e = -0.71$ since they are the likely to borrow). For newborns (i.e., 20-year-olds in our mapping to the data), the comparison is easy, as all newborns begin life in the same observable state in both economies (and it is common knowledge they do), that is, they all have zero assets, are in the low earnings class, and

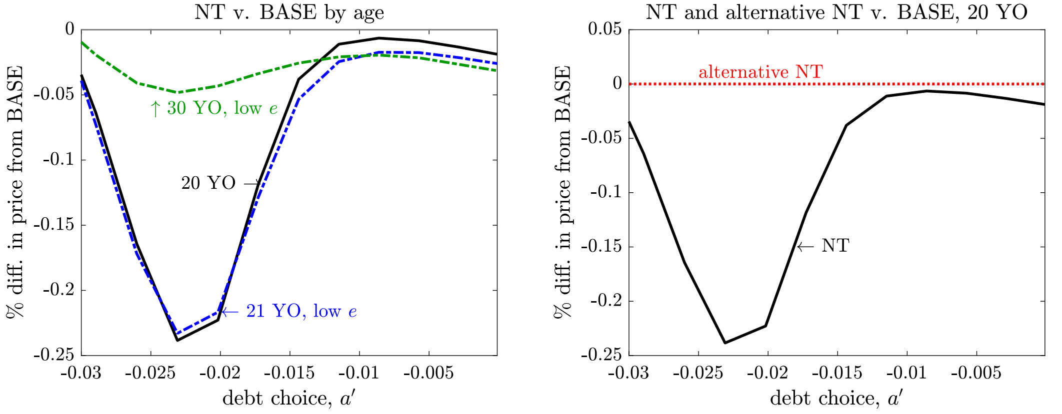

FIGURE 13.—Loan price comparison between BASE and NT economies. *Notes*: Let s_j denote the average type score for an agent of age j, and let a_j^{NT} be the average wealth of an agent of age j in the NT economy. Each line in each panel represents $100 \cdot (q^{a'}_{NT}(j, e)/q^{a'}_{BASE}(e, a_j^{NT}, s_j) - 1)$. The black lines in the left and right panels are the same by construction. The "alternative" NT line in the right panel replaces σ in equation (44) with σ_{BASE}, the decision rules from the BASE economy. All price schedules are for the lowest $e = -0.71$.

are high types with probability 0.32. The price difference comes only from the different probabilities of repayment across the two economies owing to differences in dynamic incentives. Prices are comparable up to a loan size of 0.01, and for larger loans the prices are lower in NT, reflecting higher default probabilities at each loan size. This is due to the lower incentives to repay in the NT economy. These incentive effects, though mitigated, are present even at older ages.

A Decomposition Exercise. In order to highlight the role of dynamic reputational incentives, we construct an alternative price schedule for the NT economy by replacing the decision rule $\sigma(\cdot)$ in the definition of the static inference function $\psi^2_{\mathrm{NT}}(\cdot)$ defined in equation (44) with the decision rule from the baseline economy, $\sigma_{\mathrm{BASE}}(\cdot)$. Having obtained this alternative $\tilde{\psi}^2_{\mathrm{NT}}(\cdot)$, we then compute repayment probabilities (and therefore prices) according to (45) with $\psi^2_{\mathrm{NT}}(\cdot)$ replaced by $\tilde{\psi}^2_{\mathrm{NT}}(\cdot)$. The alternative price schedule is depicted—relative to the analogous price schedule for the baseline economy—for the youngest cohort in the red dashed line in the right panel of Figure 13.[51] For convenience, we also present the standard NT price schedule (solid black line) in this figure. The alternative price schedule is virtually indistinguishable from the baseline price schedule, while the NT prices differ significantly from the baseline. What drives this? In the alternative, the static inference of type reflects the dynamic reputational incentives of the baseline model by construction. The fact that the alternative and baseline so closely resemble each other while the NT and baseline differ markedly highlights the role of dynamic reputational incentives.

B.6.2. *Full Information (FI)*

Since there is no incentive to infer one's type, there is no type score in this model. Therefore, an agent's full state is (β, e, z, a), and the set of equilibrium functions does

[51]We choose the youngest cohort to avoid integrating over a given the assumption across all models that all individuals start with no wealth and the different arguments to the pricing functions in the NT and BASE economies.

not include ψ. For comparability, and since it is purely i.i.d. and contains no information for inference, we maintain the assumption that z is unobservable. Therefore, the lender can observe $\omega_{\text{FI}} = (\beta, e, a)$ for each individual.

The household problem and equilibrium stationary distribution are exactly the same as in the main text, with the state variable s removed. The only substantial change is in the pricing and repayment probability equations. The repayment probability function in this case is $p_{\text{FI}}^{(0,a')}(\omega_{\text{FI}}) = \Pr(\text{repay } a'|\omega_{\text{FI}})$. Since ω_{FI} directly includes β and z is i.i.d., there is no further inference to be done. Therefore, a has no impact on pricing, and we obtain

$$p_{\text{FI}}^{a'}(\beta, e) = \sum_{\beta', e', z'} \left[1 - \sigma_{\text{FI}}^{(1,0)}(\beta', e', z', a')\right] Q^{\beta}(\beta'|\beta) Q^{e}(e'|e) H(z'). \tag{46}$$

The loan pricing function, $q_{\text{FI}}^{a'}(\beta, e)$, adjusts for the interest rate as in the baseline model.

In Figure 14, we compare the prices that individuals of a given age face in the FI economy with their counterpart in the BASE economy who has the average type score for that age and the FI economy's average asset holdings for that age. Since we use the average type score for a given age, the price comparison does take into account the learning that naturally occurs in the Base economy (i.e., type H (L) have a higher (lower) type score than the average type score for their given age).

As one might expect, Figure 14 shows that for each age, type L in the FI economy face lower loan prices (higher interest rates) and type H face higher prices (lower interest rates) than the BASE economy where there is some cross-subsidization. The figure also shows that these price differences change with age. Recall that current asset holdings do not affect debt prices in the FI model but do in the BASE model. As individuals age and accumulate assets, this has an impact on $q_{\text{BASE}}^{(0,a')}$. Any type in the BASE economy who borrows by age 30 having accumulated precautionary assets is very likely assessed to be type β_L. Thus, there is not much difference between the economies for a 30-year-old type β_L, which explains the imperceptible price difference, but since type β_H is pooled

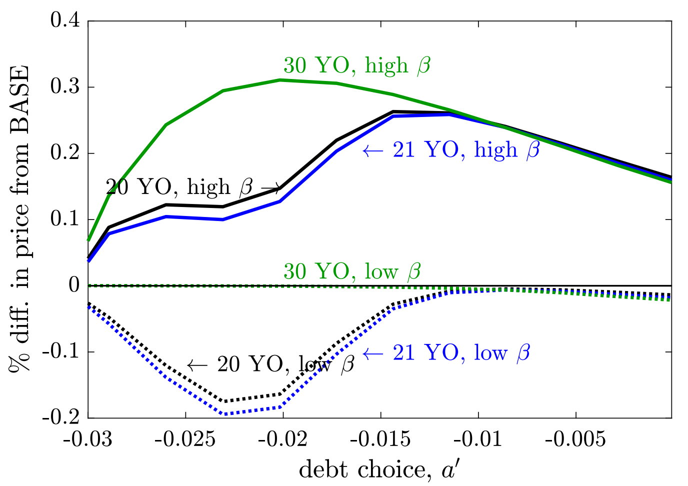

FIGURE 14.—Loan price comparison between full information and baseline economies. *Notes*: Let s_j and a_j^{FI} denote the average type score for an age-j agent in BASE and FI, respectively. Each line in the figure represents $100 \cdot (q_{FI}^{a'}(\beta, e)/q_{BASE}^{a'}(e, a_j^{FI}, s_j) - 1)$. All price schedules are for the lowest e.

with type β_L by their borrowing, hence facing much lower $q^{(0,a')}_{\text{BASE}}$, the price difference is magnified.

APPENDIX C: DATA APPENDIX

This appendix describes the construction of the data underlying the life-cycle credit ranking moments reported in Table II and Figures 1, 2, and 3. We begin with a 2 percent random sample of the FRBNY CCP/Equifax anonymized panel containing an individual's birth year and an individual's credit score in each quarter of 2003, 2004, and 2005. The credit-score measure is the Equifax Risk Score (hereafter Risk Score), which is a proprietary credit score similar to other risk scores used in the industry. We consider only living individuals who were between the ages of 21 and 60 years in 2004 and had a Risk Score value in each quarter of the three years. This yields our *base sample*.

For this sample, we compute the within-quarter percentile ranking of the individual's Risk Scores in each quarter. We call this the individual's *credit ranking*—it is a number that gives the fraction of people who had Risk Scores not exceeding the individual's score in that quarter. We then placed individuals in 5-year age bins according to their age in 2004. We compute the mean and standard deviations of the credit rankings in each bin, averaged over the four quarters of 2004. These moments were used in the regressions that determine the coefficients in the first four rows of the middle panel of Table II. To obtain the autocorrelations, we computed, for each quarter of 2004, the changes in an individual's credit ranking from the same quarters in 2003 and 2005. For each age bin, we then computed the correlation between these pairs of individual changes for each quarter of 2004.

Turning next to the default event study in Figure 3, we first isolated individuals 26 years or older who filed for Chapter 7 bankruptcy in 2004 in our base sample. This yielded our *base sample of bankrupts*. For each individual in this sample, we recorded birth year and Risk Score in the filing quarter and in the 16 quarters preceding and following the filing quarter. We converted each Risk Score into a credit ranking by computing the percentile of each Risk Score in the overall distribution of Risk Scores. We then placed each individual in the appropriate 5-year age bin based on her age in 2004. We computed the average credit ranking (percentile) in each age bin for each of the 33 quarterly observations.

APPENDIX D: DELINQUENCY

Default can arise either through delinquency, whereby agents neither repay their debts nor file for bankruptcy, as well as bankruptcy. Here we modify our model to include this option. In our modified model, the unobservable income loss from default can depend on one's type by a factor of proportionality $\tau(\beta)$, where $\tau(\beta_H) \geq \tau(\beta_L) = 1$ so that a default can be weakly more costly for high types than low.[52] Under the former choice, the household's income net of any costs associated with delinquency (which we take to be $y(e, z)(1 - \kappa_2 \cdot \tau(\beta))$ with $\kappa_2 < \kappa_1$ so there are lower costs than in bankruptcy) is used for consumption and its obligation next period is the face value of its current debt plus a penalty specified in the contract that we take to be a factor $\eta > 0$ of the debt. Upon becoming delinquent, a household can pay back its debts, file for bankruptcy, or become delinquent again. Lenders with delinquent debt are required (by law) to remove (charge

[52]Corbae and Glover (2018) provided an adverse selection labor matching model with pre-employment credit screening which generates a larger income loss for type β_H than type β_L.

off) such debts from their books which they do by selling delinquent debt to third-party collectors. For simplicity, we assume all delinquent debt is pooled and sold to third-party collection agencies at an equilibrium price $\bar{q}_\delta$ to be described below. Buyers of delinquent debt operate at a per-unit cost γ and are competitive.

D.1. *The Household Problem*

We modify the problem in Section 3.1 by expanding the set $\mathcal{D} = \{0, 1, 2\}$ where $d = 2$ signifies delinquency. Delinquency adds a new option and allows a household to avoid repaying its debt without incurring a bankruptcy fee but saddling it with more debt next period. In this case, (3) becomes

$$c^{(d,a')}(z, \omega|f) = \begin{cases} y\big(e(\omega), z\big) + a(\omega) - q^{a'}(\omega) \cdot a' \\ \quad \text{if } \big(d, a'\big) = \big(0, a'\big), \\ y\big(e(\omega), z\big)\big(1 - \kappa_1 \cdot \tau(\beta)\big)\big) - \kappa \\ \quad \text{if } a(\omega) < 0 \text{ and } \big(d, a'\big) = (1, 0), \\ y\big(e(\omega), z\big)\big(1 - \kappa_2 \cdot \tau(\beta)\big) \\ \quad \text{if } a(\omega) < 0 \text{ and } \big(d, a'\big) = \big(2, a(\omega)(1+\eta) \geq a_1\big). \end{cases} \tag{47}$$

The addition is the last line of (47). For any $a(\omega) \in [a_1, 0)$ such that $a(\omega)(1 + \eta) < a_1$ (i.e., a delinquency would take the agent past the lowest grid point), we assume the agent cannot go delinquent and must choose either bankruptcy or repayment (both of which are feasible by Assumption 1). These assumptions imply that delinquency can only happen a finite number of times in a row.[53]

Recall that earlier a household first chose whether to file for bankruptcy or not, and if not, how much to save. We now pose that the household chooses whether to default or not *and* the mode of default. If the household chooses to default, it also chooses whether to file for bankruptcy or to become delinquent; if it does not, it chooses how much to save, receiving a vector of shocks ϵ attached to each a' choice exactly as in the baseline model according to (22). To allow for correlation between the shocks associated with the default actions, we posit a nested logit structure for the shocks no default/bankruptcy/delinquency shocks. That is, rather than the independent draws from (21) as in the baseline, the vector ν is now drawn from

$$F_\nu(\nu) = \exp\left\{-\exp\left(-\frac{\nu^{d=0} - \overline{\nu}}{\alpha}\right) - \left[\exp\left(-\frac{\nu^{d=1} - \overline{\nu}}{\phi\alpha}\right) + \exp\left(-\frac{\nu^{d=2} - \overline{\nu}}{\phi\alpha}\right)\right]^\phi\right\}, \tag{48}$$

where the new parameter ϕ specifies the correlation between the shocks associated with bankruptcy ($d = 1$) and delinquency ($d = 2$).[54] The value functions conditional on each one of the choices in the feasible set $\mathcal{F}(z, \omega)$ follow trivially.

[53]When $a(\omega)(1 + \eta)$ is not on the grid $\mathcal{A}$, similarly to what we did with type scores, we distribute $a(\omega)(1 + \eta) \in [a_j, a_{j+1}]$ with probability w to a_j and probability $1-w$ to a_{j+1} where $w = (a_{j+1} - a(\omega)(1+\eta))/(a_{j+1} - a_j)$.

[54]The adjustment to kill the bonus associated with debtors' extra options in this setting is now $\overline{\nu} = -\alpha\gamma_E - \alpha \ln(1 + 2^\phi)$.

D.2. *Pricing*

All that remains is to determine how lenders price debt given the two types of default. Regulation requires that banks charge off loans that are severely past due.[55] Hence, unlike Athreya, Mustre-del-Rio, and Sanchez (2019) where delinquent debt is held on a lender's balance sheet as long as the individual is delinquent, we assume all delinquent debt is pooled after a period and sold at price $\bar{q}_\delta$ per unit. Competition ensures that debt collectors obtain zero profits net of the transaction (collection) costs to the lending process.

Turning first to the new pricing equation of loans by the financial intermediary, the probability of repayment on a new loan of size a' is altered from that given in equation (17) to

$$p^{a'}(\omega) = \sum_{\beta', z', e', s'} H(z') \cdot Q^e(e'|e) \cdot Q^s(s'(\beta')|\psi_{\beta'}^{(0,a')}(\omega)) \cdot s'(\beta') \cdot \left[1 - \sigma^{(1,0)}(\beta', z', e', a', s') - \sigma^{(2,(1+\eta)a')}(\beta', z', e', a', s')\right]. \tag{49}$$

The probability of delinquency on that new loan is

$$\delta^{a'}(\omega) = \sum_{\beta', z', e', s'} H(z') \cdot Q^e(e'|e) \cdot Q^s(s'(\beta')|\psi_{\beta'}^{(0,a')}(\omega)) \cdot s'(\beta') \cdot \sigma^{(2,(1+\eta)a')}(\beta', z', e', a', s'). \tag{50}$$

Consequently, the competitive price of a new loan offered by lenders is altered from (15) to

$$q^{a'}(\omega) = \frac{\rho}{(1+r)}\left[p^{a'}(\omega) + \delta^{a'}(\omega) \cdot \bar{q}_\delta \cdot (1+\eta)\right], \tag{51}$$

where the second term on the right is the recovery from selling the delinquent debt to a collector.

Turning next to the value of debt held by a collection agency, the probability of repayment on delinquent debt a of a household in state ω held by a collector is

$$p_\delta^{(1+\eta)a}(\omega) = \sum_{\beta', z', e', s'} H(z') \cdot Q^e(e'|e) \cdot Q^s(s'(\beta')|\psi_{\beta'}^{(2,(1+\eta)a)}(\omega)) \cdot s'(\beta') \cdot \left[1 - \sigma^{(1,0)}(\beta', z', e', (1+\eta)a, s') - \sigma^{(2,(1+\eta)^2 a)}(\beta', z', e', (1+\eta)a, s')\right], \tag{52}$$

noting the key differences between (49) and (52) are the type updates $\psi_{\beta'}^{(d,a')}$ and the future debt obligations a'. Equation (52) makes clear that punishment associated with delinquency arises from being saddled with penalties augmenting what is owed and delinquency's impact on type score.

We assume a collector does not need to discharge its own debt holdings if a person becomes delinquent again, but it pays collection costs γ each period. Denoting by $q_\delta^{(1+\eta)a}(\omega)$

[55] From https://en.wikipedia.org/wiki/Charge-off, in the United States, federal regulations require creditors to charge off installment loans after 120 days of delinquency, while revolving credit accounts must be charged off after 180 days.

the value per unit of delinquent debt a of a person in state ω held by a collector, we have

$$q_{\delta}^{(1+\eta)a}(\omega) = \frac{\rho}{(1+r)(1+\gamma)}\Big[p_{\delta}^{(1+\eta)a}(\omega) + \sum_{\beta',z',e',s'} H(z') \cdot Q^e(e'|e) \cdot Q^s(s'(\beta')|\psi_{\beta'}^{(2,(1+\eta)a)}(\omega)) \cdot s'(\beta') \cdot \sigma^{(2,(1+\eta)^2 a)}(\beta', z', e', (1+\eta)a, s') \cdot q_{\delta}^{(1+\eta)^2 a}(e', (1+\eta)a, s') \cdot (1+\eta)\Big]. \tag{53}$$

The zero profit condition for debt collectors is then

$$\bar{q}_{\delta} = \frac{\sum_{\beta,z,\omega} q_{\delta}^{(1+\eta)a}(\omega) \cdot a \cdot \sigma^{(2,(1+\eta)a)}(\beta, z, \omega) \cdot \mu(\beta, z, \omega)}{\sum_{\beta,z,\omega} a \cdot \sigma^{(2,(1+\eta)a)}(\beta, z, \omega) \cdot \mu(\beta, z, \omega)}. \tag{54}$$

Substituting (53) into (54) yields γ residually given an observed $\bar{q}_{\delta}$. We require that $\gamma \geq 0$.

D.3. *Parameterization*

To illustrate our model with both bankruptcy and delinquency, we supplement the estimated parameters from the BASE model with parameters chosen to approximate certain moments like credit card recovery rates, delinquency rates and penalties, and certain restrictions implied by the model on the data. We set the recovery rate $\overline{q}_{\delta}$ to 0.22 as in Chatterjee and Gordon (2012). We set the penalty rate in delinquency η to 30% consistent with industry averages.[56] The extreme value parameter $\phi = 0.2$ and variable cost in delinquency $\kappa_2 = 0.03$ are set to be roughly consistent with the bankruptcy and delinquency rate (measured as being delinquent for four quarters in a row consistent with our annual model period). The collection cost $\gamma = 2.07$ satisfies (54) given (53). Finally, the proportional income loss from default $\tau(\beta_H)$ is 25% higher for type β_H than β_L.

D.4. *How Does a Delinquency Option Change Equilibrium Outcomes?*

In Table IX, we provide the moments from our delinquency extension of the BASE model (adding the 4-quarter delinquency rate that was absent from Table II). While there are some differences, perhaps the most noteworthy result is that the addition of the delinquency option yields model moments not very different from their data counterparts despite not re-estimating the model.

As in the BASE model, type β_L default (i.e., choose either delinquency or bankruptcy) more than type β_H as evident in the top right panel of Figure 15 since their likelihood ratio for default exceeds 0.5 (similar to the earlier results in Figure 4). The novel aspects stem from the fact that, as evident in the budget sets of equation (47), delinquency provides a low current resource cost way to default at the expense of incurring more future debt

[56] See for example, https://www.thebalancemoney.com/credit-card-default-and-penalty-rates-explained-960643.

TABLE IX

TARGET MOMENTS, DELINQUENCY VERSUS BASELINE.

Moment (%)	Data	Baseline	Delinquency
Aggregate credit market moments			
Bankruptcy rate	1.00	1.02	0.74
Average interest rate	11.9	11.5	16.6
Interest rate dispersion	7.00	7.08	2.76
Fraction of HH in debt	7.92	9.16	11.4
Debt-to-income ratio	0.40	0.26	0.36
Delinquency rate	1.54	N.A.	1.11
Credit ranking age profile moments			
Intercept, mean credit ranking	0.278	0.325	0.394
Slope, mean credit ranking	0.038	0.037	0.022
Intercept, std. dev. credit ranking	0.215	0.219	0.267
Slope, std. dev. credit ranking	0.011	0.010	0.003
Average autocorrelation of change in credit ranking	−0.220	−0.204	−0.238

Note: Our model is yearly, so we classify delinquency as for 4 consecutive quarters of delinquency.

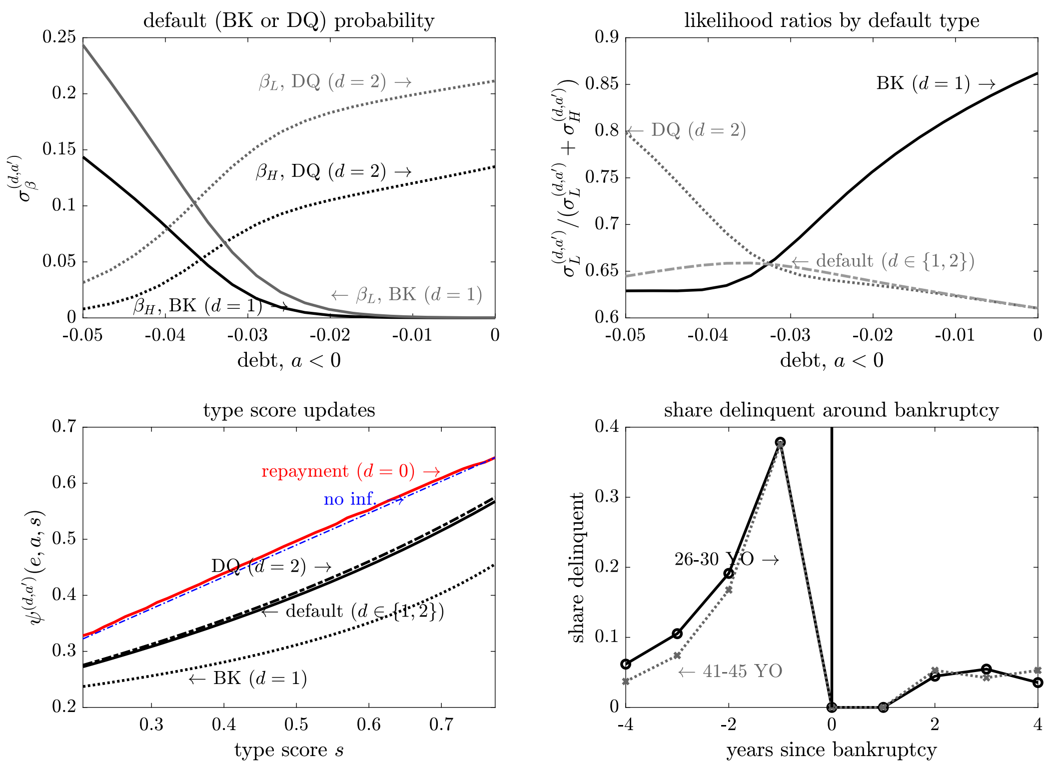

FIGURE 15.—Bankruptcy and delinquency choice probabilities. *Notes*: The individual state for the top panels of this figure is $s=0.48$ and $e=z=0$. The left panel presents the probability of either type of default for each type while the right panel presents the likelihood ratio for each type of default. The bottom left panel The bottom right panel plots the share of agents from a simulated panel who file for bankruptcy in year 0 who are delinquent in year t. This share is zero in the year of the bankruptcy (declaring bankruptcy precludes delinquency) and the year after (bankruptcy in year 0 implies $a=0$ in year 1, so delinquency is infeasible).

and lowering one's future reputation. Since type β_L care less about the future and more about current consumption than type β_H and have lower costs of default, they are more likely to choose delinquency and bankruptcy. This difference is clearly evident in the top row of Figure 15. Since type β_L is more likely to go delinquent and bankrupt, the bottom left panel of Figure 15 shows that such default decisions lead to a fall in their type scores similar to the earlier results in Figure 6.[57] It also shows that a bankruptcy leads to a bigger downward revision of type score than a delinquency.[58] As one might expect, the bottom right panel of Figure 15 shows that bankruptcies often follow delinquencies; a little more than 20% of the individuals who choose bankruptcy are already delinquent (for a model period of one year).[59]

The extended model provides testable predictions. For example, the top left panel shows that both types are more likely to choose delinquency for low debt levels and more likely to choose bankruptcy for high debt levels (for a given earnings and type score). This generates a pattern where, for both types, as debt grows, they substitute out of delinquency into bankruptcy as a form of default. This is intuitive as the future debt cost of delinquency is more severe with higher debt. If one integrates across all individuals who default, this provides a prediction that those who go bankrupt have higher debt levels than first-time delinquents. Our model generates a 17% higher level of debt held by bankrupts than first-time delinquents, while the data generate a 37% higher level.[60]

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[57]The bottom left panel also shows that repayment raises one's score relative to the no-inference case.

[58]This ordering is sensitive to $\tau(\beta_H)$. A bankruptcy can hurt one's type score less than delinquency for $\tau(\beta_H)$ close to 1.

[59]There is a smaller spike in delinquencies preceding bankruptcy for the older cohort since they have lower debts.

[60]The average credit card debt of delinquents (bankrupts) is \$5564 (\$7625) for a ratio of 1.37 (estimates are for individuals with positive credit card debt in 2004 (authors' calculations using FRBNY CCP/Equifax data)).