Supplement to "The origins and effects of macroeconomic uncertainty"

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Appendix A: First-order conditions from the estimated model

Household's problem

Household solves the following constrained optimization problem. It maximizes its value function

$$V(\overline{K}_{t-1}, I_{t-1}, B_t) = \max_{C_t, L_t, B_{t+1}, I_t, \overline{K}_t, U_t} u(C_t, L_t, B_{t+1})^{(1-\beta_t)} \big(E_t\big[V(\overline{K}_t, I_t, B_{t+1})^{1-\gamma}\big]\big)^{\frac{\beta_t}{1-\gamma}},$$

where

$$u(C_t, L_t, B_{t+1}) = (C_t - h\overline{C}_{t-1}) e^{-\tau_0 \frac{L_t^{1+\tau}}{1+\tau}} e^{\zeta_{B,t} \frac{B_{t+1}}{R_t P_t Z_t^*}},$$

subject to the following constraints:

$$\overline{K}_t = \overline{K}_{t-1}\big(1 - \delta(U_t)\big) + \big[1 - S(I_t/I_{t-1})\big] I_t,$$

$$S(I_t/I_{t-1}) = \frac{\varphi_I}{2}\big(I_t/I_{t-1} - e^{\mu^*}\Upsilon\big)^2,$$

$$\delta(U_t) = \delta_0 + \delta_1(U_t - U_{ss}) + \frac{\delta_2}{2}(U_t - U_{ss})^2,$$

$$P_t C_t + P_t\big(e^{\zeta_{\Upsilon,t}} \Upsilon^t\big)^{-1} I_t + B_{t+1}/R_t = P_t D_t + P_t W_t L_t + B_t + P_t \overline{K}_{t-1} r_t^k U_t - P_t T_t.$$

From the household's optimization problem, we can derive the following first-order intertemporal condition:

$$1 = E_t\left[M_{t+1} \frac{P_t}{P_{t+1}}\right] R_t + \frac{1}{Z_t^*} \overline{\zeta}_B e^{\widetilde{\zeta}_{B,t}} (C_t - h\overline{C}_{t-1}), \tag{18}$$

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where

$$M_{t+1} = \frac{1-\beta_{t+1}}{1-\beta_t}\beta_t\left(\frac{V_{t+1}}{(E_t V_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}}}\right)^{1-\gamma} \times \left(\frac{u(C_{t+1}, L_{t+1}, B_{t+2})}{u(C_t, L_t, B_{t+1})}\right)^{-1}\left(\frac{u_1'(C_{t+1}, L_{t+1}, B_{t+2})}{u_1'(C_t, L_t, B_{t+1})}\right) \tag{19}$$

is the stochastic discount factor.

The intratemporal condition is

$$W_t = \tau_0 L_t^{\tau}(C_t - h\overline{C}_{t-1}).$$

The first-order conditions of the household with respect to a capital utilization choice and investment decision result in the following two equations, respectively:

$$\frac{r_t^k}{\delta'(U_t)}\left[1-\frac{\varphi_I}{2}\left(\frac{I_t}{I_{t-1}}-e^{\mu^*}\Upsilon\right)^2-\varphi_I\left(\frac{I_t}{I_{t-1}}-e^{\mu^*}\Upsilon\right)\frac{I_t}{I_{t-1}}\right] + E_t\left[M_{t+1}\frac{r_{t+1}^k}{\delta'(U_{t+1})}\varphi_I\left(\frac{I_{t+1}}{I_t}-e^{\mu^*}\Upsilon\right)\frac{I_{t+1}^2}{I_t^2}\right] = \left(e^{\zeta_{\Upsilon,t}}\Upsilon^t\right)^{-1},$$

and

$$\frac{r_t^k}{\delta'(U_t)} = E_t\left[M_{t+1}\left(r_{t+1}^k U_{t+1} + \frac{r_{t+1}^k}{\delta'(U_{t+1})}\left(1-\delta(U_{t+1})\right)\right)\right].$$

Intermediate firm's problem

Intermediate firm i maximizes the present value of current and future cash flows:

$$V^{(i)}(P_{i,t-1}) = \max_{P_{i,t},K_{i,t},L_{i,t}}\left\{D_{i,t} + E_t\left[M_{t+1}V^{(i)}(P_{i,t})\right]\right\},$$

subject to the following constraints:

$$P_t D_{i,t} = P_{i,t}X_{i,t} - P_t W_t L_{i,t} - P_t r_t^k K_{i,t} - P_t G(P_{i,t}, P_{i,t-1}, Y_t),$$

$$X_{i,t} = Y_t(P_{i,t}/P_t)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}},$$

$$X_{i,t} = K_{i,t}^{\alpha}\left(e^{n_t}L_{it}\right)^{1-\alpha},$$

$$G(P_{i,t}, P_{i,t-1}, Y_t) = \frac{\phi_R}{2}\left(\frac{P_{i,t}}{\Pi_{ss}^{\kappa_\pi}\Pi_{t-1}^{1-\kappa_\pi}P_{i,t-1}}-1\right)^2 Y_t.$$

The first-order condition of the intermediate firm with respect to the price setting decision is given by

$$\left(1-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}\right)\left(\frac{P_{i,t}}{P_t}\right)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}}\frac{Y_t}{P_t} + W_t\frac{L_{i,t}}{1-\alpha}\left(\frac{1+\lambda_{p,t}}{\lambda_{p,t}}\right)\left(\frac{P_{i,t}}{P_t}\right)^{-1}\frac{1}{P_t}$$

$$- \phi_R \left(\frac{P_{i,t}}{\Pi_{ss}^{\kappa_\pi} \Pi_{t-1}^{(1-\kappa_\pi)} P_{i,t-1}} - 1 \right) \frac{Y_t}{\Pi_{ss}^{\kappa_\pi} \Pi_{t-1}^{(1-\kappa_\pi)} P_{i,t-1}}$$
$$+ E_t \left[M_{t+1} \phi_R \left(\frac{P_{i,t+1}}{\Pi_{ss}^{\kappa_\pi} \Pi_{t}^{(1-\kappa_\pi)} P_{i,t}} - 1 \right) \frac{Y_{t+1} P_{i,t+1}}{\Pi_{ss}^{\kappa_\pi} \Pi_{t}^{(1-\kappa_\pi)} P_{i,t}^2} \right] = 0.$$

Combining the first-order conditions of the intermediate firm with respect to the capital and labor choice, we get

$$r_t^k = \frac{\alpha}{1-\alpha} W_t \frac{L_{i,t}}{K_{i,t}}.$$

Appendix B: Equilibrium conditions

1. Household's value function:

$$V_t = u(C_t, L_t, B_{t+1})^{(1-\beta_t)} \left(E_t \left[V_{t+1}^{1-\gamma} \right] \right)^{\frac{\beta_t}{1-\gamma}},$$
$$u(C_t, L_t, B_{t+1}) = (C_t - h\overline{C}_{t-1}) e^{-\tau_0 \frac{L_t^{1+\tau}}{1+\tau}} e^{\zeta_{B,t} \frac{B_{t+1}}{R_t P_t Z_t^*}},$$

where $\beta_t = (1 + \hat{\beta} e^{\tilde{b}_t})^{-1}$.

2. Stochastic discount factor:

$$M_{t+1} = \frac{1-\beta_{t+1}}{1-\beta_t} \beta_t \left(\frac{V_{t+1}}{\left(E_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}}} \right)^{1-\gamma}$$
$$\times \left(\frac{u(C_{t+1}, L_{t+1}, B_{t+2})}{u(C_t, L_t, B_{t+1})} \right)^{-1} \left(\frac{u_1'(C_{t+1}, L_{t+1}, B_{t+2})}{u_1'(C_t, L_t, B_{t+1})} \right).$$

3. Household's first-order intertemporal condition:

$$1 = E_t \left[M_{t+1} \frac{P_t}{P_{t+1}} \right] R_t + \frac{1}{Z_t^*} \overline{\zeta}_B e^{\tilde{\zeta}_{B,t}} (C_t - h\overline{C}_{t-1}).$$

4. Household's first-order condition with respect to the labor supply choice:

$$W_t = \tau_0 L_t^\tau (C_t - h\overline{C}_{t-1}).$$

5. Household's first-order conditions with respect to the capital utilization choice and the investment decision:

$$\frac{r_t^k}{\delta'(U_t)} \left[1 - \frac{\varphi_I}{2} \left(\frac{I_t}{I_{t-1}} - e^{\mu^*} \Upsilon \right)^2 - \varphi_I \left(\frac{I_t}{I_{t-1}} - e^{\mu^*} \Upsilon \right) \frac{I_t}{I_{t-1}} \right]$$
$$+ E_t \left[M_{t+1} \frac{r_{t+1}^k}{\delta'(U_{t+1})} \varphi_I \left(\frac{I_{t+1}}{I_t} - e^{\mu^*} \Upsilon \right) \frac{I_{t+1}^2}{I_t^2} \right] = \left(e^{\zeta_{\Upsilon,t}} \Upsilon^t \right)^{-1},$$
$$\frac{r_t^k}{\delta'(U_t)} = E_t \left[M_{t+1} \left(r_{t+1}^k U_{t+1} + \frac{r_{t+1}^k}{\delta'(U_{t+1})} \left(1 - \delta(U_{t+1}) \right) \right) \right].$$

6. Capital accumulation:

$$\overline{K}_t = \overline{K}_{t-1}\big(1-\delta(U_t)\big) + \big[1 - S(I_t/I_{t-1})\big]I_t,$$

$$S(I_t/I_{t-1}) = \frac{\varphi_I}{2}\big(I_t/I_{t-1} - e^{\mu^*}\Upsilon\big)^2,$$

$$\delta(U_t) = \delta_0 + \delta_1(U_t - U_{ss}) + \frac{\delta_2}{2}(U_t - U_{ss})^2.$$

7. Intermediate firms' first-order condition with respect to the price setting decision:

$$\begin{aligned}&\left(1 - \frac{1+\lambda_{p,t}}{\lambda_{p,t}}\right)\frac{Y_t}{P_t} + W_t\frac{L_t}{1-\alpha}\left(\frac{1+\lambda_{p,t}}{\lambda_{p,t}}\right)\frac{1}{P_t}\\ &- \phi_R\left(\frac{\Pi_t}{\Pi_{ss}^{\kappa_\pi}\Pi_{t-1}^{(1-\kappa_\pi)}} - 1\right)\frac{Y_t}{\Pi_{ss}^{\kappa_\pi}\Pi_{t-1}^{(1-\kappa_\pi)}P_{t-1}}\\ &+ E_t\left[M_{t+1}\phi_R\left(\frac{\Pi_{t+1}}{\Pi_{ss}^{\kappa_\pi}\Pi_{t}^{(1-\kappa_\pi)}} - 1\right)\frac{Y_{t+1}\Pi_{t+1}}{\Pi_{ss}^{\kappa_\pi}\Pi_{t}^{(1-\kappa_\pi)}P_t}\right] = 0.\end{aligned}$$

8. Intermediate firms' first-order condition with respect to the capital and labor choice:

$$r_t^k = \frac{\alpha}{1-\alpha}W_t\frac{L_t}{K_t},$$

where $K_t = U_t\overline{K}_{t-1}$.

9. Production function:

$$Y_t = K_t^\alpha\big(e^{n_t}L_t\big)^{1-\alpha},$$

where $\Delta n_t = \mu + x_t$.

10. Modified Taylor rule:

$$\ln\left(\frac{R_t}{R^*}\right) = \rho_r\ln\left(\frac{R_{t-1}}{R^*}\right) + (1-\rho_r)\left(\rho_\pi\ln\left(\frac{\Pi_t}{\Pi_{ss}e^{\overline{\pi}^*}}\right) + \rho_y\ln\left(\frac{\widehat{Y}_t}{\widehat{Y}_{ss}}\right)\right) + \sigma_R\varepsilon_{R,t},$$

where $\widehat{Y}_t \equiv Y_t/Z_t^*$ is the detrended output.

11. Resource constraint:

$$Y_t = C_t + \big(e^{\zeta_{\Upsilon,t}}\Upsilon^t\big)^{-1}I_t + \frac{\phi_R}{2}\big(\Pi_t/\big(\Pi_{ss}^{\kappa_\pi}\Pi_{t-1}^{1-\kappa_\pi}\big) - 1\big)^2 Y_t + G_t.$$

12. Government spending:

$$\log G_{t+1} - \log G_{ss} = \rho_g(\log G_t - \log G_{ss}) + \sigma_g\varepsilon_{g,t+1}.$$

13. Preference shock:

$$\tilde{b}_{t+1} = \rho_\beta\tilde{b}_t + \sigma_{\beta,\xi_{t+1}^D}\varepsilon_{\beta,t+1}, \quad \varepsilon_{\beta,t+1} \sim N(0,1).$$

14. Liquidity shock:

$$\widetilde{\zeta}_{B,t+1} = \rho_{\zeta_B}\widetilde{\zeta}_{B,t} + \sigma_{\zeta_B}\varepsilon_{\zeta_B,t+1}, \quad \varepsilon_{\zeta_B,t+1} \sim N(0,1).$$

15. Shock to the relative price of the investment good:

$$\zeta_{\Upsilon,t+1} = \rho_{\Upsilon}\zeta_{\Upsilon,t} + \sigma_{\zeta_\Upsilon}\varepsilon_{\zeta_\Upsilon,t+1}, \quad \varepsilon_{\zeta_\Upsilon,t+1} \sim N(0,1).$$

16. Markup shock:

$$\log \lambda_{p,t} - \log \overline{\lambda_p} = \rho_\chi(\log \lambda_{p,t-1} - \log \overline{\lambda_p}) + \sigma_\chi \varepsilon_{\chi,t}, \quad \varepsilon_{\chi,t} \sim N(0,1).$$

17. TFP growth shock:

$$x_t = \rho_x x_{t-1} + \sigma_{x,\xi_t^S}\varepsilon_{x,t}, \quad \varepsilon_{x,t} \sim N(0,1).$$

Appendix C: Details about the solution method

This section provides more details about our log-linearization approach. As explained in the main text, our approach is quite common in the asset pricing and macro-finance literatures (e.g., Jermann (1998), Lettau (2003), Backus, Routledge, and Zin (2010), Uhlig (2010), Dew-Becker (2012), Malkhozov (2014), and Bianchi, Ilut, and Schneider (2018)). This Appendix is meant to provide more details about the method in order to make the paper self-contained. In particular, we aim to make the following points:

1. The method can be characterized as a guess-and-verify approach. This is because once the model is log linearized and solved, *with or without risk-adjustment*, the variables of the model follow a linear process in logs and are therefore log normal in levels. Thus, the method exploits this property of the solution when log linearizing the model and implements a risk-adjusted log linearization. This affects only the equilibrium conditions in which an expectational term appears. Note that log normality does not affect the rest of the log-linearized equations. When introducing stochastic volatility, the process becomes conditionally log normal. We explain how this affects the method and the quality of the approximation below.

2. To understand why the solution without risk adjustment already implies log normality, it is important to notice that all shocks are specified in logs. Thus, when taking a log-normal approximation, the solution of the model implies a linear process in logs with Gaussian innovations. Note that when the variance of a shock increases, the mean of the shock is unchanged. The mean of the exponential of the shock would change, but this is not what is used in the log-linear approximation. Thus, without the risk-adjusted log linearization, the increase in the variance of the shocks would translate into an increase in the variance of the variables expressed in logs, but it would not have first-order effects. The mean of the level of the variables would change, but this is not how we measure the effects of uncertainty. For example, the mean of log consumption would not be affected, so we would conclude that there are no effects of uncertainty on consumption. Importantly, the mean

of consumption in levels would change independently from using or not the risk-adjusted log linearization.

3. The solution with risk-adjustment allows us to take into account the effects of uncertainty on the economy. As explained above, the method exploits the fact that even without risk adjustment, the log-linearized solution implies that the variables have a log-normal distribution (i.e., they are linear in log deviations from the deterministic steady state). While the risk-adjusted log linearization allows us to take into account the effects of uncertainty, the effects of uncertainty are not automatically large in this setting. Instead, the effects of uncertainty depend on the model and the estimated parameters. In the paper, we show that nominal rigidities and Epstein–Zin preferences are important. Below we consider a very simple example to make the same point in an even simpler setting.

C.1 *A simple model*

To illustrate the points above and the approximation method used in the paper, consider the simple Fisherian model:

$$R_t = E_t[I_t/\Pi_{t+1}],$$

where R_t is the gross real interest rate (the notation here is different with respect to the paper), I_t is the gross nominal interest rate, and $\Pi_{t+1} = P_{t+1}/P_t$ is the gross inflation rate. Assume a Taylor rule for the nominal interest rate:

$$I_t/I = (\Pi_t/\Pi)^{\psi_\pi},$$

and a normal process for the log of the real interest rate:

$$\log(R_t) = r_t \sim N\big(0, \sigma_r^2\big).$$

Thus, in this simple model, the real interest rate follows an exogenous process. Furthermore, r_t is approximately equal to the net real interest rate: $\log(R_t) = \log(1 + r_t) \cong r_t$. The assumption that the exogenous shock specified in logs follows a Normal distribution is standard in the applied macro literature, where all shocks are specified as log deviations from a steady state. In this case, the steady state for the log of the real interest rate is zero. The mean of the gross real interest R_t depends on σ_r^2, however, note that σ_r^2 does not affect the mean of r_t.

In the zero (net) inflation deterministic steady state, we have

$$\Pi = 1, \qquad R = 1, \qquad I = 1.$$

The standard log approximation would give us:

$$r_t = i_t - E_t[\pi_{t+1}],$$
$$i_t = \psi_\pi \pi_t,$$

where all variables are now expressed in logs. Given that all variables are zero in the steady state, the lower case letters also denote log deviations from steady state. The solution to the model is given by

$$\begin{aligned}\pi_t &= \psi_\pi^{-1} r_t + \psi_\pi^{-1} E_t[\pi_{t+1}] \\ &= \psi_\pi^{-1} r_t + \psi_\pi^{-2} E_t[r_{t+1}] + \cdots \\ &= \psi_\pi^{-1} r_t,\end{aligned}$$

where we have used the fact that the one-step-ahead expected value of the real interest rate is zero. Note that in this case, changes in the variance of the exogenous shock (σ_r^2) do not affect the solution. However, given that π_t is a linear transformation of the normally distributed shock, r_t, it also has a normal distribution. Thus, Π_t is log normal and its mean depends on the variance of r_t. Note that this is true even if we have used the standard log linearization without risk-adjustment. But, again, this is not how we assess the effects of uncertainty. We work with logs and we look at the behavior of π_t, not Π_t. With standard log linearization, there are no effects of σ_r^2 on the mean of π_t. Thus, we conclude that in the standard log-linear approximation approach, we cannot capture the effects of uncertainty on inflation, despite that the mean of gross inflation varies with σ_r^2.

Now, consider the risk-adjusted log linearization used in the paper. As explained above, π_t is a linear transformation of a normal variable (r_t), so it also has a normal distribution. Thus, Π_t has a log-normal distribution. We can then use a guess-and-verify approach and use a risk-adjusted log linearization that takes into account that the solution satisfies log normality. We then have

$$\begin{aligned}r_t &= i_t - E_t[\pi_{t+1}] - 0.5 V_t[\pi_{t+1}], \\ i_t &= \psi_\pi \pi_t.\end{aligned}$$

Note that $V_t[\pi_{t+1}] = \sigma_\pi^2$ is a *constant* that depends on the volatility of the real interest *and* the policy parameter ψ_π. We can then start with a guess on its value, solve the model, and then replace σ_π^2 with the value implied by the solution. The solution now becomes

$$\pi_t = \psi_\pi^{-1} r_t + \psi_\pi^{-1} E_t[\pi_{t+1}] + 0.5 \psi_\pi^{-1} \sigma_\pi^2.$$

Solving forward, we have

$$\begin{aligned}\pi_t &= \psi_\pi^{-1} r_t + 0.5 \frac{\psi_\pi^{-1}}{1 - \psi_\pi^{-1}} \sigma_\pi^2 \\ &= \psi_\pi^{-1} r_t + 0.5 \frac{\psi_\pi^{-1}}{1 - \psi_\pi^{-1}} V_t\big[\psi_\pi^{-1} r_{t+1}\big] \\ &= \psi_\pi^{-1} r_t + 0.5 \frac{\psi_\pi^{-1}}{1 - \psi_\pi^{-1}} \big[\psi_\pi^{-2} \sigma_r^2\big],\end{aligned}$$

where we have used the fact $\sigma_\pi^2 = V_t[\pi_{t+1}] = V_t[\psi_\pi^{-1} r_{t+1}] = \psi_\pi^{-2}\sigma_r^2$. Now, if we vary σ_r^2, the mean of net inflation, π_t, also varies. But, interestingly, this is not because we are varying the level of the real interest rate: The shock to r_t only presents a change in variance, while its mean is still zero. In other words, the mean of the log of the gross real interest rate, r_t, is not changing.

Instead, the effect on the level of inflation is endogenous and depends on how strongly the nominal interest rate reacts to inflation. To see this, note that as we increase the response to inflation in the Taylor rule, the variance of the real interest rate becomes less and less relevant for average inflation. Consistent with the fact that shocks to r_t are in levels, while shocks to σ_r^2 is a second moment shock, the importance of the latter decays faster. As an example, suppose that we double the size of the response to inflation from 2 to 4:

$$\begin{aligned}\pi_t &= \psi_\pi^{-1} r_t + 0.5\frac{1}{\psi_\pi - 1}\left[\psi_\pi^{-2}\sigma_r^2\right] \\ &= 0.5 r_t + 0.125\sigma_r^2, \\ \pi_t &= 0.25\psi_\pi^{-1} r_t + 0.125\frac{1}{2\psi_\pi - 1}\left[\psi_\pi^{-2}\sigma_r^2\right] \\ &= 0.25\psi_\pi^{-1} r_t + 0.015625\left[\sigma_r^2\right]\end{aligned}$$

the response to r_t is cut in half, while the response is divided by 8.

C.2 *Adding regime changes*

In the presence of Markov-switching volatility regimes, the model solution is log normal conditional on the regime. In this subsection, we discuss in more detail how our approximation method compares to the one by Bansal and Zhou (2002).

To study the difference between the two approaches, consider a univariate Markov-switching process:

$$z_{t+1} = c_{\xi_{t+1}} + a z_t + \sigma_{\xi_{t+1}} \varepsilon_{t+1}, \tag{20}$$

where ξ_{t+1} denotes the volatility regime at time $t+1$. The solution of the model, presented in the main text, has this form. When we log linearize the system of model equations, we are facing log linearization equations of the following form:

$$E_t\left[e^{z_{t+1}}\right].$$

We first summarize the approach in Bansal and Zhou (2002), where they utilize conditional log normality of the process in equation (20). In particular,

$$E_t\left[e^{z_{t+1}} | \xi_{t+1}\right] = e^{E_t[z_{t+1}|\xi_{t+1}] + 0.5\,\mathrm{Var}_t[z_{t+1}|\xi_{t+1}]}.$$

Therefore, using the law of iterated expectations,

$$E_t\left[e^{z_{t+1}}\right] = E_t\left[E_t\left[e^{z_{t+1}} | \xi_{t+1}\right]\right]$$

$$= E_t\big[e^{E_t[z_{t+1}|\xi_{t+1}]+0.5\,\mathrm{Var}_t[z_{t+1}|\xi_{t+1}]}\big]$$
$$= E_t\big[e^{c_{\xi_{t+1}}+az_t+0.5\sigma^2_{\xi_{t+1}}}\big].$$

To proceed forward, Bansal and Zhou (2002) use an approximation: $e^{c_{\xi_{t+1}}+az_t+0.5\sigma^2_{\xi_{t+1}}} \approx 1 + c_{\xi_{t+1}} + az_t + 0.5\sigma^2_{\xi_{t+1}}$. This procedure implies a linearization under the expectation sign. Due to this approximation, the expectation becomes linear in the MS constant, $c_{\xi_{t+1}}$, and the final expression does not depend on the volatility of $c_{\xi_{t+1}}$. Indeed,

$$\begin{aligned} E_t\big[e^{z_{t+1}}\big] &= E_t\big[E_t\big[e^{z_{t+1}}|\xi_{t+1}\big]\big] \\ &\approx E_t\big[1 + c_{\xi_{t+1}} + az_t + 0.5\sigma^2_{\xi_{t+1}}\big] \\ &= 1 + E_t[c_{\xi_{t+1}}] + az_t + 0.5E_t\big[\sigma^2_{\xi_{t+1}}\big]. \end{aligned} \tag{21}$$

Next, we compare this procedure with our log-linearization and risk-adjustment approach. We approximate $E_t[e^{z_{t+1}}]$ as if z_{t+1} is log normally distributed (note that process in equation (20) implies only conditional log normality, so our procedure is an approximation):

$$E_t\big[e^{z_{t+1}}\big] \approx e^{E_t[z_{t+1}]+0.5\,\mathrm{Var}_t[z_{t+1}]} \approx 1 + E_t[z_{t+1}] + 0.5\,\mathrm{Var}_t[z_{t+1}].$$

Then, using law of total covariance, we compute the risk adjustment term, $\mathrm{Var}_t[z_{t+1}]$:

$$\begin{aligned} \mathrm{Var}_t[z_{t+1}] &= E_t\big[\mathrm{Var}_t[z_{t+1}|\xi_{t+1}]\big] + \mathrm{Var}_t\big[E_t[z_{t+1}|\xi_{t+1}]\big] \\ &= E_t\big[\sigma^2_{\xi_{t+1}}\big] + \mathrm{Var}_t[c_{\xi_{t+1}} + az_t] \\ &= E_t\big[\sigma^2_{\xi_{t+1}}\big] + \mathrm{Var}_t[c_{\xi_{t+1}}]. \end{aligned}$$

As a result,

$$\begin{aligned} E_t\big[e^{z_{t+1}}\big] &\approx 1 + E_t[z_{t+1}] + 0.5\big(E_t\big[\sigma^2_{\xi_{t+1}}\big] + \mathrm{Var}_t[c_{\xi_{t+1}}]\big) \\ &= 1 + E_t[c_{\xi_{t+1}}] + az_t + 0.5E_t\big[\sigma^2_{\xi_{t+1}}\big] + 0.5\,\mathrm{Var}_t[c_{\xi_{t+1}}]. \end{aligned}$$

The difference with the approach described in Bansal and Zhou (2002) (see equation (21)) is the presence of the term, $0.5\,\mathrm{Var}_t[c_{\xi_{t+1}}]$. So, our log-linearization and risk- adjustment procedure takes into account the uncertainty that comes from the Markov-switching constant. If we were to disregard this term, the two solutions would be identical. The presence of this term affects the level of the risk adjustment terms, but it has very small effect on the model dynamics. To demonstrate this point, we solve the model ignoring the uncertainty that comes from the Markov-switching constant. Table C.1 reports the moments obtained from such solution and compares them to our benchmark solution. Figure C.1 plots a simulation of the model and compares it to a simulation of the model that was solved using our benchmark solution method. It is easy to see that the two methods return very similar results, especially when it comes to the model dynamics at business cycle frequencies, the focus of our paper. More generally, both methods have their pros and cons. In one case, the effects of what we call endogenous uncertainty, captured by the MS constant are lost. In the other case, conditional log normality

TABLE C.1. This table compares moments from the model, solved using our benchmark approximation method (column 2), and an approximation method, that ignores uncertainty about MS constant (column 3). The table reports volatilities of output (Δy), investment (Δi), and consumption growth (Δc); moments of inflation π, Fed fund rate r and nominal slope of the yield curve. All variables are annualized.

	Benchmark	No unc. MS const
Std(Δy)	3.19	3.19
Std(Δi)	11.53	11.59
Std(Δc)	2.66	2.66
E(π)	2.02	2.76
Std(π)	2.41	2.41
E(r)	2.09	3.32
Std(r)	3.25	3.26
Slope	0.93	1.03

only holds approximately. We decided to retain the effects of endogenous uncertainty, but it is important to verify that the approaches do not lead to very different conclusions.

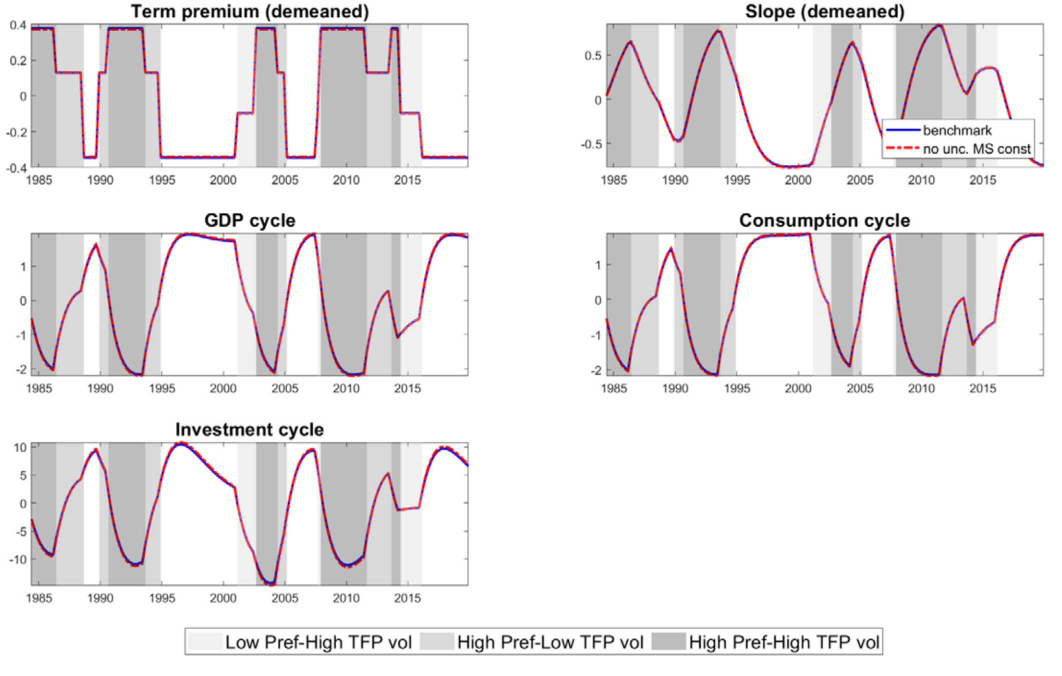

FIGURE C.1. This figure plots simulation of the model. Blue solid line corresponds to the benchmark log-linearization approach, red dotted line corresponds to the approximate solution, that ignores uncertainty about the MS constant.

TABLE D.1. This table reports the proportion of realized Den Haan–Marcet (1994) test statistics below 5% and above 95% critical values of χ^2 distribution. We simulate 20,000 economies for 3500 periods and discard the first 500 observations.

	Below 5%	Above 95%
Approximate solution	4.47%	6.58%

APPENDIX D: ACCURACY TEST

To assess the accuracy of the log-linear solution with risk adjustment employed in this paper, we conduct a Den Haan and Marcet (1994) test for the estimated model. We simulate 20,000 economies for 3500 periods and drop the first 500 observations using the posterior mode for the parameter values. We use the conditionally linear policy functions for consumption, the value function, and the nominal interest rate to compute the time path of the corresponding variables. We then use the original nonlinear Euler equation (18) to compute the realized Euler equation errors:

$$\text{err}_{t+1} = M_{t+1}\frac{P_t}{P_{t+1}}R_t + \overline{\zeta}_B e^{\widetilde{\zeta}_{B,t}}\left(\widehat{C}_t - \frac{1}{\Delta Z_t^*}h\widehat{C}_{t-1}\right) - 1,$$

where the stochastic discount factor M_{t+1} is given by equation (19) and $\widehat{C}_t = C_t/Z_t^*$. Under the null hypothesis that the approximation is exact, the Euler equation (equation (18)) implies $E_t(\text{err}_{t+1}) = 0$.

We then compute the Den Haan–Marcet statistic:

$$\text{DM} = \left[T\left(\sum_{s=1}^{T}(\text{err}_s)/T\right)^2\right] \Big/ \left[\sum_{s=1}^{T}(\text{err}_s^2)/T\right].$$

Under the null hypothesis, this statistic has a chi-squared distribution. We obtain 20,000 statistics, one for each simulated economy and we check how many of them are above the 95% and below the 5% chi-squared critical values. Table D.1 shows that the percentages of realized test statistics below 5% and above 95% critical values of a χ^2 distribution are very close to the theoretical ones. This result shows that our log-linearization approach with risk adjustment terms provides a good approximation of the model solution.

APPENDIX E: INFORMATIONAL CONTENT OF THE TERM STRUCTURE

Given the importance of demand- and supply-side uncertainty for term premia movements, the use of bond yield data in our estimation is crucial for identifying the overall effects of uncertainty and distinguishing between the two types of uncertainty. Figures E.1 and E.2 plot the impulse response functions for demand and supply uncertainty shocks from our benchmark estimation using term structure data (solid line) and an estimation without using term structure data (dashed line). Interestingly, the estimated

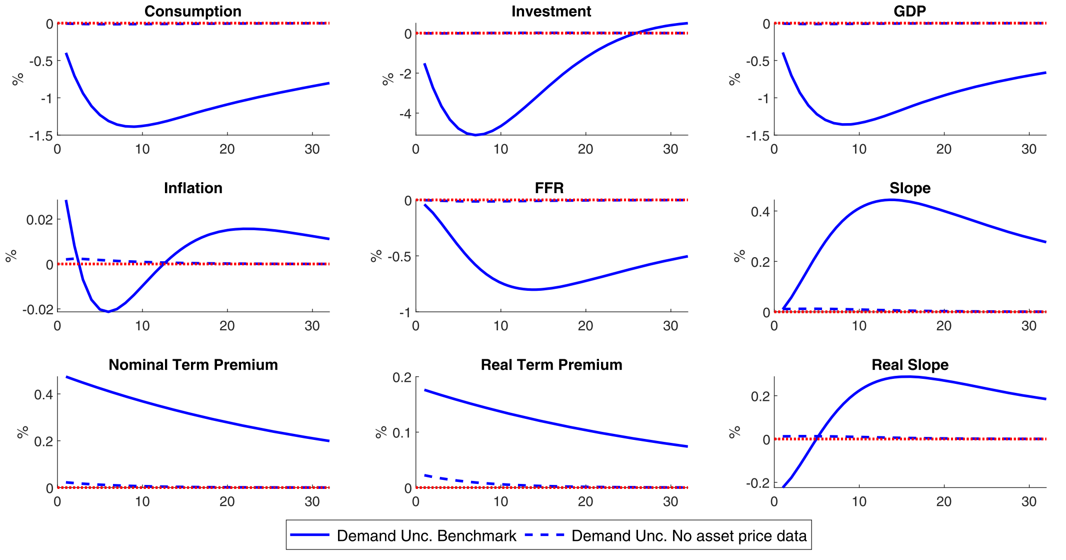

FIGURE E.1. Effects of demand-side uncertainty when removing the term structure. This figure plots the impulse responses to a demand-side uncertainty shock based on the benchmark estimation (solid line) and in an alternative estimation without the term structure (dashed line).

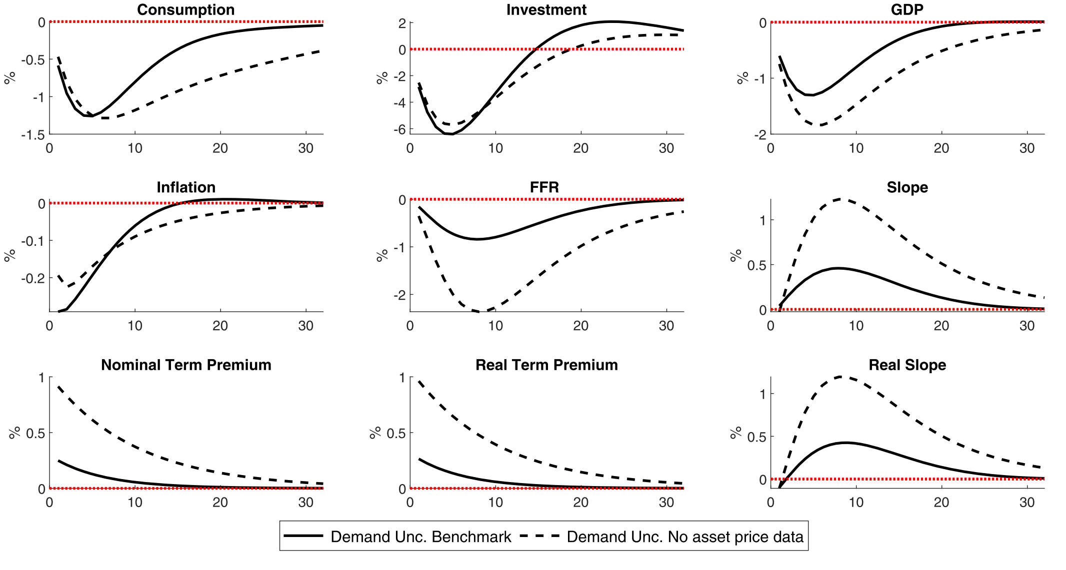

FIGURE E.2. Effects of supply-side uncertainty when removing the term structure. This figure plots the impulse responses to a supply-side uncertainty shock based on the benchmark estimation (solid line) and in an alternative estimation without the term structure (dashed line).

effects of demand-side uncertainty are significantly amplified using term structure data than without, while the effects are muted for supply-side uncertainty. As demand-side uncertainty is more important for nominal term premia compared to supply-side uncertainty, including term structure data in the estimation therefore increases the relative importance of demand-side uncertainty.

Figure E.3 illustrates that the inclusion of term structure data in the estimation affects the timing, duration, and importance of uncertainty shocks. In particular, comparing this figure with Figure 6, it is evident that using term structure data provides valuable information for the role of uncertainty in explaining business cycle fluctuations. When the term structure is not included, periods of high uncertainty have a shorter du-

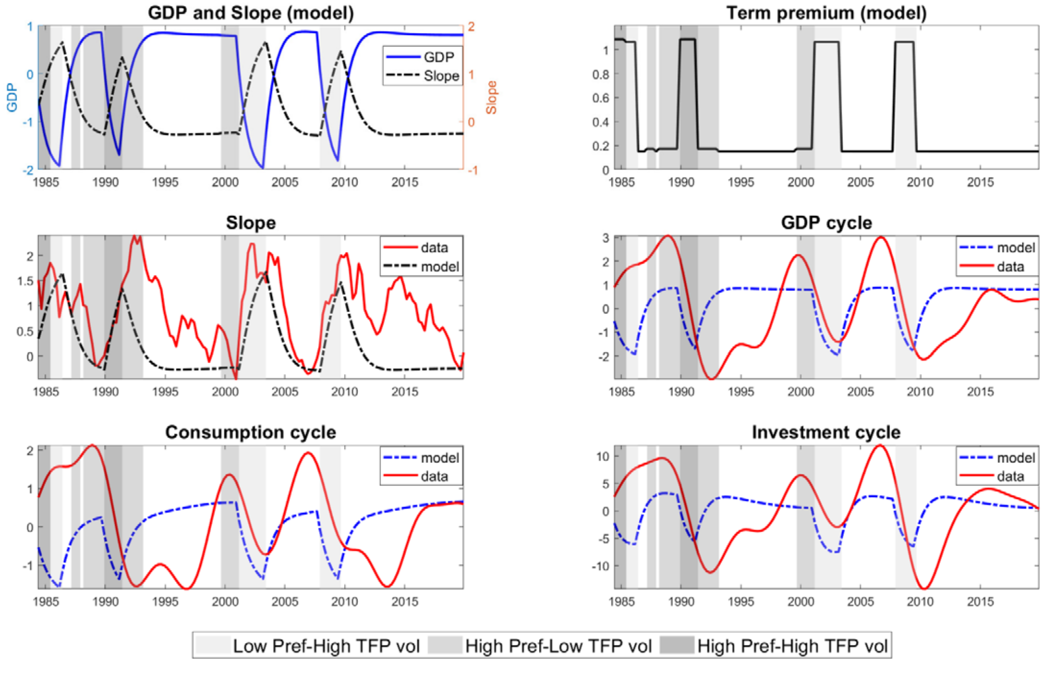

FIGURE E.3. Uncertainty-driven fluctuations in an estimated model without the term structure. The figure plots selected variables from the simulation of the model estimated without asset pricing data. The simulation only considers the effects of uncertainty based on the estimated regime sequence (all Gaussian shocks are set to zero in this simulation). Top left panel: simulated path of GDP, expressed in log deviations from steady state, and slope of the yield curve, expressed as a difference between 5-year yield and 1-year yield. Top right panel: simulated dynamic of nominal term premium in the model, expressed as a difference between 5-year nominal yield and an expected average yield on 1-quarter nominal bond over the next 20 quarters. Middle left panel: simulated slope of the yield curve and slope of the yield curve observed in the data. The subsequent panels plot the model-implied path of GDP, consumption, and investment in response to changes in uncertainty and the cyclical components of the corresponding series in the data (obtained using bandpass filter). Units on the y-axis for macro variables are percentage points (model and data). Units on the y-axis for term premium and slope are annualized percent (data and model).

TABLE E.1. This table reports results from the model estimated without using asset price data. The left panel reports nominal and real term premia conditional on the uncertainty regime. The term premium in the model is computed as the difference between 5-year yield and the expected average yield on the 1-quarter bond over the next 20 quarters. The right panel reports the unconditional slopes of the corresponding term structures, defined as the difference between yields on 5-year and 1-quarter bonds. Values are annualized percent.

	Term Premia				
Preference Unc.	Low	Low	High	High	
TFP growth Unc.	Low	High	Low	High	Average Slope
Nominal Term Premium	0.15	1.06	0.17	1.09	0.46
Real Term Premium	0.04	1.01	0.07	1.03	0.33

ration and produce smaller effects. Furthermore, in the 1991 recession there is no visible effect from the increase in demand-side uncertainty, consistent with the impulse responses from Figure E.1. Overall, when the term structure is not included, liquidity shocks become more important for explaining business cycle fluctuations as they account for around 18% and 4% of investment and consumption volatility, respectively, compared to 2.36% and 0.79% in the benchmark estimation. On the other hand, the estimation excluding the term structure also implies a counterfactual yield curve, as the unconditional nominal slope is only 0.46%, with most of the nominal spread coming from the real curve (see Table E.1). This result is due to the fact that demand shocks are a key source of inflation risk premia, but when term structure data is excluded, the role of demand shocks is significantly reduced as illustrated in Figure E.1. Thus, the term structure encodes important information about uncertainty and macroeconomic fluctuations while disciplining the relative importance of liquidity shocks. The joint estimation exploits the strong relation between the slope of the yield curve, business cycle fluctuations, and uncertainty.

Appendix F: Effects of government spending uncertainty

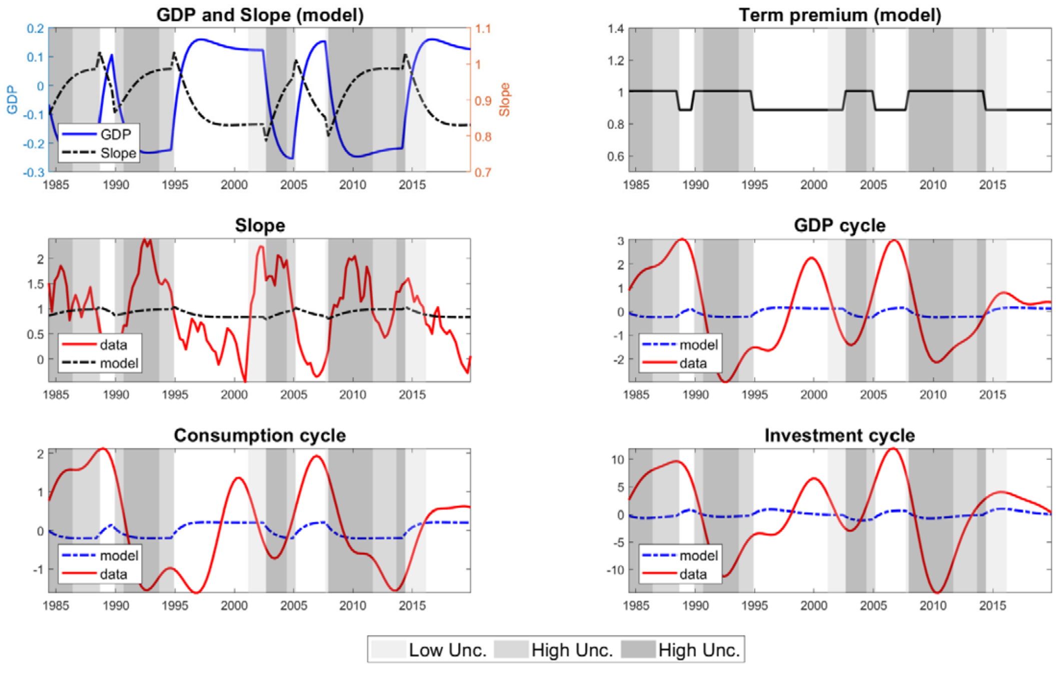

Figure F.1. The figure plots selected variables from the counterfactual simulation of the model with government spending uncertainty shocks (all Gaussian shocks, preference uncertainty shocks, and TFP uncertainty shocks are set to zero in this simulation). The government spending uncertainty in this simulation follows the estimated regime sequence for the preference uncertainty. In the low uncertainty regime, the volatility of government spending shock is assumed to be equal to its unconditional value obtain in the estimation of the model. In the high uncertainty regime, the volatility of government spending shock is assumed to be two times higher than the estimated value. Top left panel: simulated path of GDP, expressed in log deviations from steady state, and slope of the yield curve, expressed as a difference between 5-year yield and 1-year yield. Top right panel: simulated dynamic of nominal term premium in the model, expressed as a difference between 5-year nominal yield and an expected average yield on 1-quarter nominal bond over the next 20 quarters. Middle left panel: simulated slope of the yield curve and slope of the yield curve observed in the data. The subsequent panels plot the model-implied path of GDP, consumption, and investment in response to changes in uncertainty and the cyclical components of the corresponding series in the data (obtained using bandpass filter). Units on the y-axis for macro variables are percentage points (model and data). Units on the y-axis for term premium and slope are annualized percent (data and model).

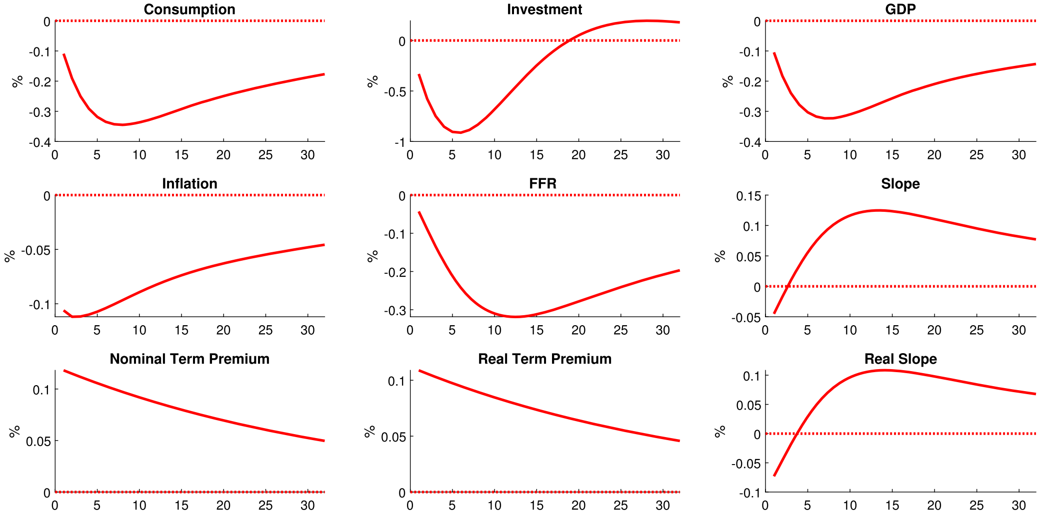

FIGURE F.2. Responses to government spending uncertainty shock. This figure plots impulse responses to a two-times increase in the volatility of government spending shocks. The government spending uncertainty is assumed to follow the estimated regime sequence for the preference uncertainty. The gray areas represent 90% credible sets. The impulse responses are computed as the change in the expected path of the corresponding variables when the volatility regime changes. The figure plots impulse responses of consumption, investment, GDP, inflation, Fed funds rate (1-quarter nominal interest rate), the slope of the yield curve expressed as the difference between 5-year and 1-year nominal yields, nominal term premium defined as the difference between 5-year nominal yield and an expected average yield on 1-quarter nominal bond over the next 20 quarters, the real term premium defined as the difference between 5-year real yield and an expected average yield on 1-quarter real bond over the next 20 quarters, the real slope expressed as the difference between 5-year and 1-year real yields. The units of the y-axis are percentage deviations from a steady state (values for inflation, interest rates, and term premia are annualized). Units on the x-axis are quarters.

Appendix G: Summary statistics

Table G.1. Mean and standard deviation of macroeconomic variables and bond yields in the model and in the data. The model moments are computed analytically from the model solution. Values are annualized percentages. Reported standard deviations for the model take into account observational errors.

	Model		Data	
	Mean	Std. Dev.	Mean	Std. Dev.
GDP	0.53	3.33	1.08	2.29
Inflation	2.02	2.42	2.62	1.63
FFR	2.09	3.25	3.73	2.93
Investment	0.53	12.16	0.82	6.48
Consumption	0.53	2.91	1.42	1.55
Price of investment	−2.10	1.62	−2.03	1.78
1-year yield	2.18	3.13	3.82	2.84
2-year yield	2.37	2.99	4.10	2.88
3-year yield	2.60	2.88	4.34	2.85
4-year yield	2.82	2.78	4.57	2.81
5-year yield	3.02	2.69	4.73	2.74

Table G.2. Mean and standard deviation of macroeconomic variables and bond yields in the model and in the data. The model moments are computed analytically from the model solution. Values are annualized percentages. Reported standard deviations for the model do not take into account observational errors.

	Model		Data	
	Mean	Std. Dev.	Mean	Std. Dev.
GDP	0.53	3.19	1.08	2.29
Inflation	2.02	2.41	2.62	1.63
FFR	2.09	3.25	3.73	2.93
Investment	0.53	11.53	0.82	6.48
Consumption	0.53	2.66	1.42	1.55
Price of investment	−2.10	1.62	−2.03	1.78
1-year yield	2.18	3.13	3.82	2.84
2-year yield	2.37	2.99	4.10	2.88
3-year yield	2.60	2.88	4.34	2.85
4-year yield	2.82	2.78	4.57	2.81
5-year yield	3.02	2.69	4.73	2.74

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