Supplement to "A Quest for Knowledge"

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B Different Universe of Questions

Our baseline model assumes that the universe of questions can be represented on the real line. That is, we implicitly assume an order on questions. In this part, we show that all our results extend to a more general question space.

To begin with, consider our baseline model and fix some knowledge $\mathcal{F}_m$. As described in Section 2, knowledge pins down $\mathcal{X}_k$—a set composed of (half-)open intervals: bounded intervals $[x_i, x_{i+1})$ of length X_i each, and two unbounded intervals $(-\infty, x_1)$ and $[x_k, \infty)$ of length ∞. All our results survive if knowledge partitions the question space into a set of intervals on the real line (possibly of infinite length).

To see this, consider any set $\widehat{\mathcal{X}}_m = \widehat{\mathcal{X}}_k \cup \widehat{\mathcal{X}}_n$ that contains $k+n$ elements: $k \geq 0$ convex-valued and bounded intervals on $\mathbb{R}$ with Euclidean distance between its upper and lower bound, $X_{i\in\widehat{\mathcal{X}}_k}$, and $n > 0$ convex-valued but unbounded intervals on $\mathbb{R}$ of infinite length, $X_{i\in\widehat{\mathcal{X}}_n} = \infty$. For any tuple (d, X) with $X \in \widehat{\mathcal{X}}_m$ and $d \in [0, X/2]$ all our definitions and expressions for benefits and cost are well-defined regardless of how $\widehat{\mathcal{X}}_m$ was generated.

For any given set $\widehat{\mathcal{X}}_m$ generated by some existing knowledge $\mathcal{F}_m$, suppose that the truth-generating process Y is such that the answer to question x characterized by (d, X) is normally distributed with a variance of $\sigma^2(d; X)$.[1] Then, all of our results continue to hold.

Using this formulation, it becomes clear which formal requirements we impose on the set of questions: (i) There are no circular paths in the set of questions $\widehat{\mathcal{X}}_m$, (ii) the set of questions is piecewise convex-valued, (iii) there is at least one unbounded area. One way to interpret these requirements is to assume a forest network in which

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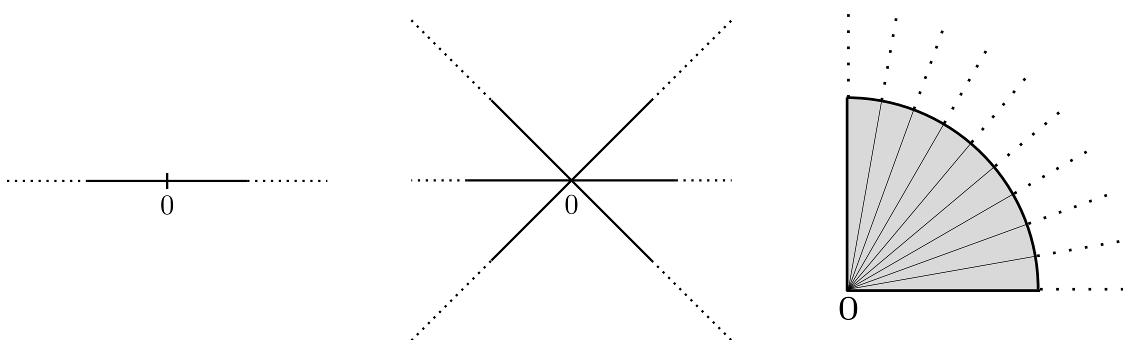
[1]Note that the dependence of the variance of the conjecture depends solely on d and X. Thus, the truth-generating process has to satisfy a Markov property as the Brownian motion on the real line in our main model. Moreover, note that the specification of the expected value of the answer is not relevant for our results as long as it is well-defined given $\mathcal{F}_m$.

the set of nodes represents knowledge and each edge represents an area. We augment this network with (at least) one "frontier"—a standard Wiener process, and define Brownian bridges over each edge of the network.

Figure 7 depicts different question spaces. While the left panel is our baseline setting, the other two provide alternatives, in the middle there are a number of different directions at which we could expand the frontier starting from the origin. In each direction, the truth would be defined by an independent Wiener process starting at $(0, y(0))$. The right panel shows the limit in which the number of directions becomes a continuum. Still, in each direction there is an independent Wiener process governing the truth over this part of the question space such that circles are excluded.



(a) Baseline (b) Tree (c) Ocean

Figure 7: *Different Question Spaces.*

For our static considerations (Sections 3 and 4), all of these models are equivalent because at least one area of infinite length exists at all times allowing the researcher to expand knowledge. Because there are no circles, knowledge partitions the question space into (conditionally) independent segements, just like in the baseline version.

For our dynamic considerations (Section 5), all models are identical as well as long as we focus on symmetric *pure* strategies. Because researchers ignore previous failures and always pick the same direction, the number of additional directions is not important. If we allow for *mixed* strategies instead, the models differ. For example, when randomizing over the direction from the continuum of directions of the ocean model, the probability that the researcher draws a direction that a previous generation had selected (and failed at) is zero. As we discuss in Section 6, such a model may provide a microfoundation for our assumption of "non-observed failures."

We now describe two specific extensions to our baseline setting to illustrate the abstract discussion above.

B.1 General Universe of Questions

Here, we show a mapping from a model with an $n-$dimensional question space. Suppose that the set of research questions consists of n real lines, $\mathcal{I} = \{I_1, \ldots, I_n\}$. In addition, the answers on each of these real lines are determined by a realized path of a one-dimensional standard Brownian motion, such that the truth-generating process is an n-dimensional independent Brownian motion $W_z = (W_z^1, ..., W_z^n)$.[2] Suppose $\mathcal{F}^i_{j(i)}$ is the finite set of $j(i)$ known realizations of the Brownian path in dimension i and $\mathcal{F}_k = \cup_{i=1}^n \mathcal{F}^i_{j(i)}$ is knowledge. As described in Section 2, each $\mathcal{F}^i_{j(i)}$ determines a partition of the domain of W_z^i denoted by $\mathcal{X}^i_{j(i)}$ with $j(i)+1$ elements. As in the baseline case, the knowledge in dimension i decomposes the dimension-i process into $j(i)-1$ independent Brownian bridges each associated with a length X_l^i, $l = \{1, ..., j(i)-1\}$ and two independent Brownian motions. Therefore, the union $\mathcal{F}_k$ determines $k = (\sum_{i=1}^n j(i)) - n$ independent Brownian bridges of length X_l^i each and $2n$ Brownian motions. By the martingale property of the Brownian motion and the fact that realizations are not directly payoff relevant, the setting is isomorphic to one in which we have k independent *standard* Brownian bridges of length X_l^i each and $2n$ *standard* Brownian motions. Thus, the set $\widehat{\mathcal{X}_k} = \{X^i_{l(i)}\} \cup \{\infty\}$ is a sufficient statistic to calculate any of the results in the text. However, the set $\widehat{\mathcal{X}_k} = \{X^i_{l(i)}\} \cup \{\infty\}$ can also be generated with an appropriate realized path of a one-dimensional Brownian motion with a corresponding $\mathcal{F}_k$.

B.2 Seminal Discoveries

We conclude this part by presenting a model with *seminal discoveries*—discoveries that open new fields of research—that builds on the multidimensional universe of questions described above. For example, Friedrich Miescher's isolation of the "nuclein" in 1869 was initially intended to contribute to the study of neutrophils, yet, in addition, it opened up the new and, to a large extent, orthogonal field of DNA biochemistry.

Formally, consider the following model of the evolution of knowledge. Initially, there

[2] Each process starts at an initial point $(0, 0)$, has a drift of zero, a variance of one, and independent, normal increments.

is a single field of research A and a single known question-answer pair, $(x_0, y(x_0)) = (0, 0)$. The set of all questions in field A is known to be one-dimensional and represented by $\mathbb{R}$. The truth is known to be generated by a standard Brownian path Y passing through $(0, 0)$. However, with an exogenous probability $p \in [0, 1]$ any discovery $(x, y(x))$ is *seminal* and opens a new, independent field of research B_x. A seminal discovery is a question-answer pair $(x, y(x))$ that is an element of two independent Brownian paths crossing only at $(x, y(x))$. Thus, upon occurrence, a seminal discovery generates knowledge in multiple dimensions. Because it is a priori unknown whether a discovery is seminal, the payoff from generating knowledge in another dimension is constant in expected terms—it does not influence a researcher's (or designer's) choices. After the seminal discovery, the updated model of truth and knowledge is the one described above with the multi-dimensional universe of questions. As we argued above, that model can, in turn, be mapped into our baseline. The special case with $p = 0$ is our baseline model.

It should become clear from our discussion that even the case in which the probability of a seminal discovery depends on the question is qualitatively similar to what we discuss in the baseline model. The quantitative differences in such a model come from the fact that questions which are likely to be a seminal discovery are more attractive to address for all parties involved.

C Comparison to Kuhn (1962)

In this part, we compare our model and findings in detail to the work of Kuhn (1962). We demonstrate which aspects we cover and where we differ from his seminal ideas.

Similarities. Kuhn himself claims that the

> paradigm as shared example[s] is the central element of the most novel and least understood aspect of 'The Structure of Scientific Revolutions.'
>
> (Kuhn, 1962, p. 187).

Our concept of "conjectures" based on existing knowledge offers an economic interpretation of this idea. The paradigm in a specific field is determined by discoveries (that constitute knowledge in our model) and their implications (which follow from the conjectures derived from knowledge). The discoveries and derived conjectures in our model provide comprehensive information on how society and researchers approach questions.

Furthermore, we believe that our framework and the reoccuring phases of expanding research in our model (see Proposition 5) closely resemble an economic approach to Kuhn (1962)'s idea of *normal science* (described in Kuhn (1962), Chapter 3). Normal science in the Kuhnian sense consists of researchers solving puzzles in the context of a given paradigm. Similarly, researchers in our model build on existing knowledge (and the truth following a Brownian motion) to form conjectures about the location of answers and search for answers where they expect them to be. Whenever a researcher finds an answer, she finds it close to where it was expected to be.

Our analysis of this model then gives rise to a formalized economic theory that explains "how little [researchers] aim to produce major novelties, conceptual or phenomenal" (Kuhn, 1962, p. 35).

Differences. Much in the spirit of Kuhn (1962), there are times at which normal science in our model fails to advance knowledge: The Brownian motion eventually takes an unexpected turn, and researchers will fail to recognize them when expanding knowledge step-by-step building on their conjectures. Kuhn (1962)'s somewhat radical idea then is that anomalies (Kuhn, 1962, Chapter 6) pile up to a crisis (Kuhn, 1962, Chapter 7) in which normal science desperately tries to connect seemingly contradictory information to an old model of the world. At least in our baseline model, such chaos is absent. Instead, while research produces discoveries during the step-by-step expansion of knowledge, conjectures shift only gradually because the discoveries realize within the search intervals chosen by the researcher, and therefore, close to where they were expected. This gradual revision of conjectures appears closer to Toulmin (1970)'s evolutionary model of science, with constant adaptation and revision of theories. When a researcher fails finding an answer building on her conjecture, science gets stuck as researchers keep repeating the same mistakes applying the old, misleading conjecture. Only some exogenous shock (for example, a moonshot or a serendipitous discovery) can take them out of their misery by providing new guidance. Such exogenous shocks may be closer to what Kuhn (1962) has in mind as it follows a period of little progress and sparks a sudden appearance of highly productive research. Thus, while Kuhn

(1962) depicts “revolutions” as settings where researchers go out of their way and wildly experiment with no discipline to uncover the new paradigm, such phases in our setting are either directed (through moonshots) or happen by chance when normal science seizes to advance knowledge.

Summary. Our model can capture many of the observations Kuhn (1962) makes. Yet, there are important differences in the mechanics: While Kuhn (1962) diagnoses that “[w]ithout commitment to a paradigm there could be no normal science” (Kuhn, 1962, p.100), our paradigms evolve as researchers go along. While our findings are in line with Kuhn (1962)’s statement that “surprises, anomalies, or crises . . . are just the signposts that point the way to extraordinary science, (p.101), our framework does not describe that researchers respond to crisis by “searching at random, trying experiments just to see what will happen” (Kuhn, 1962, p.87).

In our world, it is not that there is a missing link in the set of problems to cover, but a missing connection to the problems down the line that needs to be discovered. In Kuhn (1962)’s world, crisis is the (endogenous) driver behind radical change. He claims that:

> Confronted with anomaly or with crisis, scientists take a different attitude toward existing paradigms, and the nature of their research changes accordingly.
>
> Kuhn (1962, p. 90f).

Because the researcher is freed from paradigmatic discipline when in crises, his mind evolves more freely, yet the resolution is then modeled as the “stroke of genius” or, as he puts it:

> More often no such structure is consciously seen in advance. Instead, the new paradigm, or a sufficient hint to permit later articulation, emerges all at once, sometimes in the middle of the night, in the mind of a man deeply immersed in crisis. What the nature of that final stage is-how an individual invents (or finds he has invented) a new way of giving order to data now all assembled-must here remain inscrutable and may be permanently so.

(Kuhn, 1962, p. 89f).

While the consequences of a stroke of genius are similar in our model to what we believe Kuhn (1962) has in mind, we do not model the crisis-plagued researcher. Partially, that is because it remains also opaque within Kuhn (1962) how such thinking comes about. Instead, we ask whether and how well-designed funding institutions (absent in Kuhn (1962)) can help to start research cycles that reduce the risk of crisis and the need for a genius. Here, we connect to Merton (1957) or Partha and David (1994) and take the general freedom of scientists as given, but also acknowledge that they respond to incentives (see, e.g., Myers, 2020), a notion completely absent in Kuhn (1962).

D The Cost of Research and Proof of Lemma 1

In this section, we provide a detailed derivation of the cost of research. Lemma 1 is a corollary to the results we obtain. The cost implies an endogenous measure of the productivity of research. We model research as sampling a set of candidate answers to question x with the goal of discovering the actual answer, $y(x)$.

Formally, we assume that, conditional on a question x, the sampling decision consists of selecting an interval $[a, b] \in \mathbb{R}$. If the true answer lies inside the chosen interval, such that $y(x) \in [a, b]$, research succeeds and a discovery is made. If $y(x) \notin [a, b]$, research fails and no discovery is made. Thus, the choice of the research interval entails an ex-ante probability of successful research. Restricting the sampling decision to a single interval $[a, b]$ comes without loss for our purposes, as conjectures $G_x(y|\mathcal{F}_k)$ follow a normal distribution.

We now characterize the cost of research in terms of the three variables of interest: the research area, X, the novelty of the question, d, and the expected output, ρ.

We begin by defining a prediction interval and characterize it based on the conjecture $G_x(y|\mathcal{F}_k)$.

Definition 4 (Prediction Interval)**.** The prediction interval $\alpha(x, \rho)$ is the shortest interval $[a, b] \subseteq \mathbb{R}$ such that the answer to question x is in the interval $[a, b]$ with probability ρ.

Proposition 8. *Suppose $\alpha(x, \rho)$ is the prediction interval for probability ρ and question*

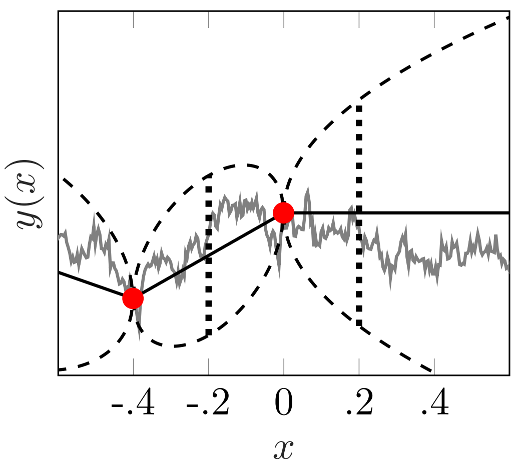


Figure 8: *Cost of research and interference.* The dotted vertical lines represent the 95% prediction intervals for the answers to questions $x = -0.2$ and $x' = 0.2$, assuming the answer to questions 0 and -0.4 are known.

x when answer $y(x)$ is normally distributed with mean μ and standard deviation σ. Then, any prediction interval has the following two features:

1. *The interval is centered around μ.*
2. *The length of the prediction interval is $2^{3/2}\operatorname{erf}^{-1}(\rho)\sigma$, where erf^{-1} is the inverse of the Gaussian error function.*

Proof. The normal distribution is symmetric around the mean with a density decreasing in both directions starting from the mean. It follows directly that the smallest interval that contains the realization with a particular likelihood is centered around the mean.

Take an interval $[z_l, z_r]$ of length $Z < \infty$ that is symmetric around the mean μ and let it be such that it contains a total mass of $\rho < 1$ in the interval. Then, a probability mass of $(1-\rho)/2$ lies to the left of the interval by symmetry of the normal distribution. Moreover, the left bound z_l of the interval has (by symmetry of the interval around the mean μ) a distance $\mu - Z/2$ from the mean. From the properties of the normal distribution,

$$\Phi(z_l) = 1/2\left(1 + \operatorname{erf}\left(\frac{z_l - \mu}{\sigma\sqrt{2}}\right)\right) = 1/2\left(1 + \operatorname{erf}\left(\frac{-Z/2}{\sigma\sqrt{2}}\right)\right).$$

Solving, using the symmetry of erf, yields

$$1/2\left(1 - \operatorname{erf}\left(\frac{Z}{\sigma 2^{3/2}}\right)\right) = \frac{1-\rho}{2} \Leftrightarrow \operatorname{erf}\left(\frac{Z}{\sigma 2^{3/2}}\right) = \rho \Leftrightarrow Z = 2^{3/2}\operatorname{erf}^{-1}(\rho)\sigma. \qquad \square$$

Figure 8 indicates that the prediction interval depends on the location of the question. Two questions with the same distance from existing knowledge (that is, distance from question $x = 0$) have different 95% prediction intervals depending on whether research deepens knowledge or expands it. That difference translates into different costs.

Proposition 8 implies that if the cost function is homogeneous of any degree in interval length $(b-a)$, we can represent it with an alternative cost function proportional to $c(\rho, d, X)$ that is multiplicatively separable in (d, X) and ρ without having to keep track of the exact location of the search interval $[a, b]$, which proves to be convenient.

It also implies that, fixing ρ, the changes in the cost with respect to distance d and area length X vary in their effect on $\sigma(d; X)$ only. Similarly, holding distance and area length constant, changes in ρ translate into cost changes according to a function of $\text{erf}^{-1}(\rho)$—a convex increasing function.

Proposition 8 intuitively links the cost of research effort to the probability of a discovery. Because the inverse error function is increasing and convex, the cost of finding an answer with probability ρ is increasing and convex in ρ.

In the paper we assume the cost to be proportional to $(a-b)^2$. As should be clear from Proposition 8, the quadratic formulation is for convenience only. What matters for our results qualitatively is that the cost is (i) homogeneous, (ii) increasing, and (iii) convex in the sampling interval $(a-b)$. Under the quadratic assumption, the cost function is characterized by a simple corollary to Proposition 8: Lemma 1.

Corollary 1. *For knowledge $\mathcal{F}_k$, probability ρ, and question x, the minimal cost of obtaining an answer to question x with probability ρ is proportional to $c(\rho, d; X) = \tilde{c}(\rho)\sigma^2(d; X)$.*

E Properties of $\tilde{c}$

Summary. The function $\tilde{c}(\rho)$ is convex and increasing on $[0, 1)$ with $\tilde{c}(0) = 0$ and $\lim_{\rho\to 1} \tilde{c}(\rho) = \infty$.[3] The derivative $\tilde{c}_\rho(\rho) = \sqrt{\pi}\,\text{erf}^{-1}(\rho)e^{\tilde{c}(\rho)}$ is increasing and convex with the same limits.

We use that, for $\rho \in (0, 1)$, $\tilde{c}(\rho)$ has a convex and increasing elasticity bounded

[3]Due to this limit and the researcher's ability to choose $\rho = 1$, we augment the support of the cost function to include $\rho = 1$ with $\tilde{c}(1) = \infty$. However, the optimal ρ is always strictly interior unless the cost parameter η is chosen to be zero in which case we assume that $\eta\tilde{c}(\rho = 1) = 0$.

below by 2 and unbounded above. Its derivative $\tilde{c}_\rho(\rho)$ has an increasing elasticity bounded below by 1 and unbounded above. We want to emphasize that these properties are not special to our quadratic cost assumption. To the contrary, $\text{erf}^{-1}(x)^k$ for any $k \geq 2$ admits similar properties with only the lower bounds changing. The following properties are invoked in the proofs:

$$\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} \in (2,\infty) \text{ and increasing,} \qquad \rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho) \in (0,\infty) \text{ and increasing,}$$
$$\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} \in (1,\infty) \text{ and increasing,} \qquad \tilde{c}_\rho^{-1}(x) = \text{erf}\left(\sqrt{\frac{W(2x^2/\pi)}{2}}\right).$$

with $W(\cdot)$ the principal branch of the Lambert W function. We formally prove the properties that do not directly follow from the definition of the inverse of the error function below.

E.1 Proofs of properties of $\tilde{c}(\rho)$

Here, we provide the formal proofs. To simplify notation, we suppress the argument ρ and denote the inverse error function by $\iota := \text{erf}^{-1}(\rho)$.

Lemma 22. *The derivatives of the inverse error function satisfy (i)* $\frac{d}{d\rho}\iota = \frac{1}{2}\sqrt{\pi}e^{\left(\iota^2\right)}$, *(ii)* $\frac{d^2}{d\rho^2}\iota = 2\iota\iota'^2$, *and (iii)* $\frac{d^3}{d\rho^3}\iota = 2\iota'^3\left(1+4\iota^2\right)$.

Proof. See Dominici (2008). □

Lemma 23. *(i)* $\lim_{\rho\to 0}\rho\frac{\iota'}{\iota} = 1$, *(ii)* $\lim_{\rho\to 1}\rho\frac{\iota'}{\iota} = \infty$, *(iii)* $\lim_{\rho\to 0}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = 0$, *and (iv)* $\lim_{\rho\to 0}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right) = \frac{\pi}{3}$.

Proof. We will make use of L'Hôpital's rule and the properties from Lemma 22.
(i) follows from

$$\lim_{\rho\downarrow 0}\rho\frac{\iota'}{\iota} = \lim_{\rho\downarrow 0}\frac{\iota'+\rho\iota''}{\iota'} = \lim_{\rho\downarrow 0}\frac{\iota'+2\rho\iota\iota'^2}{\iota'} = \lim_{\rho\downarrow 0}(1+\rho\iota\iota') = 1.$$

(ii) follows from

$$\lim_{\rho\uparrow 1}\rho\frac{\iota'}{\iota} = \lim_{\rho\uparrow 1}\frac{\iota'+\rho\iota''}{\iota'} = \lim_{\rho\uparrow 1}\frac{\iota'+2\rho\iota\iota'^2}{\iota'} = \lim_{\rho\uparrow 1}(1+2\rho\iota\iota') = \infty.$$

(iii) follows from

$$\lim_{\rho\to 0}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = \lim_{\rho\to 0}\frac{\iota'}{\iota}\left(1-\rho\frac{\iota'}{\iota}\right) + \lim_{\rho\to 0}\rho\frac{\iota''}{\iota} = \underbrace{\lim_{\rho\to 0}\iota'}_{=\sqrt{\pi}/2}\lim_{\rho\to 0}\frac{\iota-\rho\iota'}{\iota^2} + \underbrace{\lim_{\rho\to 0}2\rho\iota'^2}_{=0}$$

$$= -\lim_{\rho\to 0}\frac{\sqrt{\pi}}{2}\frac{\rho\iota''}{2\iota\iota'} = -\lim_{\rho\to 0}\frac{\sqrt{\pi}}{2}\frac{\rho\iota(\iota')^2}{2\iota\iota'} = -\lim_{\rho\to 0}\frac{\sqrt{\pi}}{2}\rho\iota' = 0.$$

(iv) follows from[4]

$$\lim_{\rho\to 0}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right) = \lim_{\rho\to 0} 2\frac{\iota''\iota-\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) + \underbrace{\lim_{\rho\to 0} 4\rho \underbrace{\iota'\iota''}_{=2(\iota')^3\iota}}_{=0}$$

$$= \lim_{\rho\to 0} 2\frac{\iota''\iota-\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) = \lim_{\rho\to 0} 2\frac{\iota''\iota}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) - 2\lim_{\rho\to 0}\frac{\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right)$$

$$= \underbrace{\lim_{\rho\to 0} 4\iota'^2\left(1-\rho\frac{\iota'}{\iota}\right)}_{=0} - 2\lim_{\rho\to 0}\frac{\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) = -2\lim_{\rho\to 0}\left(\rho\frac{\iota'}{\iota}\right)^2\frac{\iota-\rho\iota'}{\rho^2\iota}$$

$$= 2\lim_{\rho\to 0}\frac{\rho\iota''}{2\rho\iota+\rho^2\iota'} = 4\lim_{\rho\to 0}\frac{\iota'^2}{2+\rho\frac{\iota'}{\iota}} = \frac{4}{3}\lim_{\rho\to 0}\iota'^2 = \frac{\pi}{3}. \qquad \square$$

Lemma 24. *The following hold: (i) For all $\rho \in (0,1)$, $\frac{d}{d\rho}\left(\rho\tilde{c}_\rho(\rho)-\tilde{c}(\rho)\right) > 0$. (ii) For all $\rho\in(0,1)$, $\rho\tilde{c}_\rho(\rho)-\tilde{c}(\rho) > 0$. (iii) $\lim_{\rho\to 0}\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} = 2$. (iv) $\lim_{\rho\to 1}\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} = \infty$.*

Proof. (i) holds because $\frac{d}{d\rho}\left(\rho\tilde{c}_\rho(\rho)-\tilde{c}(\rho)\right) = \rho\tilde{c}_{\rho\rho}(\rho) > 0$ by convexity of the inverse error function. (ii) holds due to (i) and $\left(\rho\tilde{c}_\rho(\rho)-\tilde{c}(\rho)\right)|_{\rho=0} = 0$. (iii) holds as the elasticity is equal to $2\rho\frac{\iota'}{\iota}$ and (i) in Lemma 23. (iv) holds by the same observations and (ii) in Lemma 23. $\square$

Lemma 25. *The elasticity of $\tilde{c}(\rho)$, $\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)}$, is increasing in ρ.*

Proof. Recall that $\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} = 2\rho\frac{\iota'}{\iota}$. Therefore, it is sufficient to prove that the inverse error function has an increasing elasticity.

Note that $\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = \frac{\iota'}{\iota} + \rho\frac{\iota''\iota-\iota'^2}{\iota^2}$. From Lemma 23 know that

$$\lim_{\rho\to 0}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = 0 \qquad \lim_{\rho\to 0}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right) = \frac{\pi}{3}.$$

Thus, there exists an $\varepsilon > 0$ such that the elasticity is increasing for $\rho\in(0,\varepsilon)$. To show that it is increasing for all $\rho\in(0,1)$ suppose –toward a contradiction– that the derivative of the elasticity crosses 0. In this case, it has to hold that $\frac{\iota''\iota-\iota'^2}{\iota^2} = -\frac{\iota'}{\rho\iota}$.

[4] To arrive at the first line let $\lambda := \iota'/\iota$ and observe that $(\rho\lambda)'' = (\lambda+\rho\lambda')' = 2\lambda' + \rho\lambda''$ and $\lambda' = 2(\iota')^2 - \lambda^2$ which implies $\lambda'' = 4\iota'\iota'' - 2\lambda\lambda'$.

Consider the second derivative of the elasticity at such a critical point

$$\begin{aligned}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right)|_{\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right)=0} &= 2\frac{\iota''\iota-\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right)+\rho\frac{\iota'''\iota-\iota''\iota'}{\iota^2}\\ &= -2\frac{\iota'}{\iota\rho}\left(1-\rho\frac{\iota'}{\iota}\right)+\rho\frac{\iota'''\iota-\iota''\iota'}{\iota^2} = 2\frac{\iota'}{\iota\rho}\left(\rho\frac{\iota'}{\iota}-1\right)+2\rho\frac{\iota'^3}{\iota}4\iota^2>0\end{aligned}$$

where the last inequality follows because the elasticity is weakly greater than one and all other terms are positive.

Thus, any critical point must be a minimum. However, the elasticity is continuous and increasing at $\rho\in(0,\varepsilon)$. Thus, there is no interior maximum and the elasticity is increasing throughout. $\square$

Lemma 26. *The elasticity of* $\tilde{c}_\rho(\rho)$*,* $\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)}$*, is increasing in* ρ*.*

Proof. Note that the elasticity of $\tilde{c}_\rho(\rho)$ is equal to

$$\begin{aligned}\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} &= \rho\frac{\frac{d}{d\rho}\left(2\iota\iota'\right)}{2\iota\iota'}\\ &= \rho\frac{2\iota\iota''+2\iota'^2}{2\iota\iota'}\\ &= \rho\frac{\iota'}{\iota}\left(2\iota^2+1\right),\end{aligned}$$

where the last equality follows by replacing $\iota''=2\iota\iota'^2$ from Lemma 22 and factoring out ι'^2. The derivative of this elasticity is

$$\frac{d}{d\rho}\left(\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)}\right)=\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right)\left(2\iota^2+1\right)+\frac{d}{d\rho}\left(2\iota^2+1\right)\rho\frac{\iota'}{\iota}.$$

Note that the second term is unambiguously positive as $\iota'>0$ and $\iota>0$. The sign of the first term is determined by the sign of $\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right)$: the derivative of the inverse error function elasticity. It is

$$\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right)=\frac{\iota''}{\iota'}+\rho\frac{\iota'''\iota'-\iota''^2}{\iota'^2}=\frac{\iota''}{\iota'}+2\rho\iota''^2(1+2\iota(2\iota-1)).$$

We know that $\iota''>0$ and $\iota'>0$. Thus, we only need to show that $1+2\iota(2\iota-1)>0$. Note that this is a convex function of ρ with a minimum at $\iota\iota'=\frac{1}{4}$ which is solved by $\rho=\mathrm{erf}\left(\sqrt{\frac{W\left(\frac{1}{2\pi}\right)}{2}}\right)\approx 0.29$ where W denotes the principal branch of the Lambert-W function. Evaluating $1 + 2\iota(2\iota - 1)$ at this minimum yields

$$1 + \left(\sqrt{2W\left(\frac{1}{2\pi}\right)} - 1\right)\sqrt{2W\left(\frac{1}{2\pi}\right)} \approx 0.75.$$

Hence, we can conclude that $\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right)$ is positive and the result follows. $\square$

F Comparative Statics of Expanding Knowledge

Recall that the optimal distance $d^\eta(\infty)$ and the optimal success probability $\rho^\eta(\infty)$ when expanding knowledge are implicitly defined by the system of first-order conditions. The comparative statics then follow from applying the implicit function theorem. In particular, we obtain

$$\begin{pmatrix} \frac{d}{d\eta}d^\eta(\infty) \\ \frac{d}{d\eta}\rho^\eta(\infty) \end{pmatrix} = -\frac{1}{det(\mathcal{H})}\begin{pmatrix} f_{d\eta}f_{\rho\rho} - f_{\rho\eta}f_{d\rho} \\ f_{\rho\eta}f_{dd} - f_{d\eta}f_{d\rho} \end{pmatrix} \tag{8}$$

where we use the shorthand $f = u_R(d, \rho; X)$ for the researcher's value from expanding knowledge with distance d and success probability ρ. We obtain

$$\begin{aligned} f_{d\eta} &= -\tilde{c}(\rho) \\ f_{\rho\eta} &= -\tilde{c}_\rho(\rho)d \\ f_{dd} &= \rho V_{dd}(d; \infty) \\ f_{\rho\rho} &= -\eta\tilde{c}_{\rho\rho}(\rho)d \\ f_{d\rho} &= V_d(d; \infty) - \eta\tilde{c}_\rho(\rho). \end{aligned}$$

Suppressing the point of evaluation and plugging in, the comparative statics yields at the optimal distance $d^\eta(\infty)$ and success probability $\rho^\eta(\infty)$

$$\begin{aligned} \frac{d}{d\eta}d^\eta(\infty) &= -\frac{1}{det(\mathcal{H})}\left(\tilde{c}(\rho)\eta\tilde{c}_{\rho\rho}(\rho)d + \tilde{c}_\rho(\rho)d(V_d(d; \infty) - \eta\tilde{c}_\rho(\rho))\right) \\ &= -\frac{\eta d}{det(\mathcal{H})}\left(\tilde{c}(\rho)\tilde{c}_{\rho\rho}(\rho) + \tilde{c}_\rho(\rho)(\tilde{c}(\rho)/\rho - \tilde{c}_\rho(\rho))\right) \\ &< 0, \end{aligned}$$

where the first equality follows from the first-order condition with respect to d and factoring out ηd. The inequality follows from the fact that the determinant of the Hessian is positive and that the elasticity of $\tilde{c}(\rho)$ is increasing in ρ (Lemma 25).

For the optimal success probability, we obtain at the optimal distance $d^\eta(\infty)$ and success probability $\rho^\eta(\infty)$

$$\begin{aligned}\frac{d}{d\eta}\rho^\eta(\infty) &= -\frac{1}{det(\mathcal{H})}\left(-\tilde{c}_\rho(\rho)d\rho V_{dd}(d;\infty)+\tilde{c}(\rho)(V_d(d;\infty)-\eta\tilde{c}_\rho(\rho))\right)\\ &= -\frac{d/(3q)\tilde{c}(\rho)}{det(\mathcal{H})}\left(\frac{\tilde{c}_\rho(\rho)\rho}{\tilde{c}(\rho)}+\frac{V_d(d;\infty)-V(d;\infty)/d}{d/(3q)}\right)\\ &< 0,\end{aligned}$$

where the equality follows from the first-order condition with respect to ρ, plugging in the expression for $V_{dd}(d;\infty)$ (noticing that $d^\eta(\infty) < 4q$) and factoring out $d/(3q)\tilde{c}(\rho)$. The inequality follows from the fact that the determinant of the Hessian is positive, that the elasticity of $\tilde{c}(\rho)$ increases in ρ and that it is greater than two (Lemma 24), and that the term in parentheses is equal to negative one half after plugging in.

Hence, both the optimal distance and the optimal success probability decrease in the cost parameter η when expanding knowledge.

G Omitted Proofs

Lemma 27. $\frac{\partial V(d;\infty|d>4q)}{\partial d} < 0$.

Proof.

$$\frac{\partial V(d;\infty|d>4q)}{\partial d} = -\frac{d}{3q}+1+\sqrt{\frac{d-4q}{d}}\frac{d-q}{3q}$$

Letting $\tau := d/q(> 4$ by assumption) the statement is negative if $\frac{3-\tau}{3}+\sqrt{\frac{\tau-4}{\tau}}\frac{\tau-1}{3} < 0$. The left-hand side is increasing in τ and converges to 0 as $\tau \to \infty$. □

Lemma 28. $V_d(d;X) > 0$ *if* $d \in [0, X-4q]$ *and* $X \in (4q, 6q]$.

Proof. Note that for $X \in (4q, 6q]$ and $d \in [0, X-4q]$,

$$V_d = \frac{1}{3q}\left(X-2d-(X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right).$$

Assume towards a contradiction that there is a feasible combination of d and X such that $V_d(d;X) \leq 0$. Then, the following inequality must hold

$$\frac{X-2d}{X-d-q} \leq \sqrt{\frac{X-d-4q}{X-d}}.$$

Observe that this inequality cannot hold at the bounds $d = 0$ and $d = X - 4q$: If $d = 0$, then the left-hand side is strictly greater than one while the right-hand side is strictly less than one. If $d = X - 4q$, then the left-hand side reduces to $(8q - X)/(3q)$ which is strictly positive as $X \leq 6q$, while the right-hand side is equal to zero.

Hence, if the inequality is ever satisfied for some feasible (X, d), then by continuity and the intermediate value theorem, there must be a feasible $(\underline{X}, \underline{d})$ combination such that $V_d(\underline{d}; \underline{X}) = 0$. Thus, for a feasible $\underline{d}$, there must be a solution $\underline{X}$ to the quadratic equation (in $\underline{X}$)

$$\frac{X - 2\underline{d}}{X - \underline{d} - q} = \sqrt{\frac{X - \underline{d} - 4q}{X - \underline{d}}}.$$

The solution to this equation is

$$\underline{X}_{1,2} = \frac{1}{4}\left(5\underline{d} + 3q \pm (\underline{d} - q)\sqrt{\frac{\underline{d} + 5q}{\underline{d} - 3q}}\right).$$

However, no feasible solution exists, as $\underline{d} \leq 2q$ (due to the upper bound $X - 4q$ on $\underline{d}$ and the upper bound $6q$ on X), leading to no solution for $\underline{X}$ in the real domain. A contradiction. $\square$

Lemma 29. *$V_X(d^0(X); X) < 0$ if $X \geq 4q$ and $d \in [0, X - 4q]$.*

Proof. Observe that for any $X \geq 4q$ and $d \leq X - 4q$

$$V_{Xd} = \frac{1}{24q}\left(8 - 3\sqrt{\frac{X - d}{X - d - 4q}} - (5(X - d) + 4q)\frac{\sqrt{X - d - 4q}}{(X - d)^{3/2}}\right).$$

Denote $a := X - d$, this is an increasing function in a as $\frac{dV_{Xd}}{da} = \frac{4q^2}{a^{5/2}(a-4q)^{3/2}} > 0$. Hence, the highest value of V_{Xd} is attained for $a \to \infty$ and

$$\lim_{a\to\infty} \frac{1}{24q}\left(8 - 3\underbrace{\sqrt{\frac{a}{a - 4q}}}_{\to 1} - 5\underbrace{\frac{a\sqrt{a - 4q}}{a^{3/2}}}_{\to 1} - 4q\underbrace{\frac{\sqrt{a - 4q}}{a^{3/2}}}_{\to 0}\right) = 0.$$

It follows that the V_{Xd} converges to zero from below implying that $V_{Xd} < 0$. Thus, $V_X(d^0(X), X) < V_X(d = 0, X)$ and we obtain

$$V_X(d, X | d \leq 4q, X - d \geq 4q) = \frac{d + (X - d - q)\sqrt{\frac{X-d-4q}{X-d}} - (X - q)\sqrt{\frac{X-4q}{X}}}{3q}$$

$$< V_X(d=0,X|d\leq 4q, X-d\geq 4q) = \frac{(X-q)\sqrt{\frac{X-4q}{X}}-(X-q)\sqrt{\frac{X-4q}{X}}}{3q} = 0. \quad \square$$

Lemma 30. *If* $X \in (4q, 8q)$, $\mathrm{d}^2V(X/2,X)/\mathrm{d}X^2 < 0$ *and* $\mathrm{d}^2V(d^0(X),X)/(\mathrm{d}X)^2 > 0$.

Proof. Considering the boundary solution we obtain

$$\frac{d^2V(X/2,X)}{dX^2} = -\frac{X^2-2qX-2q^2}{3qX^{3/2}\sqrt{X-4q}}+\frac{1}{6q} \quad \frac{d^3V(X/2,X)}{dX^3} = \frac{4q^2}{X^{5/2}(X-4q)^{3/2}} > 0$$

implying that $\frac{d^2V(X/2,X)}{dX^2} \leq \frac{d^2V(4q,8q)}{dX^2}$ with

$$\frac{d^2V(4q,8q)}{dX^2} = -\frac{64q^2-16q^2-2q^2}{3q8^{3/2}q^{3/2}2q^{1/2}}+\frac{1}{6q} = -\frac{46q^2}{96\sqrt{2}q^3}+\frac{1}{6q} = \frac{8-23/\sqrt{2}}{48q} < 0.$$

Next, consider the value of any interior solution and apply the envelope and implicit function theorem to obtain

$$\begin{aligned}
\frac{dV(d^0(X),X)}{dX} &= V_X + d'(X)\underbrace{V_d}_{=0 \text{ by optimality of } d} = V_X \\
\frac{d^2V(d^0(X),X)}{dX^2} &= V_{XX} + d'(X)V_{dX} + d'(X)\underbrace{(V_{Xd}+V_{dd}d'(X))}_{=0 \text{ by IFT on FOC}} + d''(X)\underbrace{V_d}_{=0 \text{ by optimality}} \\
&= V_{XX}(d^0(X),X) + d'(X)V_{dX} = V_{XX}(d^0(X),X) \underbrace{-\frac{V_{dX}^2}{V_{dd}}}_{>0 \text{ as } V_{dd}<0}.
\end{aligned}$$

Observing that $V_{XXd}(d,X|d\leq 4q, X-d\geq 4q) = \frac{4q^2}{(X-d)^{5/2}(X-d-4q)^{3/2}} > 0$, we can compute as lower bound for

$$\begin{aligned}
V_{XX}(d^0(X),X) = \frac{1}{24q}\Bigg(3\left(\sqrt{\frac{X-d}{X-d-4q}}-\sqrt{\frac{X}{X-4q}}\right)+6\left(\sqrt{\frac{X-d-4q}{X-d}}-\sqrt{\frac{X-4q}{X}}\right) \\
+\left(\frac{X-4q}{X}\right)^{3/2}-\left(\frac{X-d-4q}{X-d}\right)^{3/2}\Bigg) \geq V_{XX}(d=0,X) = 0
\end{aligned}$$

implying that $\mathrm{d}^2V(d^0(X),X)/(\mathrm{d}X^2) \geq 0$. $\square$

Lemma 31. *Assume* $X \in [4q, 8q]$, *then* $\mathrm{d}^2U_R(d=X/2;X)/(\mathrm{d}X)^2 < 0$.

Proof. Take the case of the boundary solution: we are analyzing a one-dimensional optimization problem with respect to ρ. Denote the objective $f(\rho;X)$ and the optimal value by $\varphi(X) = \max_\rho f(\rho; X)$. Then, the optimal ρ solves $f_\rho = 0$. We obtain

$$
\begin{aligned}
\varphi'(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + f_X \\
\varphi''(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) + \underbrace{(f_{\rho\rho}\rho'(X) + f_{X\rho})}_{=0 \text{ by total differentiation of FOC}} \rho'(X) + f_{XX} + \rho'(X) f_{X\rho} \\
&= f_{XX} - \frac{f_{X\rho}^2}{f_{\rho\rho}} = \rho^\eta(X) V_{XX}(X/2; X) + \frac{(V_X - \frac{V}{X})^2}{V\frac{c''}{c'}}
\end{aligned}
$$

where we used that $\sigma^2_{XX}(X/2; X) = 0$. The expression yields as condition for the value to be strictly concave $\rho^\eta(X)c''/c' > -(V_X - \frac{V}{X})^2/(V_{XX}V)$ where the inequality sign changed direction as $V_{XX}(X/2; X) < 0$ by Lemma 30 in the region considered. Further, note that the left-hand side larger than two by the properties of the inverse error function. We will show that the right-hand side is below one which is sufficient for concavity. We therefore consider the right-hand side at the boundary solution, which simplifies to

$$
\frac{\sqrt{X - 4q}\left(X^{3/2} - 2(X + 2q)\sqrt{X - 4q}\right)^2}{4\left(X^2 - 2q^2 - 2qX\right)\left(X^{3/2} - 2(X - 4q)\sqrt{X - 4q}\right)}.
$$

We now show that it is also smaller than one. Because the denominator is positive, a necessary and sufficient condition is

$$
\sqrt{X - 4q}\left(X^{3/2} - 2(X + 2q)\sqrt{X - 4q}\right)^2 - 4\left(X^2 - 2q^2 - 2qX\right)\left(X^{3/2} - 2(X - 4q)\sqrt{X - 4q}\right) < 0.
$$

Factoring out $\sqrt{\frac{X-4q}{X}}$, dividing by $X^{3/2}$, and simplifying the condition becomes

$$
\sqrt{\frac{X - 4q}{X}}\left(13X^2 - 48qX\right) - 8X^2 + 16qX + 40q^2 < 0. \tag{9}
$$

Notice that $\sqrt{\frac{X-4q}{X}}$ increases in X and thus attains its upper bound for $X = 8q$ at $1/\sqrt{2}$. Moreover, since $13X > 48q$ for any $X \in [4q, 8q]$, this implies that (9) holds if

$$
\begin{aligned}
\frac{13X^2 - 48qX}{\sqrt{2}} - 8X^2 + 16qX + 40q^2 < 0 \\
\left(\frac{13}{\sqrt{2}} - 8\right) X^2 + (16 - \frac{48}{\sqrt{2}}) qX + 40q^2 < 0
\end{aligned}
$$

Now notice that $13/\sqrt{2} > 8$ and thus the LHS is strictly convex meaning a maximum must be at one of the boundaries in X. But then at $X = 4q$ we have

$$\left(\frac{13}{\sqrt{2}} - 8\right) 16q^2 + (16 - \frac{48}{\sqrt{2}})4q^2 + 40q^2 = 8(\sqrt{2} - 3)q^2 < 0$$

and at $X = 8q$

$$\left(\frac{13}{\sqrt{2}} - 8\right) 64q^2 + (16 - \frac{48}{\sqrt{2}})8q^2 + 40q^2 = 8(28\sqrt{2} - 43)q^2 < 0.$$

which implies that condition (9) holds, thus closing the proof. □

Lemma 32. *Let $d^i < X/2$ be a local maximum of $u_R(\rho, d, X)$. If $d^i(X)$ exists on $X \in [4q, 8q]$, then $\mathrm{d}^2 U_R(d = d^i(X); X)/(\mathrm{d}X)^2 > 0$.*

Proof. The implicit function theorem yields for $d'(X)$ and $\rho'(X)$

$$\begin{pmatrix} d'(X) \\ \rho'(X) \end{pmatrix} = -\frac{1}{f_{dd} f_{\rho\rho} - f_{\rho d}^2} \begin{pmatrix} f_{dX} f_{\rho\rho} - f_{\rho X} f_{d\rho} \\ f_{\rho X} f_{dd} - f_{dX} f_{d\rho} \end{pmatrix}.$$

Note that $-\frac{1}{f_{dd} f_{\rho\rho} - f_{\rho d}^2} < 0$ as this is $-\frac{1}{det(\mathcal{H})}$ and the determinant of the second principal minor being positive is a necessary second order condition for a local maximum given that the first ($f_{\rho\rho}$) is negative.

Denote the objective $f(\rho, d; X)$ and the optimal value by $\varphi(X) = \max_{\rho,d} f(d, \rho; X)$. Then, the optimal (d, ρ) solves $f_\rho = 0$ and $f_d = 0$. Differentiating the value of the researcher twice with respect to X yields

$$\begin{aligned}
\varphi'(X) = &\underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + \underbrace{f_d}_{=0 \text{ by optimality}} d'(X) + f_X \\
\varphi''(X) = &\underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) + \underbrace{f_d}_{=0 \text{ by optimality}} d'(X) \\
&+ d'(X) \underbrace{(f_{dX} + f_{dd} d'(X) + f_{d\rho}\rho'(X))}_{=0 \text{ by total differentiation of foc wrt } d} + \rho'(X) \underbrace{(f_{\rho X} + f_{\rho d} d'(X) + f_{\rho\rho}\rho'(X))}_{=0 \text{ by total differentiation of foc wrt } \rho} \\
&+ f_{dX} d'(X) + f_{\rho X}\rho'(X) + f_{XX} = f_{dX} d'(X) + f_{\rho X}\rho'(X) + f_{XX}.
\end{aligned}$$

Observe first that $f_{XX} > 0$ as $f_{XX} = \rho V_{XX}(d; X) - \eta\tilde{c}(\rho)\sigma^2_{XX}(d; X)$ and $V_{XX} > 0$ by proof of Proposition 2 (in particular, Lemma 30) and $\sigma^2_{XX}(d; X) = -\frac{2d^2}{X^3}$. Next, we show $f_{dX} d'(X) + f_{\rho X}\rho'(X) > 0$ using the implicit function theorem together with the property of the local maximum that $f_{\rho\rho}f_{dd} > f_{\rho d}^2$.

$$f_{dX}d'(X) + f_{\rho X}\rho'(X) = -f_{dX}\left(\frac{f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho}}{f_{dd}f_{\rho\rho} - f_{\rho d}^2}\right) - f_{\rho X}\left(\frac{f_{\rho X}f_{dd} - f_{dX}f_{d\rho}}{f_{dd}f_{\rho\rho} - f_{\rho d}^2}\right).$$

As we only need to sign this expression, we can ignore the denominator to verify

$$-f_{dX}(f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho}) - f_{\rho X}(f_{\rho X}f_{dd} - f_{dX}f_{d\rho}) > 0 \Leftrightarrow \frac{f_{dX}}{f_{\rho X}}\frac{f_{\rho\rho}}{f_{d\rho}} + \frac{f_{\rho X}}{f_{dX}}\frac{f_{dd}}{f_{\rho d}} > 2.$$

where we used the signs of the terms that follow because

$$f_{\rho\rho} = -\eta\tilde{c}_{\rho\rho}(\rho)\sigma^2 < 0 \qquad f_{d\rho} = V_d - \eta\tilde{c}_\rho(\rho)\sigma_d^2 < V_d - \eta\frac{\tilde{c}(\rho)}{\rho}\sigma_d^2 = 0$$

$$f_{dX} = \rho V_{dX} - \eta\tilde{c}(\rho)\sigma_{dX}^2 < 0 \qquad f_{\rho X} = V_X - \eta\tilde{c}_\rho(\rho)\sigma_X^2 < V_X - \eta\sigma_X^2\tilde{c}(\rho)/\rho < 0$$

which in turn follow from the first-order conditions and Proposition 2.

Because $f_{\rho\rho}f_{dd} - f_{\rho d}^2 > 0$, we can replace $\frac{f_{\rho\rho}}{f_{d\rho}}$ with $\frac{f_{d\rho}}{f_{dd}}$ as $\frac{f_{\rho\rho}}{f_{d\rho}} > \frac{f_{d\rho}}{f_{dd}}$ yielding

$$2 < \frac{f_{dX}}{f_{\rho X}}\frac{f_{d\rho}}{f_{dd}} + \frac{f_{\rho X}}{f_{dX}}\frac{f_{dd}}{f_{\rho d}}$$

which is true as the right-hand side can be written as $g(a) = a + \frac{1}{a}$. $g(a)$ is a strictly convex function for $a > 0$ and minimized at $a = 1$ with $g(a = 1) = 2$. □

Lemma 33. *$d^\eta(\infty)$ is linear in q and $\rho^\eta(\infty)$ is constant in q.*

Proof. The lemma follows because $\sigma^2(mq;\infty) = mq$ and thus (by Proposition 1) the functions $f(m,q) := V(mq;\infty)/\sigma^2(mq;\infty)$ and $g(m,q) := V_d(mq;\infty)$ are homogeneous of degree 0 in q. It is then immediate from (FOC^d) and (FOC$^\rho$) that $d^\eta(\infty)$ is homogeneous of degree 1 in q and $\rho^\eta(\infty)$ is homogeneous of degree 0. Noticing that $d^\eta(\infty)(q = 0) = 0$ implies the result. □

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