SUPPLEMENT TO "CAP-AND-TRADE AND CARBON TAX MEET ARROW-DEBREU"

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This supplementary material consists of a section discussing the limitations of partial equilibrium analysis and disposal cone, a section on *Rebate Walras Law*, proofs of Propositions 1 and 2 in Anderson and Duanmu (2025), a special case of Theorem 2 in Anderson and Duanmu (2025), and detailed analysis of the examples from Anderson and Duanmu (2025).

1. Additional Literature Review

In this section, we compare the formulation in Anderson and Duanmu (2025) to partial equilibrium analysis and the disposal cone formulation proposed in Florenzano (2003).

1.1. **Limitations of Partial Equilibrium Analysis.**

Partial equilibrium arguments are unsatisfactory, for three fundamental reasons:

(1) Introducing a quota or tax on some commodities can result in substantial changes in sectors that appear to have little to do with the regulated commodities. These changes are not predictable through partial equilibrium arguments. Indeed, Examples 4 and 5 in Anderson and Duanmu (2025) show that, due to multiplicity of equilibria, it *may be impossible to achieve an emissions target by setting an emission tax rate*;

(2) Once the government sets the quotas or taxes, and the rebate scheme, partial equilibrium arguments cannot guarantee that an equilibrium price exists. Inded, we show in Examples 4 and 5 in Anderson and Duanmu (2025), that emission tax equilibria in fact *need not exist*;

(3) The formulation of Pareto Optimality requires the definition of the set of all feasible consumption-production pairs, but these cannot be defined in partial equilibrium;

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(4) Pigou (1920) proposed levying an add-on tax equal to the difference between the social and private costs, in order to restore the first order conditions. There is considerable literature on Pigouvian taxation through partial equilibrium analysis. However, levying a tax results in a new equilibrium with different prices, productions, and consumptions. At the new equilibrium, the marginal costs and benefits are different, and there is no reason to think that the same tax will restore the first order conditions for Pareto optimality at the new equilibrium.[1]

1.2. **Disposal Cone.** Free-disposal equilibrium (excess demand must be nonnegative) and non-free-disposal equilibrium (excess demand must be zero) are classical equilibrium notions.[2] "Free disposal" of bads is a euphemism for unconstrained emissions. Florenzano (2003) formulated a hybrid of the two notions, via a disposal cone that specifies a subset of commodities that may be freely disposed. Florenzano's work is a substantial improvement on the pre-existing literature, but it has three significant limitations:

(1) The disposal cone formulation does not allow a positive cap on emissions;
(2) A "complementary slackness" condition rules out a positive tax on emissions;[3]
(3) Florenzano (2003) establishes the existence of a price for which the excess demand lies in the disposal cone. However, her proof does not establish the existence of equilibria with positive disposal. Since it is not practical to eliminate all pollution in the near term, one should demonstrate the existence of equilibrium with positive emissions.

Our quota equilibrium model, on the other hand, defines a quota-compliance region $\mathcal{Z}(m)$ that reflects the society's choice on what *quantities* of pollution may be emitted. Any revenue generated from emissions via a quota or tax is rebated to agents, which allows us to avoid complementary slackness so that nonzero emissions and nonzero prices may coexist.

2. Miscellaneous Results

In this section, we present Rebate Walras Law, an example comparing fuel tax and emissions tax regulatory schemes, a special case of Theorem 2 in Anderson and Duanmu (2025), and proofs of Propositions 1 and 2 in Anderson and Duanmu (2025).

[1]Goulder and Williams III (2003) point out that there is a substantial bias from ignoring general equilibrium effects in estimating excess burden.

[2]While Arrow and Debreu (1954) used free-disposal equilibrium as the equilibrium concept, McKenzie (1981) used non-free-disposal equilibrium as the equilibrium concept.

[3]Complementary slackness is a consequence of budget balance in the Arrow-Debreu and related models. If a tax on pollution generates positive revenues, these revenues would evaporate from the model, leaving consumers without sufficient income to buy the goods produced by the firms.

2.1. **Rebate Walras' Law.** Walras' Law states that the value of excess demand at any price vector p must be 0 for each agent. This is true whether p is an equilibrium price or not. Walras' Law implies that if prices are strictly positive and there is positive excess demand in one market, then there is negative excess demand in another market. As noted on page 42 of Florenzano (2003), the equilibrium definition requires that, if any commodity is in excess supply, its price must be 0. However, emission tax equilibrium usually involves nonzero prices and nonzero excess supply of regulated commodities, and zero excess supply among unregulated commodities. Thus, Walras' Law is inconsistent with our definition of emission tax equilibrium. However, a modification, *Rebate Walras' Law*, does hold at the aggregate level.[4] It states that the value of excess demand for the *unregulated* commodities is 0.

Definition 2.1. Given a price vector p and a production vector y of all firms, let x be the demand vector for all agents. *Rebate Walras' Law* states that

$$\sum_{n=k+1}^{\ell} p_n \bar{x}_n = \sum_{n=k+1}^{\ell} p_n(\bar{e}_n + \bar{y}_n)$$

where $\bar{x} = \sum_{\omega \in \Omega} x(\omega)$, $\bar{e} = \sum_{\omega \in \Omega} e(\omega)$ and $\bar{y} = \sum_{j \in J} y(j)$ are the aggregate demand, the aggregate endowment and the aggregate production.

Theorem 2.2. *Let $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, \mathcal{V}, t, \lambda\}$ be an emission tax production economy. Suppose, for all $\omega \in \Omega$ and $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, the own-consumption preference $P_\omega(x_{-\omega}, y, p)$ is locally nonsatiated. Then Rebate Walras' Law holds.*

Proof. Let p be a price vector and y be a production vector of all firms. Let x be the demand vector for all agents. We have $\sum_{n=1}^{\ell} p_n \bar{x}_n = \sum_{n=1}^{k} p_n \bar{x}_n + \sum_{n=k+1}^{\ell} p_n \bar{x}_n$. The aggregate budget of agents consists of the value of aggregate supply and the total tax revenue. So we have $\sum_{n=1}^{\ell} p_n(\bar{y}_n + \bar{e}_n) - \sum_{n=1}^{k} p_n(\bar{y}_n + \bar{e}_n - \bar{x}_n) = \sum_{n=1}^{k} p_n \bar{x}_n + \sum_{n=k+1}^{\ell} p_n(\bar{y}_n + \bar{e}_n)$. As $P_\omega(x, y, p)$ is locally non-satiated for all $\omega \in \Omega$ and $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, we have

$$\begin{aligned} & \sum_{n=1}^{\ell} p_n \bar{x}_n = \sum_{n=1}^{\ell} p_n(\bar{y}_n + \bar{e}_n) - \sum_{n=1}^{k} p_n(\bar{y}_n + \bar{e}_n - \bar{x}_n) \\ \Longleftrightarrow & \sum_{n=1}^{k} p_n \bar{x}_n + \sum_{n=k+1}^{\ell} p_n \bar{x}_n = \sum_{n=1}^{k} p_n \bar{x}_n + \sum_{n=k+1}^{\ell} p_n(\bar{y}_n + \bar{e}_n) \\ \Longleftrightarrow & \sum_{n=k+1}^{\ell} p_n \bar{x}_n = \sum_{n=k+1}^{\ell} p_n(\bar{y}_n + \bar{e}_n). \end{aligned}$$

[4]Because of the rebate scheme, the Rebate Walras' Law does not hold at an individual level.

Hence, the Rebate Walras' Law holds. □

Rebate Walras Law and its Effect on unregulated Commodities

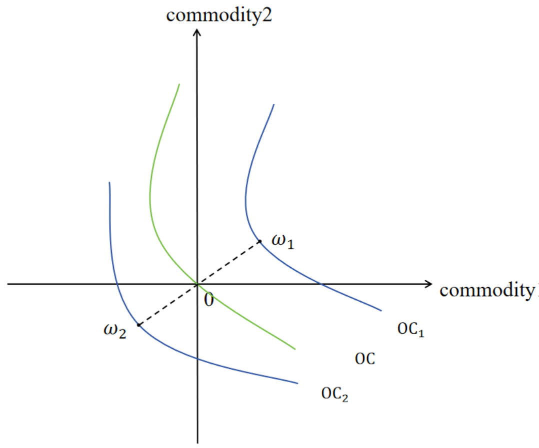

FIGURE 1. The figure plots the tax rebate effect on demands of unregulated commodities. The figure shows net trades, i.e., the endowments are placed at the origin. The curve OC_1 is the offer curve for agent 1, and it goes through ω_1. The curve OC_2 is the offer curve for agent 2, and it goes through ω_2. The curve OC is the aggregate offer curve, and it goes through the origin.

We consider an exchange economy with two agents, one regulated commodity, and two unregulated commodities. Both agents derive utility from unregulated commodities, and the marginal utility of the regulated commodity is 0. Hence, agents consume the regulated commodity only to generate additional income to consume unregulated commodities. Fig. 1 shows net trades, i.e., the endowments e_i are placed at the origin. Given an emission tax rate and agents' consumption of the regulated commodity, for $i \in \{1, 2\}$, we denote agent i's available additional income for consuming commodities 1 and 2 by I_i, where I_i is the sum of the value of agent i's excess consumption of the regulated commodity and agent i's rebate from the government's emission tax revenue. We implement I_i through an increment ω_i to e_i.[5] Since e_i is placed at the origin in the figure, the augmented endowment $e_i + \omega_i$ is placed at ω_i. Notice that $\omega_1 = -\omega_2$.

[5]Since the emission tax rate is fixed, by the normalization of prices, we have $p_1 + p_2 = c$. Let $\omega_i = \frac{1}{c}(I_i, I_i)$ be the additional endowment of commodities 1 and 2 for agent i. Then ω_i implements the additional available income I_i for agent i.

Deriving the Aggregate Offer Curve OC from OC_1 and OC_2

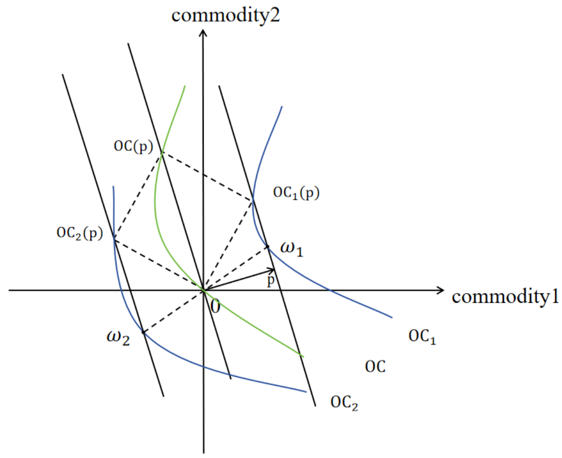

FIGURE 2. The figure plots how the aggregate offer OC is derived from the offer curves OC_1 and OC_2. For the price vector p and $i \in \{1, 2\}$, the line through ω_i perpendicular to p crosses OC_i at exactly one point $\mathrm{OC}_i(p)$. The line through the origin perpendicular to p crosses OC at exactly one point $\mathrm{OC}(p)$. The point $\mathrm{OC}(p)$ is the sum of $\mathrm{OC}_1(p)$ and $\mathrm{OC}_2(p)$. The four points $\mathrm{OC}_1(p)$, $\mathrm{OC}_2(p)$, $\mathrm{OC}(p)$, and the origin, form a parallelogram.

The offer curve OC_i goes through the augmented endowment, and hence in Fig. 1 goes through ω_i. In the usual setting with monotonic preferences and no rebate, each agent's offer curve goes through the origin. In the setting with tax rebate, the individual offer curves OC_i do *not* go through the origin. However, the aggregate offer curve OC does go through the origin. This fact is both necessary and sufficient for the existence of equilibrium. In the usual setting with monotonic preferences and no rebate, Walras' Law (for *all* commodities) holds at an individual level. In this setting, with regulated commodities and rebate, Rebate Walras' Law holds for the *unregulated* goods at an *aggregate*, rather than individual, level.

Fig. 2 provides a graphical illustration on the derivation of the offer curve OC from individual offer curves OC_1 and OC_2. At an equilibrium price p (not shown in Fig. 2), the diagonal of the parallelogram through $\mathrm{OC}_1(p)$ and $\mathrm{OC}_2(p)$ goes through the origin.

The distinction between Walras' Law, and Rebate Walras' Law, is critical to the existence of emission tax equilibrium. In the classical setting with no rebate and possibly non-monotonic preferences, Walras' Law implies complementary slackness: any good in excess supply must have a price of zero. Here, by contrast, there is excess supply of regulated commodities (i.e. the excess supply is released into the environment), but the emission price is nonzero. This is possible because Rebate Walras' Law applies only to the unregulated commodities. Thus, Rebate Walras' Law makes the existence of Emissions Tax Equilibrium conceivable. Rebate Walras's Law does not hold at the individual level, but it does hold at the aggregate level. As in the usual case, we only require (Rebate) Walras' Law at the aggregate level to show that the aggregate offer curve OC goes through the origin, i.e. equilibrium exists.

2.2. **Fuel Taxes Versus Emission Taxes.** We have so far focused on emission taxes on regulated commodities instead of fuel taxes. In this section, we present examples of economies with three commodities: CO_2, coal and electricity. However, instead of placing an emission tax on CO_2, we place a tax on total utilization of coal as a consumption good and as an input to production. Moreover, our tax on coal is an add-on to the equilibrium price. In other words, to buy coal, one must pay the equilibrium price plus the tax.[6] These examples fit the model in Shafer and Sonnenschein (1976).

Example 2.3. In this example, we consider fuel taxes in the economy described in Example 4 of Anderson and Duanmu (2025). We show that, for every given total net CO_2 emissions level $v \in (0,1)$, every fuel tax equilibrium with emissions level v is strictly Pareto dominated[7] by every emission tax equilibrium in Example 4 of Anderson and Duanmu (2025) with the same emissions level v. Let $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \lambda\}$ be a *fuel tax production economy* such that:

(1) The economy $\mathcal{F}$ has the same agent and set of commodities as the emission tax production economy $\mathcal{E}$ in Example 4 of Anderson and Duanmu (2025). The agent's endowment, consumption set and preference are also the same for $\mathcal{F}$ and $\mathcal{E}$;
(2) $\mathcal{F}$ and $\mathcal{E}$ have the same private firms. Since there is only one agent, all firms distribute profit to the single agent;
(3) $\mathcal{F}$ and $\mathcal{E}$ have the same compliance region $\mathcal{V} = \mathbb{R}_{\leq 0} \times \{0\}^2$. We allow for free disposal of CO_2 while requiring non-free-disposal at equilibrium for coal and electricity;
(4) The government sets a fuel tax rate t on coal, and rebate all the tax revenue to the single agent. We let the tax rate on coal vary to compute fuel tax equilibrium for different tax rates.

In particular, let $t \geq 0$ be the tax rate on coal. Through normalization, for a price vector p, we require that $t + \sum_{k=1}^{3} |p_k| = 1$, that is, $(t,p) \in \Delta$. Note that only the first firm uses coal as an input in production, and the agent does not desire coal.

[6]Our "tax" could be either positive or negative; in the latter case, it serves as a subsidy of the commodity.
[7]At the emissions levels 0 and 1, weak Pareto domination holds.

We now define the fuel tax equilibrium for a given tax rate t. For every price vector p such that $(t,p) \in \Delta$ and every $y \in Y$, the agent's *fuel tax budget set* $\mathcal{B}^t_\omega(y,p)$ is defined to be:

$$\left\{ z \in X : p \cdot z \leq p \cdot e + \sum_{j \in J} (p \cdot y(j) + ty(j)_2) + \sum_{j \in J} t|y(j)_2| = p \cdot e + \sum_{j \in J} p \cdot y(j) \right\}.$$

For every price vector p such that $(t,p) \in \Delta$ and every $(x,y) \in \mathcal{A} \times Y$, the agent's *fuel tax demand set* $\mathcal{D}^t_\omega(x,y,p)$ is

$$\left\{ z \in \mathcal{B}^t_\omega(y,p) : v \succ_{x_{-\omega},y,\omega,p} z \implies v \notin \mathcal{B}^t_\omega(x,y,p) \right\}.$$

For every price vector p such that $(t,p) \in \Delta$ and each $j \in J$, the firm's supply set $S^t_j(p)$ is $\operatorname{argmax}_{z \in Y_j}(p \cdot z + tz_2)$. A *$\mathcal{V}$-compliant fuel tax equilibrium* under the tax rate t is $(\bar{x}, \bar{y}, \bar{p})$:

(1) $(t, \bar{p}) \in \Delta$;
(2) $\bar{x}(\omega) \in \mathcal{D}^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for the single agent ω;
(3) $\bar{y}(j) \in S^t_j(\bar{p})$ for all $j \in J$. Every firm is profit maximizing given the fuel tax rate t and the price vector $\bar{p}$;
(4) $\bar{x} - \sum_{j \in J} \bar{y}(j) - e \in \mathcal{V}$.

We first consider the existence of fuel tax equilibrium.

Claim 2.4. *For every $t \in [0,1)$, there exists a $\mathcal{V}$-compliant fuel tax equilibrium with tax rate t. Moreover, for every $v \in [0,1]$, there exists a $\mathcal{V}$-compliant fuel tax equilibrium such that the total net CO_2 emissions level is v.*

Proof. We first show that there exists a $\mathcal{V}$-compliant fuel tax equilibrium for any tax rate $t \in [0,1)$. We break our analysis into two cases:

(1) Suppose $t < \frac{1}{2}$. Let $x = (0,0,1)$, $y = \big((1,-1,1),(0,0,0)\big)$ and $p = (0, \frac{1}{2} - t, \frac{1}{2})$. Both firms are profit maximizing with zero profit given p and t. The agent's total income is $\frac{1}{2} - t + t = \frac{1}{2}$. Since the agent only derives utility from consuming electricity, $x = (0,0,1)$ is in the agent's fuel tax demand set. Finally, we have $x - e - \sum_{j \in J} y(j) = (-1,0,0) \in \mathcal{V}$. Hence, (x,y,p) is a $\mathcal{V}$-compliant fuel tax equilibrium;
(2) Suppose $t \geq \frac{1}{2}$. Let $x = (0,1,0)$, $y = \big((0,0,0),(0,0,0)\big)$ and $p = (0,0,1-t)$. Note that no trade occurs, and no firm operates. Both firms are profit maximizing with zero profit given p and t. The agent's total income is 0. So $x = (0,1,0)$ is in the agent's fuel tax demand set. Finally, we have $x - e - \sum_{j \in J} y(j) = (0,0,0) \in \mathcal{V}$. Hence, (x,y,p) is a $\mathcal{V}$-compliant fuel tax equilibrium.

We now show that, for $v \in [0,1]$, there exists a $\mathcal{V}$-compliant fuel tax equilibrium whose total net CO_2 emissions is v. Given $v \in [0,1]$, let $x_v = (0, 1-v, v)$, $y_v = \big((v, -v, v), (0,0,0)\big)$, $t = \frac{1}{2}$ and $p_v = (0, 0, \frac{1}{2})$. Both firms are profit maximizing with zero profit given p and t. The agent's total income consists only of rebate from fuel tax revenue, which is $\frac{v}{2}$. Since the agent only derives utility from consuming electricity, $x = (0, 1-v, v)$ is in the agent's fuel tax demand set. Finally, we have $x_v - e - \sum_{j \in J} y_v(j) = (-v, 0, 0) \in \mathcal{V}$. Hence, (x_v, y_v, p_v) is a $\mathcal{V}$-compliant fuel tax equilibrium such that the total net CO_2 emissions is v. $\square$

We now study the welfare properties of $\mathcal{V}$-compliant fuel tax equilibrium, and its comparison to $\mathcal{V}$-compliant emission tax equilibrium in Example 4 of Anderson and Duanmu (2025). Coal is used to generate electricity for two purposes: consumption of electricity by the agent, and sequestration of CO_2 by the second firm. Because the fuel tax applies to coal used for both purposes, it discourages sequestration and results in an inefficient outcome. We now provide the detailed calculations in the context of this example.

As shown in Claim 5.3 in Anderson and Duanmu (2025), for every $v \in [0,1]$, the emission tax production economy $\mathcal{E}$ has a $\mathcal{V}$-compliant emission tax equilibrium such that the total net CO_2 emissions is v. By Claim 3.5, the equilibrium consumption-production pair always takes the form $(\hat{x}_v, \hat{y}_v)$ where $\hat{x}_v = (0, 0, \frac{1+v}{2})$ and $\hat{y}_v = \big((1, -1, 1), (v-1, 0, \frac{v-1}{2})\big)$. We now compare, for any given level of total net CO_2 emissions level, the agent's equilibrium consumption of electricity between fuel tax equilibrium and emission tax equilibrium.

Claim 2.5. *For $v \in [0,1]$, the agent's electricity consumption at any fuel tax equilibrium with total net CO_2 emissions v is no greater than $\frac{1+v}{2}$, with strict inequality for $v \in (0,1)$.*

Proof. Given $v \in [0,1]$, let $(\bar{x}_v, \bar{y}_v, \bar{p}_v)$ be a $\mathcal{V}$-compliant fuel tax equilibrium such that the total net CO_2 emissions is v. We break our analysis into the following two cases:

(1) We first consider the case where $\bar{p}_v(1) \geq 0$.[8] By the agent's utility function and unboundedness of the agent's consumption set, we have $\bar{p}_v(3) > 0$. Hence, the equilibrium production for the second firm is $(0,0,0)$. As the total net CO_2 emissions is v, the first firm's equilibrium production must be $(v, -v, v)$. Hence, the agent's equilibrium consumption of electricity is bounded by v. It is clear that $v \leq \frac{1+v}{2}$ for all $v \in [0,1]$, and the equality holds only when $v = 1$;

(2) We then consider the case where $\bar{p}_v(1) < 0$. If the second firm's equilibrium production is $(0,0,0)$, then the result follows from the same analysis in the above paragraph. Now

[8]We use $\bar{p}_v(i)$ to denote the i-th coordinate of $\bar{p}_v$.

suppose the second firm's equilibrium production is not $(0,0,0)$. We conclude that $\bar{p}_v(3) = -2\bar{p}_v(1)$. As the second firm operates, the first firm's equilibrium production must also not be $(0,0,0)$. Hence, we must have $\bar{p}_v(1) - (\bar{p}_v(2)+t) + \bar{p}_v(3) = 0$, which implies $\bar{p}_v(2) + t = \bar{p}_v(3) + \bar{p}_v(1) = -\bar{p}_v(1)$. Suppose the first firm's equilibrium production is $(r, -r, r)$. Then the agent's total income is given by

$$\bar{p}_v(2) + r\big(\bar{p}_v(1) + \bar{p}_v(3) - \bar{p}_v(2)\big) = \bar{p}_v(2) + rt \leq \bar{p}_v(2) + t.$$

Thus, the agent's equilibrium electricity consumption is bounded by $\frac{\bar{p}_v(2)+t}{\bar{p}_v(3)} = \frac{-\bar{p}_v(1)}{\bar{p}_v(3)} = \frac{1}{2}$. Thus, unless $r = 1$ and $v = 0$, the agent's equilibrium consumption of electricity is strictly less than $\frac{1+v}{2}$.

Combining the two cases together, we obtain the desired result. $\square$

Note that the only externality arises from the total net CO_2 emissions and the agent only derives utility from consuming electricity. For every $v \in (0,1)$, every fuel tax equilibrium with total net CO_2 emissions v is Pareto dominated by every emission tax equilibrium with total net CO_2 emissions v. For $v \in \{0,1\}$, an emission tax equilibrium with total net CO_2 emissions v is at least as good as any fuel tax equilibrium with total net CO_2 emissions v.

The next example shows that fuel tax equilibrium may also fail to exist.

Example 2.6. Let $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \lambda\}$ be a finite production economy with fuel tax on coal:

(1) The economy $\mathcal{E}$ has three commodities CO_2, coal and electricity;
(2) There are two agents with the same consumption set $X = \{0\} \times \mathbb{R}^2_{\geq 0}$ and the same endowment $e = (0,1,0)$. Given the total net emissions v of CO_2, the first agent's utility function $f_v(x_{11}, x_{21}, x_{31}) = x_{31} - \frac{v^2}{2}$ where x_{11}, x_{21}, x_{31} are the first agent's consumption of CO_2, coal and electricity. The second agent's utility function is $g_v(x_{12}, x_{22}, x_{32}) = x_{22} + x_{32} - \frac{v^2}{2}$, where x_{12}, x_{22}, x_{32} are the second agent's consumption of CO_2, coal and electricity. Note that the second agent derives utility from consuming both coal and electricity;
(3) There is one producer with production set $Y_1 = \{(r, -r, r) : r \in \mathbb{R}_{\geq 0}\}$. So the producer has the production technology to burn r units of coal to generate r units of electricity and r units of CO_2 as byproduct;
(4) The compliance region $\mathcal{V} = \mathbb{R}_{\leq 0} \times \{0\}^2$;

(5) Both agents have the same shareholding of the private firm, i.e., $\theta_1(1) = \theta_2(1) = \frac{1}{2}$. The government's rebate shares to both agents are the same, i.e., $\lambda(1) = \lambda(2) = \frac{1}{2}$.

By the agents' utility functions, the equilibrium prices for coal and electricity must both be positive. If the fuel tax is set to be greater than $\frac{1}{2}$, by normalization, the sum of absolute values of equilibrium prices of CO_2, coal and electricity must be less than $\frac{1}{2}$. Since the firm is profit maximizing, the only possible equilibrium production for the producer is $(0, 0, 0)$. However, the first agent has a positive budget and wishes to only consume electricity. Hence, there is no equilibrium.

2.3. **Proofs of Propositions 1 and 2.** In this section, we complete the proofs of Propositions 1 and 2 of Anderson and Duanmu (2025).

Proof of Proposition 1. Agent ω's budget set $B^m_\omega(\bar{y}, \bar{p})$ is

$$\{z \in X_\omega : \bar{p} \cdot z \leq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j}(\bar{p} \cdot \bar{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)})\}.$$

As $\sum_{j \in J} \theta_{\omega j} \mathrm{proj}_k(\bar{p}) \cdot m^{(j)} = \tilde{\theta}(\omega)(0)\mathrm{proj}_k(\bar{p}) \cdot m$, the budget set $\tilde{B}^m_\omega(\bar{y}, \bar{p})$ of agent ω in the economy $\mathcal{F}$ is the same as $B^m_\omega(\bar{y}, \bar{p})$. Hence, $\bar{x}(\omega)$ is an element of the quota demand set $\tilde{D}^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$ in the economy $\mathcal{F}$. As $\bar{y}(j) \in S^m_j(\bar{p})$ for all $j \in J$, we have $\bar{y}(j) \in \operatorname{argmax}_{z \in Y_j} \bar{p} \cdot z$. Finally, as $\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \in \mathcal{Z}(m)$, we conclude that $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant global quota equilibrium for $\mathcal{F}$. $\square$

Proof of Proposition 2. As $k = 1$, $\sum_{j \in J} \theta_{\omega j} \pi_k(p) \cdot m^{(j)} = \tilde{\theta}(\omega)(0)\pi_k(p) \cdot m$ for $p \in \Delta$. By the same proof of Proposition 1, a quota equilibrium in $\mathcal{E}$ is a global quota equilibrium in $\mathcal{F}$. $\square$

2.4. **A Special Case of Theorem 2.** In this section, we consider a special case of Theorem 2 in Anderson and Duanmu (2025) in which agents cannot consume any regulated commodities. Since endowments are fixed, the total net emissions of the regulated commodities depend only on the production. The following result is similar to Theorem 2 in Anderson and Duanmu (2025) except that the total net emissions of the regulated commodities are replaced by the total net production of the regulated commodities.

Theorem 2.7. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (m^{(j)})_{j \in J}, \mathcal{Z}(m)\}$ be a quota production economy as in Definition 3.2 of Anderson and Duanmu (2025), and suppose that the only externality arises from the total net emissions of the first k commodities. Suppose $\mathcal{Z}(m)_n = \{0\}$ for all $k < n \leq \ell$ and $\mathrm{proj}_k(X_\omega) = \{0\}$ for all $\omega \in \Omega$. Let $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium. Then:*

(1) $(\bar{x}, \bar{y})$ *is constrained weakly Pareto optimal, i.e.,* $(\bar{x}, \bar{y})$ *is weakly Pareto optimal among all production-consumption pairs with the same total production of the regulated commodities;*

(2) Suppose $P_\omega(\bar{x}_{-\omega}, \bar{y}, \bar{p})$ *is negatively transitive and locally non-satiated for all* $\omega \in \Omega$*. Then* $(\bar{x}, \bar{y})$ *is constrained Pareto optimal, i.e.,* $(\bar{x}, \bar{y})$ *is Pareto optimal among all production-consumption pairs with the same total production of the regulated commodities.*

3. Detailed Analysis of Examples

In this section, we provide detailed analysis and computation for Examples 1, 2, 3, and 4 in Anderson and Duanmu (2025).

3.1. Detailed Analysis of Example 1.

Claim 3.1. *Let* $\mathcal{E}$ *be the quota production economy considered in Example 1 of Anderson and Duanmu (2025). For any* $-m \in (0, 1]$*, there is a unique* $\mathcal{Z}(m)$*-compliant quota equilibrium:*

- *agents* ω_1 *and* ω_2*'s equilibrium consumption are* $(0, 0, -m)$ *and* $(0, 1+m, 0)$*, respectively;*
- *the private firm's equilibrium production is* $(-m, m, -m)$*;*
- *the equilibrium price is* $(\frac{-m-1}{2}, \frac{-m}{2}, \frac{1}{2})$*.*

Proof. Fix $-m \in (0, 1]$. Let $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3)$ be an equilibrium price associated with a $\mathcal{Z}(m)$-compliant quota equilibrium. As the consumption set for both agents is $\{0\} \times \mathbb{R}^2_{\geq 0}$, The equilibrium production of the private firm must be $(-m, m, -m)$. By the agents' utility functions and unbounded consumption sets, we have $\bar{p}_2 > 0$ and $\bar{p}_3 > 0$. Hence, agent ω_1 must sell all her endowment to consume electricity, and agent ω_2 must use all her income to consume coal. As we require non-free-disposal at equilibrium for both coal and electricity, the equilibrium consumption for agent ω_1 and ω_2 are $(0, 0, -m)$ and $(0, 1+m, 0)$, respectively.

Since all quotas are assigned to the government firm and the government firm distributes all its profit to agent ω_2, agent ω_2's total income is $-\bar{p}_1(-m) = \bar{p}_1 m$. As $\bar{p}_2 > 0$ and $-m \in (0, 1]$, agent ω_2's equilibrium consumption, which is $(0, 1+m, 0)$, has a positive value. Hence, we have $\bar{p}_1 < 0$. By the private firm's production technology, we have $\bar{p}_1 - \bar{p}_2 + \bar{p}_3 = 0$. As $\bar{p}_1 < 0$ and $\bar{p}_2 > 0$, we have $\bar{p}_3 = \bar{p}_2 - \bar{p}_1 = \bar{p}_2 + |\bar{p}_1|$. As $\bar{p} \in \Delta$, we have $\bar{p}_3 + \bar{p}_2 + |\bar{p}_1| = 1$, which implies that $\bar{p}_3 = \frac{1}{2}$. Note that agent ω_1's total income is $\bar{p}_2$. Since agent ω_1 uses all the income to consume electricity, we have $\bar{p}_2 = \frac{-m}{2}$. Hence, the equilibrium price must be $(\frac{-m-1}{2}, \frac{-m}{2}, \frac{1}{2})$. $\square$

Claim 3.2. *The unregulated economy $\mathcal{E}'$ in Example 1 of Anderson and Duanmu (2025) has a unique equilibrium with:*

- *agents ω_1 and ω_2's equilibrium consumption are $(0,0,1)$ and $(0,0,0)$, respectively;*
- *the private firm's equilibrium production is $(1,-1,1)$;*
- *the equilibrium price is $(0, \frac{1}{2}, \frac{1}{2})$*

Proof. Let $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3)$ be an equilibrium price. By the agents' utility functions and unbounded consumption sets, we have $\bar{p}_2 > 0$ and $\bar{p}_3 > 0$. In the unregulated economy $\mathcal{E}'$, the government firm's profit is 0. As the private firm has a linear production technology, its profit at any equilibrium must be 0. As agent ω_2's endowment is $(0,0,0)$, agent ω_2's income at any equilibrium is 0. Since agent ω_2's consumption set is $\{0\} \times \mathbb{R}^2_{\geq 0}$ and $\bar{p}_2, \bar{p}_3 > 0$, agent ω_2's equilibrium consumption must be $(0,0,0)$. Note that agent ω_1 derives utility from electricity and is indifferent to coal consumption. As the equilibrium price $\bar{p}_2$ of coal is positive and agent ω_1 is only endowed with coal, agent ω_1 must sell all the endowment (to the private firm) to consume electricity. Hence, the equilibrium production of the private firm must be $(1,-1,1)$. Since we require non-free-disposal at equilibrium for electricity, agent ω_1's equilibrium consumption must be $(0,0,1)$. Note that agent ω_1's total income is $\bar{p}_2$ and the value of her total consumption is $\bar{p}_3$. Thus, we must have $\bar{p}_2 = \bar{p}_3$. By the private firm's production technology, we have $\bar{p}_1 - \bar{p}_2 + \bar{p}_3 = 0$, which implies that $\bar{p}_1 = 0$. As $\bar{p} \in \Delta$, we have $\bar{p} = (0, \frac{1}{2}, \frac{1}{2})$. $\square$

3.2. **Detailed Analysis of Example 2.**

Claim 3.3. *Let $\mathcal{E}$ be the quota production economy considered in Example 2 of Anderson and Duanmu (2025). If the quota $-m$ is less than 6, then there is a unique equilibrium price $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, which is independent of quota levels and quota allocations among firms.*

Proof. As agents derive utility from electricity and their consumption sets allow them to consume an unbounded amount of electricity, the equilibrium price $\bar{p}_3$ of electricity must be positive. If the equilibrium price $\bar{p}_2$ of coal is less than or equal to 0, the first private firm's production technology implies that the equilibrium price $\bar{p}_1 < 0$, and $\bar{p}_3 + \bar{p}_1 \leq 0$.[9] By the

[9] If $\bar{p}_1 - \bar{p}_2 + \bar{p}_3 > 0$, then the first firm's profit is unbounded.

second private firm's production technology, $-\bar{p}_3 - 2\bar{p}_1 \leq 0$. So we have $\bar{p}_3 + \bar{p}_1 - \bar{p}_3 - 2\bar{p}_1 = -\bar{p}_1 \leq 0$, which implies $\bar{p}_1 \geq 0$. This is a contradiction with $\bar{p}_1 < 0$. Hence $\bar{p}_2 > 0$.

Since agents do not obtain utility from coal, the equilibrium price for coal is positive, and the second firm's production technology does not allow it to consume coal, the first firm must purchase the entire endowment of coal. Hence, the equilibrium production of the first firm must be $(6, -6, 6)$. Since the quota is less than 6, the second firm's equilibrium production must not be 0. From the profit maximization of the first private firm, we have $\bar{p}_1 - \bar{p}_2 + \bar{p}_3 = 0$, which implies that $\bar{p}_3 = \bar{p}_2 - \bar{p}_1$. As $\bar{p} \in \Delta$, we have $|\bar{p}_1| + |\bar{p}_2| + |\bar{p}_3| = -\bar{p}_1 + \bar{p}_2 + \bar{p}_3 = 2\bar{p}_3 = 1$. Hence, we have $\bar{p}_3 = \frac{1}{2}$. From the profit maximization of the second private firm, we have $\bar{p}_1 = -\frac{1}{2}\bar{p}_3$. Hence, we have $\bar{p}_1 = -\frac{1}{4}$. As $\bar{p} \in \Delta$ and $\bar{p}_2 > 0$, we have $\bar{p}_2 = \frac{1}{4}$. Hence, the equilibrium price is $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. □

Claim 3.4. *Let $\mathcal{E}$ be the quota production economy considered in Example 2 of Anderson and Duanmu (2025). For every $v = -m \in [0, 6)$ and agent $\omega \in \Omega$, her equilibrium consumption is $(0, 0, \frac{\omega}{2} + \frac{v}{6})$ in the global quota case and is $(0, 0, \frac{\omega}{2} + \frac{v\omega}{12})$ in the cap-and-trade case.*

Proof. Fix $v = -m \in [0, 6)$. By Claim 3.3, the equilibrium price is $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

We first consider the global quota case. That is, all quotas are allocated to the government firm. For $\omega \in \Omega$, her total income consists of the value of her endowment plus the dividend from the government firm,[10] which is $\frac{1}{4}\omega + \frac{1}{3}\frac{v}{4} = \frac{\omega}{4} + \frac{v}{12}$. As the agent only derives utility from consuming electricity, the agent must spend all income to consume electricity. As the equilibrium price of electricity is $\frac{1}{2}$, the equilibrium consumption of agent ω is $(0, 0, \frac{\omega}{2} + \frac{v}{6})$.

We now consider the cap-and-trade case. That is, all quotas are allocated to the first private firm. For $\omega \in \Omega$, her total income consists of the value of her endowment plus the dividend from the first private firm,[11] which is $\frac{1}{4}\omega + \frac{\omega}{6}\frac{v}{4} = \frac{\omega}{4} + \frac{v\omega}{24}$. As the agent only derives utility from consuming electricity, the agent must spend all income to consume electricity. The equilibrium consumption of agent ω is $(0, 0, \frac{\omega}{2} + \frac{v\omega}{12})$. □

3.3. **Computation of Agents' Demands in Example 3.** Recall that the government sets a tax rate t, hence the price of commodity 0, $p_0 = -t \leq 0$. Moreover, we normalize p_0, p_1, p_2 such that $p_1 + p_2 = 1$. Given a price vector $p = (p_0, p_1, p_2)$ and a consumption vector (x_{01}, x_{02}) of commodity 0, agent i's emissions of commodity 0 equal $(0.2 - x_{0i})$, so agent i must pay a

[10]The two private firms have zero profit.

[11]The government firm and the second private firm have zero profit.

tax of $t(0.2 - x_{0i}) = -p_0(0.2 - x_{0i})$. The first agent's expenditure[12] on commodities 1 and 2 is the value of agent 1's endowment of goods 1 and 2, minus the tax paid by agent 1, plus agent 1's rebate of the total taxes paid by both agents:

$$\begin{aligned} I_1 &= 2p_1 + rp_2 + p_0(0.2 - x_{01}) - \frac{p_0}{2}\left[(0.2 - x_{01}) + (0.2 - x_{02})\right] \\ &= 2p_1 + rp_2 + \frac{p_0}{2}(x_{02} - x_{01}). \end{aligned}$$

Similarly, the second agent's expenditure on commodities 1 and 2 is:

$$I_2 = rp_1 + 2p_2 + \frac{p_0}{2}(x_{01} - x_{02}).$$

We first compute agents' demand for commodities 1 and 2 as a function of p_1, p_2, I_1, I_2. Since the first agent's utility function $U_1(x_{01}, x_{11}, x_{21}) = \left(x_{11} - \frac{1}{8}x_{21}^{-8}\right) - \frac{1}{10}x_{01}^2$, we have $\left(\frac{\partial U_1}{\partial x_{11}}, \frac{\partial U_1}{\partial x_{21}}\right) = (1, x_{21}^{-9})$. Thus, the demand for commodities 1 and 2 must satisfy $\frac{1}{x_{21}^{-9}} = \frac{p_1}{p_2}$, which implies that $x_{21} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{9}}$. We also have $p_1 x_{11} + p_2 x_{21} = I_1$. Hence, we have $x_{11} = \frac{I_1}{p_1} - \frac{p_2}{p_1}x_{21} = \frac{I_1}{p_1} - \frac{p_2}{p_1}(\frac{p_1}{p_2})^{\frac{1}{9}} = \frac{I_1}{p_1} - \left(\frac{p_2}{p_1}\right)^{\frac{8}{9}}$. Thus, agent 1's demand for commodities 1 and 2, as a function of p_1, p_2, I_1, is $\left(\frac{I_1}{p_1} - \left(\frac{p_2}{p_1}\right)^{\frac{8}{9}}, \left(\frac{p_1}{p_2}\right)^{\frac{1}{9}}\right)$. To compute agent 1's demand for commodity 0, we first note that the marginal utility of consuming commodities 1 and 2 in terms of the expenditure I_1 is $\frac{1}{p_1}$. We also note that the marginal available income from consumption of commodity 0 is $-p_0$. Since agent 1's demand for commodities 1 and 2 is $\left(\frac{I_1}{p_1} - \left(\frac{p_2}{p_1}\right)^{\frac{8}{9}}, \left(\frac{p_1}{p_2}\right)^{\frac{1}{9}}\right)$, each unit of additional income generates $\frac{1}{p_1}$ additional utility to agent 1. Hence, the marginal utility of consumption of commodities 1 and 2 financed by consumption of commodity 0 is $\frac{-p_0}{p_1}$. The marginal disutility of agent 1 of consuming commodity 0 is $-\frac{\partial U_1}{\partial x_{01}} = \frac{1}{5}x_{01}$. Hence, agent 1's demand for commodity 0 must satisfy $\frac{1}{5}x_{01} = \frac{-p_0}{p_1}$, which implies that agent 1's demand for commodity 0 is $\frac{-5p_0}{p_1}$. Thus, agent 1's demand for all three commodities is $\left(\frac{-5p_0}{p_1}, \frac{I_1}{p_1} - \left(\frac{p_2}{p_1}\right)^{\frac{8}{9}}, \left(\frac{p_1}{p_2}\right)^{\frac{1}{9}}\right)$. By the same calculation, agent 2's demand for all three commodities is $\left(\frac{-p_0}{p_2}, \left(\frac{p_2}{p_1}\right)^{\frac{1}{9}}, \frac{I_2}{p_2} - \left(\frac{p_1}{p_2}\right)^{\frac{8}{9}}\right)$. Let $\eta = \frac{p_1}{p_2}$. As $p_1 + p_2 = 1$, we have $\frac{-5p_0}{p_1} = -5p_0\frac{1+\eta}{\eta}$ and $\frac{-p_0}{p_2} = -p_0(1+\eta)$. Hence, we have $I_1 = \frac{2\eta}{1+\eta} + \frac{r}{1+\eta} + \frac{p_0}{2}\left[5p_0\frac{1+\eta}{\eta} - p_0(1+\eta)\right]$ and $I_2 = \frac{r\eta}{1+\eta} + \frac{2}{1+\eta} + \frac{p_0}{2}\left[p_0(1+\eta) - 5p_0\frac{1+\eta}{\eta}\right]$. Hence, agents' demands are functions of the tax rate and $\eta = \frac{p_1}{p_2}$. In particular, for any given tax rate, agents' excess demands are functions

[12] Agents consume commodity 0 only to generate additional income to consume commodities 1 and 2. Since agents have monotone preferences for commodities 1 and 2, they will spend all of the income available on commodities 1 and 2.

of the single variable $\eta = \frac{p_1}{p_2}$. We solve numerically for the values of η such that the excess demand for good 1 is zero. By Rebate Walras' Law (Theorem 2.2), the excess demand for good 2 is also zero at those same values of η.

3.4. **Detailed Analysis of Example 4.** We provide rigorous proofs for Claim 5.1, Claim 5.2 and Claim 5.3 of Example 4 in Anderson and Duanmu (2025) to complete our analysis of this example. We also present Claim 3.5, which plays an important role in Example 2.3.

Proof of Claim 5.1. Let $t_0 \leq \frac{1}{4}$ be the tax rate on CO_2. Let $\bar{x} = (0,0,1), \bar{y} = \big((1,-1,1),(0,0,0)\big)$ and $\bar{p} = (-t_0, \frac{1}{2} - t_0, \frac{1}{2})$. We claim that $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-compliant emission tax equilibrium with tax rate t_0 on CO_2. At the equilibrium price $\bar{p}$, both firms are profit maximizing with zero profit. The budget set for the agent is:

$$\left\{ z \in X : \bar{p} \cdot z \leq \frac{1}{2} - t_0 + t_0 = \frac{1}{2} \right\}.$$

Since the agent derives utility only from consuming electricity, $\bar{x}$ is in the agent's emission tax demand set $D^t(\bar{x}, \bar{y}, \bar{p})$. Note that $\bar{x} - e - \sum_{j \in J} \bar{y}(j) = (-1,0,0) \in \mathcal{V}$. So $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-compliant emission tax equilibrium with tax rate t_0.

Now, suppose that $t > 0$ is an emission tax rate on CO_2 under which there is a $\mathcal{V}$-compliant emission tax equilibrium $(\hat{x}, \hat{y}, \hat{p})$. By definition, we know that $\hat{p}_1 = -t$. The equilibrium price $\hat{p}_3$ must not be less than $2t$, since otherwise the second firm's profit is unbounded. For the same reason, we know that $\hat{p}_2 \geq \hat{p}_3 - t > 0$. As the endowment $e = (0,1,0)$, the agent's budget at equilibrium is positive. The equilibrium production for the first firm must not be $(0,0,0)$ since the agent has a positive budget that she will spend entirely on electricity. Hence, we conclude that $\hat{p}_3 = t + \hat{p}_2$. As $\hat{p} \in \Delta$, we have $2t + 2\hat{p}_2 = 1$, which implies that $\hat{p}_3 = \frac{1}{2}$. As $\hat{p}_3 \geq 2t$, we know that $t \leq \frac{1}{4}$. Thus, we conclude that $\mathcal{F}$ has a $\mathcal{V}$-compliant emission tax equilibrium if and only if the tax rate $t \leq \frac{1}{4}$. $\square$

Proof of Claim 5.2. Pick the emission tax rate $0 \leq t_0 < \frac{1}{4}$ of CO_2. Claim 5.1 in Anderson and Duanmu (2025) indicates that there is a $\mathcal{V}$-compliant emission tax equilibrium $(\hat{x}, \hat{y}, \hat{p})$.

We first consider the case where $t_0 > 0$. Using the same argument as in the second paragraph of the proof of Claim 5.1, we conclude that $\hat{p}_3 = \frac{1}{2}$ and $\hat{p}_2 = \frac{1}{2} - t_0$. As $\hat{p}_3 > 2t_0$, the equilibrium production for the second firm is $(0,0,0)$. Suppose that the equilibrium production for the first firm is $(r, -r, r)$ for $r < 1$. The emission tax budget set is:

$$\left\{ z \in X : \hat{p} \cdot z \leq \frac{1}{2} - t_0 + rt_0 \right\}.$$

As $t_0 < \frac{1}{4}$ and $r < 1$, we have

$$\frac{1}{2}r - \left(\frac{1}{2} - t_0 + rt_0\right) = \frac{1}{2}(r-1) - t_0(r-1) < 0. \tag{3.1}$$

Equation Eq. (3.1) implies that the consumption $(0,0,r)$ is in the agent's budget set. However, the agent demands more than r units of electricity. So $(0,0,r)$ is not in the emission tax demand set $D^t(\hat{x},\hat{y},\hat{p})$. Hence, the first firm's equilibrium production is $(1,-1,1)$. So the total net emissions of CO_2 is 1. □

Proof of Claim 5.3. Pick $v_0 \in [0,1]$. Let $\hat{x}_{v_0} = (0,0,\frac{1+v_0}{2})$, $\hat{y}_{v_0} = \big((1,-1,1),(v_0-1,0,\frac{v_0-1}{2})\big)$, and $\hat{p} = (-\frac{1}{4},\frac{1}{4},\frac{1}{2})$. We claim that $(\hat{x}_{v_0},\hat{y}_{v_0},\hat{p})$ is a $\mathcal{V}$-compliant emission tax equilibrium associated with the emission tax rate $\frac{1}{4}$ such that the total net CO_2 emissions is v_0. As $\hat{p}_1 = -\frac{1}{4}$, the associated emission tax rate is $\frac{1}{4}$. Note that $\hat{x}_{v_0} - e - \sum_{j\in J}\hat{y}_{v_0}(j) = (-v_0,0,0) \in \mathcal{V}$. Hence, the total net CO_2 emissions is v_0. Both firms are profit maximizing, with zero profit, given the price vector $\hat{p}$. The emission tax budget set for the agent is $\{z \in X : \hat{p}\cdot z \leq \frac{1}{4}(1+v_0)\}$. Since the agent derives utility only from consuming electricity, $\hat{x}_{v_0}$ is an element of the emission tax demand set $D^t(\hat{x}_{v_0},\hat{y}_{v_0},\hat{p})$, completing the proof. □

The following claim is crucial for the comparison of welfare properties between the emission tax equilibrium in this example and the fuel tax equilibrium in Example 2.3.

Claim 3.5. *Given $v \in [0,1]$, let $(\hat{x}_v,\hat{y}_v,\hat{p}_v)$ be a $\mathcal{V}$-compliant emission tax equilibrium such that the total net CO_2 emissions is v. Then*

$$\hat{x}_v = \left(0,0,\frac{1+v}{2}\right) \text{ and } \hat{y}_v = \left((1,-1,1),\left(v-1,0,\frac{v-1}{2}\right)\right).$$

Proof. We first consider the case where $v = 1$. The equilibrium production $\hat{y}_v$ must be $\big((1,-1,1),(0,0,0)\big)$. The emission tax rate associated with $(\hat{x}_v,\hat{y}_v,\hat{p}_v)$ is $t = -\hat{p}_v(1)$. The agent's total income is $\frac{1}{2} - t + t = \frac{1}{2}$. Since the agent derives utility only from consuming electricity, the agent's equilibrium consumption $\hat{x}_v$ is $(0,0,1)$.

We now consider the case where $v < 1$. By Claims 5.2 and 5.3 in Anderson and Duanmu (2025), the emission tax rate associated with $(\hat{x}_v,\hat{y}_v,\hat{p}_v)$ must be $\frac{1}{4}$. Using the same argument as in the second paragraph of the proof of Claim 5.1, the equilibrium price $\hat{p}_v$ must be $(-\frac{1}{4},\frac{1}{4},\frac{1}{2})$. Since we require non-free-disposal of coal at equilibrium, the first firm's equilibrium production must be $(1,-1,1)$. So the second firm's equilibrium production is $(v-1,0,\frac{v-1}{2})$. The agent's total income is $\frac{1}{4} + \frac{v}{4}$. Since the agent derives utility only from consuming electricity, the agent's equilibrium consumption $\hat{x}_v$ is $(0,0,\frac{1+v}{2})$. □

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