Quantitative Economics 2 (2011), 455–491 1759-7331/20110455

Dating the timeline of financial bubbles during the subprime crisis

Peter C. B. Phillips
Yale University, University of Auckland, University of Southampton, and Singapore Management University

Jun Yu
Singapore Management University

A new recursive regression methodology is introduced to analyze the bubble characteristics of various financial time series during the subprime crisis. The methods modify a technique proposed in Phillips, Wu, and Yu (2011) and provide a technology for identifying bubble behavior with consistent dating of their origination and collapse. The tests serve as an early warning diagnostic of bubble activity and a new procedure is introduced for testing bubble migration across markets. Three relevant financial series are investigated, including a financial asset price (a house price index), a commodity price (the crude oil price), and one bond price (the spread between Baa and Aaa). Statistically significant bubble characteristics are found in all of these series. The empirical estimates of the origination and collapse dates suggest a migration mechanism among the financial variables. A bubble emerged in the real estate market in February 2002. After the subprime crisis erupted in 2007, the phenomenon migrated selectively into the commodity market and the bond market, creating bubbles which subsequently burst at the end of 2008, just as the effects on the real economy and economic growth became manifest. Our empirical estimates of the origination and collapse dates and tests of migration across markets match well with the general dateline of the crisis put forward in the recent study by Caballero, Farhi, and Gourinchas (2008a).

Keywords. Financial bubbles, crashes, date stamping, explosive behavior, migration, mildly explosive process, subprime crisis, timeline.

JEL classification. C15, G01, G12.

Peter C. B. Phillips: peter.phillips@yale.edu
Jun Yu: yujun@smu.edu.sg
Phillips acknowledges support from the NSF under Grants SES 06-47086 and SES 09-56687. Yu acknowledges support from the Singapore Ministry of Education AcRF Tier 2 fund under Grant T206B4301-RS. We wish to thank a co-editor, three anonymous referees, seminar participants in many universities, the 2010 International Symposium on Econometric Theory and Applications held at Singapore Management University, the World Congress of the Econometric Society in Shanghai, the second Singapore Conference on Quantitative Finance at Saw Centre for Quantitative Finance, the Workshop on Econometric and Financial Studies After Crisis at Academia Sinica, the 2011 Shanghai Econometrics Workshop, and the Frontiers in Financial Econometrics Workshop for helpful comments.

DOI: 10.3982/QE82

> There is a very real danger, fellow citizens, that the Icelandic economy in the worst case could be sucked into the whirlpool, and the result could be national bankruptcy (Prime Minister Geir Haarde, televised address to Icelandic Nation, October 8, 2008).

> Between 40 and 45 percent of the world's wealth has been destroyed in little less than a year and a half (Stephen Schwarzman, March 11, 2009).

> Federal Reserve policymakers should deepen their understanding about how to combat speculative bubbles to reduce the chances of another financial crisis (Donald Kohn, Federal Reserve Board Vice Chairman, March 24, 2010).

1. Introduction

Financial bubbles have been a longstanding topic of interest for economists, involving both theorists and empirical researchers. Some of the main issues have focused on mechanisms for modeling bubbles, reconciling bubble-like behavior in the context of rational expectations of future earnings, mechanisms for detecting bubbles and measuring their extent, exploring causes and the psychology of investor behavior, and considering suitable policy responses. While there is general agreement that financial bubbles give rise to misallocation of resources and can have serious effects on real economic activity, as yet there has been little consensus among economists and policy makers on how to address the many issues raised above.

The global financial turmoil over 2008–2009, triggered by the subprime crisis in the United States and its subsequent effect on commodity markets, exchange rates, and real economic activity, has led to renewed interest among economists in financial bubbles and their potential global consequences. There is now widespread recognition among policy makers as well as economists that changes in the global economy over the last decade, far from decoupling economic activity as was earlier believed, have led to powerful latent financial linkages that have increased risks in the event of a large common shock. The magnitude of the crisis is so large, the mechanism so complex, and the consequences so important to the real economy that understanding the phenomenon, exploring its causes, and mapping its evolution have presented major challenges to the economics profession. As the quotations that preface this article indicate, a substantial percentage of the world's accrued wealth was destroyed within 18 months of the subprime crisis, with manifold effects ranging from the collapse of major financial institutions to the near bankruptcy of national economies. There is also recognition that new empirical methods are needed to improve understanding of speculative phenomena and to provide early warning diagnostics of financial bubbles.

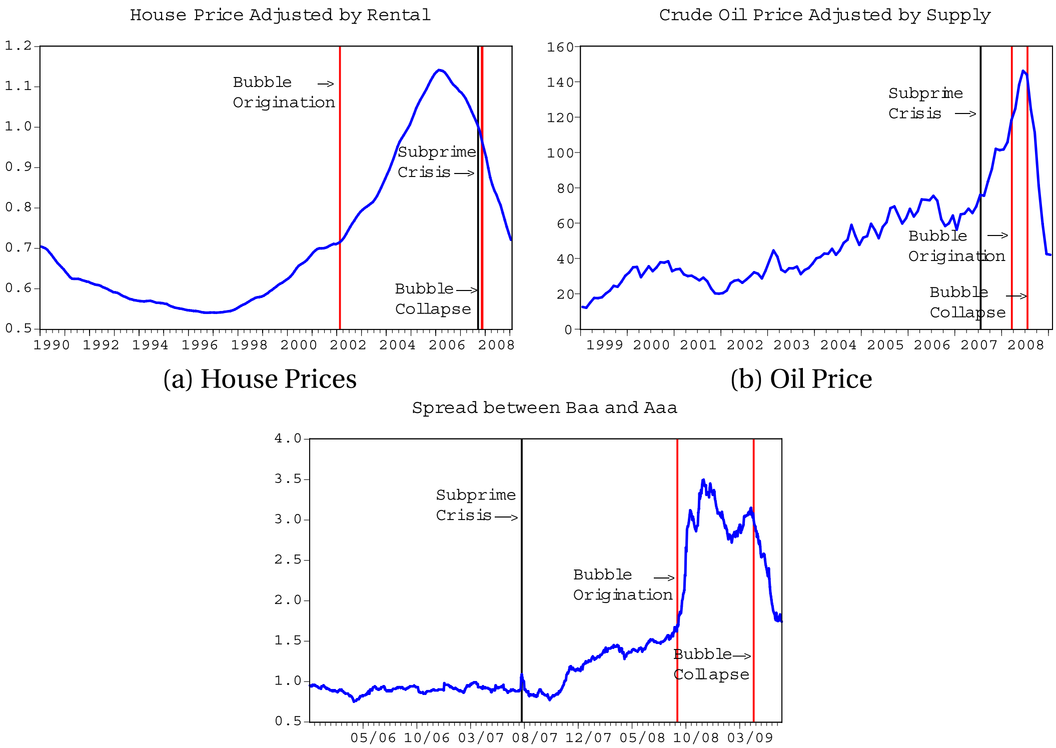
The recent background of financial exuberance and collapse with concatenating effects across markets and nations provides a rich new environment for empirical research. The most urgent ongoing questions relate to matters of fiscal, monetary, and regulatory policies for securing financial stability and buttressing real economic activity. Beyond these immediate policy issues are underlying questions relating to the emergence of the phenomenon and its evolutionary course through the financial and economic systems. It is these latter issues that form the focus of interest of the present paper.

The subprime crisis is not an isolated empirical event. In a recent article, Caballero, Farhi, and Gourinchas (CFG) (2008a) argued that the Internet bubble in the 1990s, the asset bubbles over 2005–2006, the subprime crisis in 2007, and the commodity bubbles of 2008 are all closely related. CFG go further and put forward a sequential hypothesis concerning bubble creation and collapse that accounts for the course of the financial turmoil in the U.S. economy using a simple general equilibrium model without monetary factors, but with goods that may be partially securitized. Date-stamping the timeline of the origination and collapse of the various bubbles is a critical element in the validity of this sequential hypothesis. Empirical evaluation further requires some econometric technology for testing the presence of bubble migration across markets.

The present paper uses econometric methodology to test if and when bubbles emerged and collapsed in the real estate market, the commodity market, and the bond market over the period surrounding the subprime crisis. New econometric methods are introduced for testing bubble migration across markets. Several series are studied. In particular, we investigate the bubble characteristics in the U.S. house price index from January 1990 to January 2009, the price of crude oil from January 1999 to January 2009, and the spread between Baa and Aaa bond rates from January 3, 2006 to July 2, 2009. Figure 1 shows time series plots of the three series. Our methods enable us to determine whether a bubble emerged in each series, to date-stamp the origination in that event, and correspondingly to assess whether the bubble collapsed and the date of that collapse. The empirical date stamps so determined are then matched against the hypothesized sequence of events described in the model of CFG.

The econometric methods used here are closely related to those proposed in Phillips, Wu, and Yu (2011; PWY hereafter). In particular, the methods rely on forward recursive regressions coupled with sequential right-sided unit root tests. The sequential tests assess period by period evidence for unit root behavior against mildly explosive alternatives. Mildly explosive behavior can be modeled by an autoregressive process with a root (ρ) that exceeds unity but that is still in the general vicinity of unity. Phillips and Magdalinos (PM) (2007a, 2007b) showed that this "mildly explosive" vicinity of unity can be successfully modeled in terms of deviations of the form $\rho - 1 = c/k_n > 0$, where c is a positive constant and k_n is a sequence that passes to infinity with, but more slowly than, the sample size n, so that $\rho \to 1$. These processes therefore involve only mild departures from strict (rational) martingale behavior in markets. They include submartingale processes of the type that have been used to model rational bubble behavior in finance (Evans (1991), Campbell, Lo, and McKinley (1997)). PM (2007a, 2007b) have investigated this class of process, developed a large sample asymptotic theory, and shown that these models are amenable to econometric inference, unlike purely explosive processes for which no central limit theory is applicable.

PWY applied forward recursive regression methods to Nasdaq stock prices during the 1990s, and using sequential tests against mildly explosive alternatives were able to date-stamp the origination of financial exuberance in the Nasdaq market to mid-1995, prior to the famous remark of Alan Greenspan in December 1996 about irrational exuberance in financial markets. This test therefore revealed that there was anticipatory empirical evidence supporting mildly explosive behavior in stock prices over a year prior



(c) Bond Price

FIGURE 1. Time series plots of real prices for three financial assets: (a) monthly observations of the house price index from January 1990 to January 2009 adjusted by rental; (b) monthly observations of crude oil prices adjusted by supply; and (c) daily observations of the spread between Baa bond rates and Aaa bond rates from January 3, 2006 to July 2, 2009. The estimated bubble origination and collapse dates are also shown on the figures.

to Greenspan's remarks. In ongoing work, Phillips and Yu (2009) and Phillips, Shi, and Yu (2011) developed a limit theory for this date stamping technology, explored multiple bubble detection, and checked the finite sample capability of the procedure to identify and date bubble behavior. The date stamp estimators were shown to be consistent for the origination and collapse of bubble behavior and the dating mechanism was shown to work well in finite samples.

We use this methodology to explore the sequential pattern of events of the current financial crisis. Dating helps to characterize the phenomenon by identifying the individual events and by fixing their extent and sequencing. It may be viewed as a first step in understanding the phenomenon and in searching for causes of the behavioral changes involved in bubble origination and collapse. Date stamping in conjunction with migration analysis assists in evaluating hypotheses about the concatenation of bubble activity over time and across markets, such as those developed by CFG. The forward recursive regression approach used here enables early identification of the appearance of mildly explosive behavior in asset prices, thereby providing anticipatory evidence of a (local) move away from martingale behavior. This evidence can be used as an early warning diagnostic of (financial) exuberance, and thereby can assist policy makers in surveillance and regulatory actions, as urged by Fed Vice Chairman Donald Kohn in one of the opening quotes of this article. Similarly, the approach helps to identify a subsequent switch back to martingale behavior as explosive sentiment collapses.

Empirical evidence of emergent mildly explosive behavior is found in all of the time series studied here, and in all of them that manifest mildly explosive behavior, there is further evidence of subsequent collapse. Figure 1 shows the origination and collapse dates for the bubbles identified in the three financial time series mentioned earlier. For the real estate market, the bubbles emerged prior to the subprime crisis. For the other series, the bubbles all emerged after the subprime crisis. These findings reveal a sequence of mildly explosive events, each followed by a financial collapse that corroborates the sequential hypothesis given by CFG. Consideration of a wider group of related financial series following the eruption of the subprime crisis indicates that bubbles of the type found in the series in Figure 1 are not always evident in other commodities. Accordingly, the empirical evidence supports a selective migration of the bubble activity through financial markets as the subprime crisis evolved and liquid funds searched for safe havens.

The present paper differs from PWY in three aspects. The first difference involves the treatment of initialization. In PWY, the initial condition is fixed to be the first observation in the full sample, whereas in this paper, the initial observation is selected based on an information criterion. The use of information criteria in the selection of the initial observation allows for sharper identification of the bubble origination date. As a result, when a long series is available, the new method may not necessarily use all the observations to identify the most recent bubble episode. Second, in this paper, a method for testing bubble migration is developed, a limit theory for the new procedure is obtained, and the migration test is implemented in the empirical application. Finally, the empirical focus of this paper is the subprime crisis and some of the events unfolding over the period 2002–2009.

The plan of the paper is as follows. Section 2 reviews the econometric methodology for dating bubble characteristics, discusses rational bubble and variable discount rate sources of financial exuberance, outlines some of the relevant facts concerning the subprime crisis, and relates the timeline implications of the theoretical results obtained in CFG (2008a). Section 3 describes the data that are used in the present empirical study. Section 4 presents the empirical findings and matches the estimates to the theory of CFG (2008a). Section 5 concludes. New limit theory for the migration test is developed in the Appendix.

2. Bubbles, the subprime crisis, econometric dating, and bubble migration

2.1 *Bubbles and crashes*

In the popular press, the term "financial bubble" refers to a situation where the price of a financial asset rapidly increases and does so in a speculative manner that is distinct from what is considered to be the asset's intrinsic value. The term carries the innuendo that the increase is not justified by economic fundamentals and that there is, accordingly, risk of a subsequent collapse in which the asset price falls precipitously. In such cases, the bubble phenomenon is typically confirmed in retrospect.

A common definition that makes this usage precise is that bubble conditions arise when the price of an asset significantly exceeds the fundamental value that is determined by the discounted expected value of the cash flows that ownership of the asset can generate. However, discount rates may be variable and, as demonstrated below, the time profile of the discount rate can have important effects on the characteristics of the fundamental price and may even propagate explosive price behavior.

An important secondary characteristic of the bubble phenomenon is that during both the run-up and run-down periods, the asset is subject to high volume trading in which the direction of change is widely anticipated (and relied upon), as distinct from normal market conditions in which the asset price follows a near martingale. It is this deviation from martingale behavior that provides a mechanism for identifying both the emergence of the boom phase of a bubble behavior and its subsequent crash.

This distinction is recognized in the rational bubble literature, which characterizes the boom phase of a bubble in terms of explosive dynamics or submartingale behavior. This property contrasts with the efficient market martingale property, which implies unit root time series dynamic behavior. To explain the difference in terms of the commonly used present value model, let P_t be the stock price at time t before the dividend payout, let D_t be the dividend payoff from the asset at time t, and let r be the discount rate ($r > 0$). The standard no arbitrage condition implies that

$$P_t = \frac{1}{1+r} E_t(P_{t+1} + D_{t+1}) \tag{1}$$

and recursive substitution yields

$$P_t = F_t + B_t, \tag{2}$$

where $F_t = \sum_{i=1}^{\infty} (1+r)^{-i} E_t(D_{t+i})$ and

$$E_t(B_{t+1}) = (1+r)B_t. \tag{3}$$

Hence, the asset price is decomposed into two components: a "fundamental" component, F_t, that is determined by expected future dividends, and a supplementary solution that corresponds to the "bubble" component, B_t.[1] In the absence of bubble conditions, $P_t = F_t$. Otherwise, $P_t = F_t + B_t$ and price embodies the explosive component B_t, which satisfies the submartingale property (3). Consequently, under bubble conditions, P_t will manifest the explosive behavior inherent in B_t. This explosive property is very different from the random wandering (or unit root) behavior that is present in F_t when D_t is a martingale and that is commonly found for asset prices in the empirical literature.

Over long periods of time, some asset prices like equities also tend to manifest empirical evidence of a drift component. Unit root time series with a drift can generate

[1] Extensions of the framework (1)–(3) to log linear approximations such as those in Campbell and Shiller (1988) and the validity of these approximations are considered in Lee and Phillips (2011).

periods of run-up if the variance of the martingale component is small and the drift is strong enough. But accumulated gains in such cases are at most of $O(n)$ for sample size n. In practice, of course, the drift component is usually small and is generally negligible over short periods, so the unit root behavior is the dominant characteristic and clear evidence of gains only shows up over long horizons. On the other hand, the run-up rate in an explosive process is $O((1+r)^n)$ for some $r > 0$, as in (3), and is therefore much greater. This difference between linear and exponential growth combined with the nonlinear curvature in an explosive process are testable properties that distinguish the two processes. In terms of model (1) and its solution (2), both B_t and P_t increase rapidly during the boom phase of the bubble according to $E_t(B_{t+h}) = (1+r)^h B_t$ and the initialization $B_0 > 0$. But when the bubble conditions collapse and the particular solution disappears, then $P_t = F_t$, which corresponds to a sudden collapse in the asset price. If the dividend process D_t follows a martingale, reflecting market conditions that generate cash flows, then F_t is similarly a martingale and is cointegrated with D_t. Under such conditions, the presence of an additional "rational bubble" submartingale component B_t in P_t can account for an explosive run-up in the asset price P_t.

Importantly, making the discount factor r_t either stationary or integrated of order 1 does not change our analysis qualitatively because the implications for the statistical properties of F_t, B_t, and P_t are the same as with the constant r. For example, if r_t is stationary, (3) becomes

$$E_t(B_{t+1}) = (1+r_t)B_t. \tag{4}$$

Then if (3) is fitted, $r = (\prod_{t=1}^{T}(1+r_t))^{1/T} > 1$, implying an explosive process for B_t and hence P_t, even if F_t itself is not explosive.

2.2 *The effects of a time varying discount rate*

This paper interprets explosiveness in price as sufficient evidence for bubbles and this interpretation holds true under a variety of assumptions on the discount rate. As indicated above, certain time profiles for the discount rate can have an important effect on the characteristics of the fundamental price. The present section illustrates this possibility by developing a simple propagating mechanism for explosive behavior in the fundamental price under a time varying discount rate.

If dividends grow at a constant rate r_D with $r_D < r$ in (1),[2] the fundamental value of the stock price is

$$F_t = \frac{D_t}{r - r_D}. \tag{5}$$

This is the well known Gordon growth model. It is evident that in this case the fundamental value can be very sensitive to changes in r when r is close to r_D. In fact, the fundamental value diverges as $r \searrow r_D$, so that a price run-up is evidently possible under

[2]This assumption obviously violates the assumption we adopted earlier, namely, constancy, stationarity, or integration of order 1.

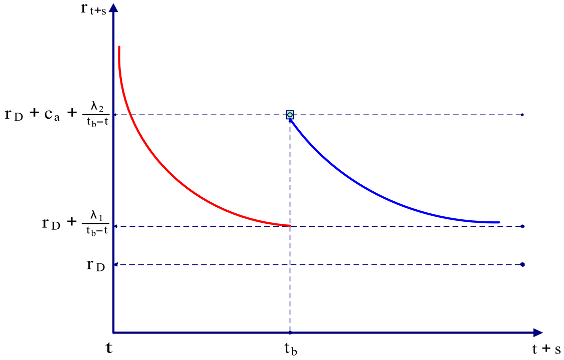


FIGURE 2. Time path of the discount rate r_{t+s} in (9).

certain time profiles for the discount rate. This simple Gordon model reveals the potential impact of a time varying discount rate, but it provides no price dynamics. The following argument provides an analytic formulation that shows how an explosive time path in fundamental values can be generated by time variation in the discount rate.

Consider a continuous time version of (5) with time varying discount rate r_t, namely

$$F_t = \int_0^\infty \exp(-sr_{t+s}) E_t D_{t+s}\, ds. \tag{6}$$

Suppose dividends have a constant expected growth rate r_D such that

$$E_t D_{t+s} = \exp(r_D s) D_t. \tag{7}$$

Then D_t is a martingale when $r_D = 0$. Combining (6) and (7) yield

$$F_t = \int_0^\infty \exp(-s(r_{t+s} - r_D)) D_t\, ds. \tag{8}$$

Given some fixed time point t_b, constants $c_a > 0$, and $\lambda_1 > \lambda_2 > 0$, let the time profile of the discount rate r_{t+s} for $t \in (0, t_b]$ be

$$r_{t+s} = \begin{cases} r_D + \dfrac{t_b - t - s}{s} c_a + \dfrac{\lambda_1}{s}, & \text{for } 0 \leq s < t_b - t, \\ r_D + c_a + \dfrac{\lambda_2}{s}, & \text{for } s \geq t_b - t. \end{cases} \tag{9}$$

Then the discount rate decreases toward some level $r_D + \frac{\lambda_1}{t_b - t}$ as $t + s \nearrow t_b$ and jumps to the level $r_D + c_a + \frac{\lambda_2}{t_b - t}$ immediately thereafter, as shown in Figure 2. Thus, the time profile of the discount factor has a structural break at t_b in which a higher rate of discounting occurs at t_b. The break itself widens asymptotically as $t \nearrow t_b$.

We then have

$$F_t / D_t = \int_0^\infty \exp(-s(r_{t+s} - r_D))\, ds$$

$$
\begin{aligned}
&= \int_0^{t_b-t} \exp(-c_a(t_b-t-s)-\lambda_1)\,ds + \int_{t_b-t}^{\infty} \exp(-c_a s-\lambda_2)\,ds \\
&= e^{-\lambda_1}\left[\frac{e^{-c_a(t_b-t-s)}}{c_a}\right]_0^{t_b-t} + e^{-\lambda_2}\left[\frac{e^{-c_a s}}{-c_a}\right]_{t_b-t}^{\infty} \\
&= \frac{e^{-\lambda_1}}{c_a}\left[1-e^{-c_a(t_b-t)}\right] + \frac{e^{-\lambda_2}}{c_a}e^{-c_a(t_b-t)} \\
&= \frac{e^{-\lambda_1}}{c_a} + \frac{(e^{-\lambda_2}-e^{-\lambda_1})}{c_a}e^{-c_a(t_b-t)} := \sigma_t
\end{aligned}
$$

and the time path of F_t/D_t is explosive over $t \in (0, t_b]$. Over this interval, F_t evolves according to the differential equation

$$dF_t = (e^{-\lambda_2}-e^{-\lambda_1})e^{-c_a(t_b-t)}D_t\,dt + \sigma_t\,dD_t.$$

Since $c_a F_t/D_t = e^{-\lambda_1} + (e^{-\lambda_2}-e^{-\lambda_1})e^{-c_a(t_b-t)}$, we have

$$dF_t = \frac{(e^{-\lambda_2}-e^{-\lambda_1})e^{-c_a(t_b-t)}}{e^{-\lambda_1}+(e^{-\lambda_2}-e^{-\lambda_1})e^{-c_a(t_b-t)}}c_a F_t\,dt + \sigma_t\,dD_t \quad \text{for } t \in (0, t_b].$$

For t close to t_b, the generating mechanism for F_t is approximately

$$
\begin{aligned}
dF_t &= \frac{(e^{-\lambda_2}-e^{-\lambda_1})}{e^{-\lambda_1}+(e^{-\lambda_2}-e^{-\lambda_1})}c_a F_t\,dt + \sigma_t\,dD_t \\
&= \{1-e^{-(\lambda_1-\lambda_2)}\}c_a F_t\,dt + \sigma_t\,dD_t,
\end{aligned}
$$

which is an explosive diffusion because

$$c_b = \{1-e^{-(\lambda_1-\lambda_2)}\}c_a > 0,$$

since $c_a > 0$ and $e^{-(\lambda_1-\lambda_2)} < 1$. The discrete time path of F_t in this neighborhood is therefore propagated by an explosive autoregressive process with coefficient $\rho = e^{c_b} > 1$.

The heuristic explanation of this behavior is as follows. As $t \nearrow t_b$ there is growing anticipation that the discount factor will soon increase. Under such conditions, investors anticipate the present to become more important in valuing assets. This anticipation in turn leads to an inflation of current valuations and price fundamentals F_t become explosive as this process continues.

On the other hand, for $t > t_b$, we have

$$r_{t+s} = r_D + c_a + \frac{\lambda_2}{s} \quad \text{for } s > 0$$

and then

$$
\begin{aligned}
\frac{F_t}{D_t} &= \int_0^{\infty} \exp(-s(r_{t+s}-r_D))\,ds \\
&= \int_0^{\infty} \exp(-c_a s-\lambda_2)\,ds
\end{aligned}
$$

$$= e^{-\lambda_2}\left[\frac{e^{-c_a s}}{-c_a}\right]_0^\infty = \frac{e^{-\lambda_2}}{c_a}.$$

So $F_t = \frac{e^{-\lambda_2}}{c_a} D_t$ for $t > t_b$ and price fundamentals are collinear with D_t. When D_t is a Brownian motion or an integrated process in discrete time, F_t and D_t are cointegrated. Thus, after time t_b, price fundamentals comove with D_t.

It follows that the time profile (9) for the discount rate r_t induces a subinterval of explosive behavior in F_t before t_b. In this deterministic setting, it is known as time t_b approaches that there will be an upward shift in the discount factor that makes present valuations more important. A more realistic model might allow for uncertainty in this time profile and for a stochastic trajectory for r_t that accommodates potential upward shifts of this type.

Econometric dating procedures of the type described below may be used to assess evidence for subperiods of explosive price behavior that are induced by such time variation in the discount factor, just as for other potential sources of financial exuberance.

2.3 *Subprime crisis and event timeline*

The subprime mortgage crisis is generally regarded as an important triggering element in the ongoing global financial crisis. The subprime event began with a dramatic rise in mortgage delinquencies and foreclosures that started in late 2006 in the United States, as easy initial adjustment rate mortgage terms began to expire and refinancing became more difficult at the same time that house prices were falling. The event had wider and, soon, global consequences because of the huge scale of mortgage backed securities (MBS) in the financial system, extending the impact of mortgage failure to the asset positions of investment and commercial banks. The crisis became apparent in the last week of July 2007 when German bank regulators and government officials organized a $5 billion bailout of IKB, a small bank in Germany. We may therefore treat the beginning of August 2007 as the public onset date of the subprime crisis, although the realities in terms of rising mortgage delinquencies commenced earlier.

Much has already been written about the causes of this crisis and a host of factors have been suggested, including poor appreciation of the risks associated with MBS, weak underwriting standards and risk assessment practices in general, increasingly complex financial products, high levels of financial leverage with associated vulnerabilities, shortfalls in understanding the impact of large common shocks on the financial system, and inadequate monitoring by policy makers and regulators of the accumulating risk exposure in the financial markets. We refer readers to Brunnermeier (2009), Greenlaw, Hatzius, Kashyap, and Shin (2008), and Hull (2008) for detailed discussions of the subprime crisis and its manifold implications. The concern of the present work is the crisis timeline and, more specifically, the issues of empirically dating the origination and collapse of the various financial bubbles that occurred as the crisis events unfolded.

Prior to the subprime crisis and following the collapse in dot.com stocks in 2000–2001, the housing market in many states of the United States sustained rapid increases in valuations fueled by a period of low interest rates, large foreign capital inflows, and high-risk lending practices of financial institutions. In the resulting boom, home ownership in the United States increased to 69.2% in 2004 from 64% in 1994 (Callis and Cavanaugh (2007)) and nominal house prices increased by more than 180% over the period 1997–2006. Household debt, as a percentage of disposable income, increased from 77% to 127% over the period 1990–2007 (*Economist*, November 22, 2008). At the same time, the MBS market, derived from residential mortgages, mushroomed, and major banks and financial institutions around the world invested in securities that were ultimately founded on the U.S. housing market. For example, the nominal outstanding amount of asset backed commercial paper (ABCP) increased by more than 80% over the period July 2004 to July 2007.

The concatenation of events that occurred after the housing market peaked in 2005 and went into decline, followed by the subprime mortgage crisis and subsequent repercussions on financial institutions over 2007–2008 and finally the impact on world trade and real economic activity, is now well known. Securities backed by subprime mortgages lost most of their value, investors lost confidence, and liquidity dried up as money flowed to assets which appeared to have inherently lower risk, such as Treasury bonds, and to other assets like commodities and currencies such as the U.S. dollar and the Japanese yen (mainly through the unwinding of the carry trade industry), generating a so-called flight to quality. In consequence, commodity prices soared. As the crisis deepened, stock markets around the world fell, and commercial banks, mortgage lenders, and insurance companies failed. Consumption and investment expenditures dropped, many Organization for Economic Cooperation and Development (OECD) economies went into serious recession, export driven economies in Asia sustained double digit percentage declines in exports, growth slowed significantly in China, and world trade declined. Concomitant with these real economic effects, global demand for commodities declined and commodity prices fell.

In a recent study, CFG (2008a) proposed a model which seeks to explain the main features of this sequence of complex interlinked financial crises. The CFG model links together global financial asset scarcity, global imbalances, the real estate bubble, the subprime crisis, and the commodity bubble in a general equilibrium macroeconomic environment without monetary factors. The model is based on CFG (2008b); it assumes that the economy has two countries (U and M) and features two goods (X and Z). A key part of the CFG framework is a sequence of hypotheses involving successive bubble creations and collapses, which we briefly review as follows.

Country U is interpreted as the United States and country M as the emerging market economies and commodity producers. Good X is a nonstorable good, a fraction of which can be capitalized, and is produced by both countries. Good Z is a storable commodity and is produced only by country M. A presumption in the model is that there exists a global imbalance at period t_0. The imbalance can be interpreted as arising from continuing capital flows from emerging markets to the United States as the United States runs a growing trade deficit with emergent economies, which in turn rely more heavily on export driven growth.

To allow country U to have both a large current account deficit and low interest rates, a fundamental assumption that CFG made is that a bubble developed initially in country U. In practical terms, this may be viewed as a bubble in the equity, housing, and mortgage markets in the United States, the latter providing financial assets that offer sufficient rewards to be attractive to the rest of the world. Another fundamental assumption is that the bubble bursts at $t = 0$, leaving investors (both local and foreign) to look for alternative stores of value. In the first stage, a flight-to-quality reaction migrates the bubble to "good" assets and so the price of commodities (notably Z) jumps, which results in a significant wealth transfer from U to M. In the second stage, under the assumption that the financial asset crisis and wealth transfer precipitates a severe growth slowdown, the excess demand for the good asset is destroyed, leading to a decrease in inventory of the good Z, and the bubble in commodity prices collapses.

Accordingly, this model can describe events in which asset bubbles emerged and subsequently collapsed, creating a sequence of bubble effects in one market after another. When the real estate bubble crashed and the value of MBS securities fell substantially, liquidity flowed into other markets, creating bubbles in commodities and oil markets as investors transferred financial assets. The deepening financial crisis then sharply slowed down economic growth, which in turn destroyed the commodity bubbles. Obviously, this story makes strong predictions concerning the timing of the origination and the collapse of various bubble phenomena in different markets. To evaluate the evidence in support of such interpretations of the events, consistent date stamping of those events is critical.

2.4 *Econometric dating of the timeline*

Bubbles can be definitively identified only in hindsight after a market correction (*Economist*, June 18, 2005).

The time path of P_t in the rational bubble model (with bubble component B_t) is explosive. Similarly, in the run-up phase of a financial bubble, a pattern of stochastically explosive or mildly explosive behavior is a characteristic feature. The econometric determination of bubble behavior therefore relies on a test procedure having power to discriminate between unit root (or martingale like) local behavior in a process and mildly explosive stochastic alternatives. The same distinction in reverse is required during a bubble collapse. Phillips and Magdalinos (PM) (2007a, 2007b) analyzed the properties of mildly explosive stochastic processes and developed a limit theory for autoregressive coefficient estimation and inference in that context.

PWY (2011) used forward recursive regression techniques and PM asymptotics to test for the presence of mildly explosive behavior in 1990s Nasdaq data and to date-stamp the origination and collapse of the Nasdaq bubble. It was shown that a sup unit root test against a mildly explosive alternative obtained from forward recursive regressions has the power to detect periodically collapsing bubbles. To improve the power and sharpen date detection, this paper modifies the sup test of PWY by selecting the initial condition based on an information criterion. The new methods are used in combination with the limit theory in Phillips and Yu (2009) and Phillips, Shi, and Yu (2011), which establishes consistency of the dating estimators.

The key idea of PWY is simple to implement and relies on recursively calculated right-sided unit root tests to assess evidence for mildly explosive behavior in the data.

In particular, for time series $\{X_t\}_{t=1}^n$, we apply standard unit root tests (such as the coefficient test or the Dickey–Fuller t test) with usual unit root asymptotics under the null against the alternative of an explosive or mildly explosive root. The test is a right-sided test and therefore differs from the usual left-sided tests for stationarity. Contrary to the quotation that heads this section, it is possible by means of these tests to identify the emergence of mildly explosive behavior as it occurs, thereby presaging bubble conditions. It is not necessary to wait for a market correction to identify bubble conditions in hindsight.

More specifically, by recursive least squares, we estimate the autoregressive specification

$$X_t = \mu + \delta X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0, \sigma^2), \tag{10}$$

allowing for the fact that the independent and identically distributed (iid) assumption may be relaxed to serially dependent errors with martingale difference primitives making the usual (possibly semiparametric) adjustments to the tests that are now standard practice in left-sided unit root tests. The null hypothesis is $H_0 : \delta = 1$ and the right-tailed alternative hypothesis is $H_1 : \delta > 1$, which allows for mildly explosive autoregressions with $\delta = 1 + c/k_n$, where $k_n \to \infty$ and $k_n/n \to 0$. The latter requirement ensures that the process X_t is mildly explosive in the sense of PM (2007a). If $k_n = O(n)$ and $\delta = 1 + c/n$, then the alternative is local to unity, and X_t has random wandering behavior and is no longer mildly explosive. In that event, consistent dating of periods of exuberance ($c > 0$) is not possible.

The regression in the first recursion uses $\tau_0 = \lfloor nr_0 \rfloor$ observations for some fraction r_0 of the total sample, where $\lfloor \cdot \rfloor$ denotes the integer part of its argument. Subsequent regressions employ this originating data set supplemented by successive observations giving a sample of size $\tau = \lfloor nr \rfloor$ for $r_0 \le r \le 1$. Denote the corresponding coefficient test statistic and the Dickey–Fuller t statistic by DF_r^δ and DF_r^t, namely

$$DF_r^\delta := \tau(\hat{\delta}_\tau - 1), \qquad DF_r^t := \left(\frac{\sum_{j=1}^{\tau} \tilde{X}_{j-1}^2}{\hat{\sigma}_\tau^2} \right)^{1/2} (\hat{\delta}_\tau - 1), \tag{11}$$

where $\hat{\delta}_\tau$ is the least squares estimate of δ based on the first $\tau = \lfloor nr \rfloor$ observations, $\hat{\sigma}_\tau^2$ is the corresponding estimate of σ^2, and $\tilde{X}_{j-1} = X_{j-1} - \tau^{-1} \sum_{j=1}^{\tau} X_{j-1}$. Obviously, DF_1^δ and DF_1^t correspond to the full sample test statistics. Under the null hypothesis of pure unit root dynamics and using standard weak convergence methods (Phillips (1987)), we have, as $\tau = \lfloor nr \rfloor \to \infty$ for all $r \in [r_0, 1]$, the limit theory

$$DF_r^\delta \Rightarrow \frac{r \int_0^r \tilde{W}_r(s)\, dW(s)}{\int_0^r \tilde{W}_r^2(s)}, \qquad DF_r^t \Rightarrow \frac{\int_0^r \tilde{W}_r(s)\, dW(s)}{\left(\int_0^r \tilde{W}_r^2(s) \right)^{1/2}}, \tag{12}$$

where W is standard Brownian motion and $\tilde{W}_r(s) = W(s) - \frac{1}{r}\int_0^r W$ is demeaned Brownian motion.[3]

If model (10) is the true data generating process for all t, then recursive regressions are unnecessary. In this case, a right-sided unit root test based on the full sample is able to distinguish a unit root null from an explosive alternative. In practice, of course, empirical bubble characteristics are much more complicated than model (10) and involve some regime change(s) between unit root (martingale) behavior with $\delta = 1$ and mildly explosive behavior with $\delta > 1$, and potential reinitialization as market temperature shifts from normal to exuberant sentiment and back again. A distinguishing empirical feature of bubble behavior is that market correction typically occurs as sentiment reverts back and mildly explosive behavior collapses. A model to capture this type of reversion was first constructed by Evans (1991), who argued that conventional unit root tests had little power to detect periodically collapsing bubbles generated in this manner. As shown in Phillips and Yu (2009), such a model which mixes a unit root process with a collapsed explosive process actually behaves like a unit root process over the full sample (in fact, with some bias toward stationarity as explained below), thereby invalidating the standard unit root test as a discriminating criterion when it is applied to the full sample.

To find evidence for the presence of a bubble in the full sample, PWY (2011) suggested using a sup statistic based on the recursive regression. This involves comparing $\sup_r DF_r^t$ with the right-tailed critical values from the limit distribution based on $\sup_{r\in[r_0,1]} \int_0^r \tilde{W}\,dW/(\int_0^r \tilde{W}^2)^{1/2}$. Similarly, for the coefficient test, one can compare the sup statistic $\sup_r DF_r^\delta$ with the right tailed critical values from the limit distribution based on $\sup_{r\in[r_0,1]} r\int_0^r \tilde{W}\,dW/\int_0^r \tilde{W}^2$.

Our approach to finding the timeline of the bubble dynamics also makes use of forward recursive regressions. We date the origination of the bubble by the estimate $\hat{\tau}_e = \lfloor n\hat{r}_e \rfloor$, where

$$\hat{r}_e = \inf_{s\geq r_0}\{s : DF_s^\delta > cv_{\beta_n}^\delta\} \quad \text{or} \quad \hat{r}_e = \inf_{s\geq r_0}\{s : DF_s^t > cv_{\beta_n}^{df}\} \tag{13}$$

$cv_{\beta_n}^\delta$ ($cv_{\beta_n}^{df}$) is the right-side $100\beta_n\%$ critical value of the limit distribution of the DF_r^δ (DF_r^t) statistic based on $\tau_s = \lfloor ns \rfloor$ observations, and β_n is the size of the one-sided test. Conditional on finding some originating date $\hat{r}_e$ for (mildly) explosive behavior, we date the collapse of the bubble by $\hat{\tau}_f = \lfloor n\hat{r}_f \rfloor$, where

$$\hat{r}_f = \inf_{s\geq \hat{r}_e+\gamma\ln(n)/n}\{s : DF_s^\delta < cv_{\beta_n}^\delta\} \quad \text{or} \quad \hat{r}_f = \inf_{s\geq \hat{r}_e+\gamma\ln(n)/n}\{s : DF_s^t < cv_{\beta_n}^{df}\}. \tag{14}$$

This dating rule for $\hat{\tau}_f$ requires that the duration of the bubble is nonnegligible — at least a small infinity as measured by the quantity $\gamma \ln n$ so that episodes of smaller order than $\gamma \ln n$ are not considered significant in the dating algorithm for τ_f. The parameter

[3]Note that $r\int_0^r \tilde{W}_r(s)\,dW(s)/\int_0^r \tilde{W}_r(s)^2 =_d \int_0^1 \tilde{W}_1(s)\,dW(s)/\int_0^1 \tilde{W}_1(s)^2$ and $\int_0^r \tilde{W}_r(s)\,dW(s)/(\int_0^r \tilde{W}_r(s)^2)^{1/2} =_d \int_0^1 \tilde{W}_1(s)\,dW(s)/(\int_0^1 \tilde{W}_1(s)^2)^{1/2}$ so that the recursive limit distributions in (12) are all equivalent to those based on a full sample of size n.

γ can be set so that the minimum duration is tuned to the sampling interval. This minimal duration requirement helps to reduce the type I error in the unit root test, so that false detections are controlled, without affecting the consistency property of the estimator.

The consistent estimation of r_e and r_f requires a slow divergence rate of critical values so that test size tends to zero as $n \to \infty$. For practical implementation, we set the critical value sequences $\{cv_{\beta_n}^{\delta}, cv_{\beta_n}^{df}\}$ according to an expansion rule such as $cv_{\beta_n}^{\delta} = -0.44 + \ln(\lfloor nr \rfloor)/C$ and $cv_{\beta_n}^{df} = -0.08 + \ln(\lfloor nr \rfloor)/C$. Both these critical values diverge at a slowly varying rate with $cv_{\beta_n}^{df} < cv_{\beta_n}^{\delta}$. For practically reasonable sample sizes, these critical values are close to the 5% critical values for DF_1^{δ} and DF_1^{t} if the constant C is chosen to be large, say 100. For example, when $n = 100$, $cv_{\beta_n}^{\delta} = -0.44 + \ln(\lfloor nr \rfloor)/C = -0.394$ and $cv_{\beta_n}^{df} = -0.08 + \ln(\lfloor nr \rfloor)/C = -0.034$. The 5% critical values for DF_1^{δ} and DF_1^{t} are -0.44 and -0.08, respectively. Similar critical value expansion rates have been trialed in extensive simulations in Phillips and Yu (2009) and found to give very satisfactory results in terms of small size and high discriminatory power. More conversative rules for these critical values are obtained by choosing smaller values of the constant C, as in the application reported later in the paper.

Under the mildly explosive bubble model,

$$
\begin{aligned}
X_t &= X_{t-1} 1\{t < \tau_e\} + \delta_n X_{t-1} 1\{\tau_e \le t \le \tau_f\} \\
&\quad + \left(\sum_{k=\tau_f+1}^{t} \varepsilon_k + X_{\tau_f}^* \right) 1\{t > \tau_f\} + \varepsilon_t 1\{t \le \tau_f\}, \\
\delta_n &= 1 + \frac{c}{n^{\alpha}}, \quad c > 0, \alpha \in (0, 1),
\end{aligned}
\tag{15}
$$

Phillips and Yu (2009) showed that $\hat{r}_e \xrightarrow{p} r_e$ and $\hat{r}_f \xrightarrow{p} r_f$ under some general regularity conditions. Model (15) mixes together two processes, a unit root process and a mildly explosive process with a root above 1 taking the form $\delta_n = 1 + \frac{c}{n^{\alpha}}$. This type of mildly explosive process over $\tau_e \le t \le \tau_f$ was originally proposed and analyzed by PM (2007a, 2007b). The above system is more complex because it involves regime switches from unit root to mildly explosive behavior at τ_e and from the mildly explosive root back to a unit root at τ_f. At τ_f, the switch also involves a reinitialization of the process and X_t collapses to $X_{\tau_f}^*$, corresponding to a bubble collapse back to fundamental values prevailing prior to the emergence of the bubble. We may, for instance, set $X_{\tau_f}^* = X_{\tau_e} + X^*$ for some $O_p(1)$ random quantity X^*, so that $X_{\tau_f}^*$ is within an $O_p(1)$ realization of the pre-bubble value of X_t.

Under this model specification (15), Phillips and Yu (2009) showed that when $\tau = [nr] \in [\tau_e, \tau_f)$,

$$
DF_r^{\delta} = \tau(\hat{\delta}_n(\tau) - 1) = n^{1-\alpha} rc + o_p(1) \to +\infty
$$

and

$$DF_r^t = \left(\frac{\sum_{j=1}^{\tau} \tilde{X}_{j-1}^2}{\hat{\sigma}_\tau^2} \right)^{1/2} (\hat{\delta}_n(\tau) - 1) = n^{1-\alpha/2} \frac{c^{3/2} r^{3/2}}{2^{1/2} r_e^{1/2}} \{1 + o_p(1)\} \to +\infty.$$

Hence, provided that $cv_{\beta_n}^{\delta}$ goes to infinity at a slower rate than $n^{1-\alpha}$ and that $cv_{\beta_n}^{df}(r)$ goes to infinity at a slower rate than $n^{1-\alpha/2}$, DF_r^{δ} and DF_r^t both consistently estimate r_e. Moreover, when $\tau = [nr] > \tau_f$,

$$DF_r^{\delta} = \tau(\hat{\delta}_n(\tau) - 1) = -n^{1-\alpha} rc \to -\infty \tag{16}$$

and

$$DF_r^t = \left(\frac{\sum_{j=1}^{\tau} \tilde{X}_{j-1}^2}{\hat{\sigma}_\tau^2} \right)^{1/2} (\hat{\delta}_n(\tau) - 1) = -n^{(1+\alpha)/2} \frac{c^{1/2} r^{1/2}}{2^{1/2}} \{1 + o_p(1)\} \to -\infty. \tag{17}$$

Hence, DF_r^{δ} and DF_r^t both consistently estimate r_f. Importantly, (16) diverges to negative infinity, so it is apparent that in the post-bubble period $\tau > \tau_f$ the autoregressive coefficient $\hat{\delta}_n(\tau)$ is biased downward, which in this case means biased toward stationarity. This bias is explained by the fact that the collapse of the bubble is sharp following τ_f in model (15) and produces a mean reverting effect in the data, which manifests in the limit theory as a slight bias toward stationarity in the estimated unit root.

We now provide some heuristic discussion about the capacity of these forward recursive regression tests to capture the timeline of bubble activity. The tests have discriminatory power because they are sensitive to the changes that occur when a process undergoes a change from a unit root to a mildly explosive root or vice versa. This sensitivity is much greater than in left-sided unit root tests against stationary alternatives, due to the downward bias and long left tail in the distribution of the autoregressive coefficient in unit root and near stationary cases. By contrast, as is apparent ex post in the data when there has been a bubble, the trajectories implied by unit root and mildly explosive processes differ in important ways. Although a unit root process can generate successive upward movements, these movements still have a random wandering quality unlike those of a stochastically explosive process where there is a distinct nonlinearity in movement and little bias in the estimation of the autoregressive coefficient. Forward recursive regressions are sensitive to the changes implied by this nonlinearity. When data from the explosive (bubble) period are included in estimating the autoregressive coefficient, these observations quickly influence the estimate and its asymptotic behavior due to the dominating effect of the signal from mildly explosive data. This difference in signal between the two periods provides identifying information and explains why the two test procedures consistently estimate the origination date. When the bubble bursts and the system switches back to unit root behavior, the signal from the explosive period continues to dominate that of the unit root period. This domination, which at this point is effectively a domination by initial conditions, is analogous to the domination by distant initializations that can occur in unit root limit theory, as shown recently by PM (2009). More than this, the crash and reinitialization give the appearance in the data of a form of mean reversion to an earlier state, so that the estimated autoregressive coefficient is smaller than unity and the classical unit root test statistics diverge to minus infinity, as shown in (16) and (17) above.

2.5 *Initialization*

To improve the power of the PWY procedure, we modify the methods by selecting the initial condition based on the Schwarz (1978) Bayesian information criterion (BIC). In PWY, the initial observation in each recursive regression was fixed to the first observation of the full sample. While this choice is convenient, when time series mix a nonexplosive regime with an explosive regime, a more powerful test is obtained if the recursive statistics are calculated using sample data from a single regime for bubble detection. This observation motivates us to use the data to choose the initialization. The method follows an approach to endogenous initialization in time series regression that was suggested in Phillips (1996).

Suppose an origination date $\hat{\tau}_e$ has been identified by the procedure of PWY.[4] Let $n_{\min}$ be the number of observations in a base sample of the observations $\{X_{\tau_e - n_{\min}+1}, \dots, X_{\tau_e}\}$. The base sample may be constructed by taking some percentage of the sample before $\hat{\tau}_e$. In our applications below, we use 10%. For the base sample, we compare the BIC value of two competing models: a unit root model and an autoregressive model. If the BIC value of the unit root model is smaller and the point estimate of δ is larger than 1, we reset the initial condition to $\tau_e - n_{\min} + 1$. Otherwise, we expand the base sample to $\{X_{\tau_e - n_{\min}}, \dots, X_{\tau_e}\}$, so that another observation is added to the beginning of the sample. Based on the new sample, we again compare the BIC value of the competing models. If the BIC value of the unit root model is smaller and the point estimate of δ is larger than 1, we reset the initial condition to $\tau_e - n_{\min}$. This exercise is repeated until the BIC value of the unit root model is smaller. If the sample eventually becomes $\{X_1, \dots, X_{\tau_e}\}$ and the BIC value of the unit root model is still larger, we set the initial condition to $t = 1$, which is the same as that used in PWY. If the initialization emerging from this procedure is $\hat{\tau}_0$, then the recursive testing methodology of PWY is applied from $\hat{\tau}_0$. With this initialization, denote the estimate of the origination date $\hat{\tau}_e(\hat{\tau}_0)$ and the estimate of the collapse date $\hat{\tau}_f(\hat{\tau}_0)$. Obviously, $\hat{\tau}_e(1) = \hat{\tau}_e$ and $\hat{\tau}_f(1) = \hat{\tau}_f$. However, if $\hat{\tau}_0 > 1$, it is possible that $\hat{\tau}_e(\hat{\tau}_0) \neq \hat{\tau}_e$ and $\hat{\tau}_f(\hat{\tau}_0) \neq \hat{\tau}_f$. In general, it is expected that $\hat{\tau}_e(\hat{\tau}_0) \leq \hat{\tau}_e$ since the backward recursion to locate the initialization $\hat{\tau}_0$ begins from $\hat{\tau}_e$.

[4] In case no bubble is found, no change in the PWY procedure is required. However, a flexible moving window recursive approach is also possible, which allows for variable initializations and may be more effective in assessing evidence for multiple bubbles. See Phillips, Shi, and Yu (2011).

Assume the sample is $\{X_{\tau_e - n_{\min} - n_k + 1}, \ldots, X_{\tau_e}\}$. The BIC value of the unit root model is

$$\ln\left(\frac{\sum_{t=\tau_e - n_{\min} - n_k}^{\tau_e} (\Delta X_t - \overline{X})^2}{n_k + n_{\min}}\right) + \frac{\ln(n_k + n_{\min})}{n_k + n_{\min}},$$

whereas the BIC value of the autoregression is

$$\ln\left(\frac{\sum_{t=\tau_e - n_{\min} - n_k}^{\tau_e} (X_t - \hat{\mu} - \hat{\delta} X_{t-1})^2}{n_k + n_{\min}}\right) + \frac{2\ln(n_k + n_{\min})}{n_k + n_{\min}},$$

where $\overline{X} = \frac{1}{n_{\min}+n_k}\sum_{t=\tau_e - n_{\min} - n_k + 1}^{\tau_e} X_t$, and $\hat{\delta}$ and $\hat{\mu}$ are the ordinary least squares (OLS) estimators of δ and μ from the autoregressive model

$$X_t = \mu + \delta X_{t-1} + \varepsilon_t.$$

It is known that when the criterion is applied in this way, BIC can consistently (i.e., almost surely as $n \to \infty$) distinguish a unit root model from a stationary model without specifying transient dynamics (see Phillips (2008)). Using similar methods, it can be shown that BIC consistently distinguishes a unit root model from a model with an explosive root. In essence, the use of BIC to select the initialization is equivalent to the use of BIC to choose a break point, although in the present case, it is not necessary to specify transient behavior.

2.6 *Testing bubble migration*

As discussed earlier, the CFG model links together bubbles from different markets in a migration mechanism. Using the econometric dating algorithm, bubble periods in different markets can be formally dated as we described above. Then, with recursive statistics that measure the existence and intensity of bubble phenomena in different markets, we may empirically test whether a bubble migrates from one market to another market. This section outlines a new reduced form procedure for testing such bubble migration from one series X_t to another series Y_t.

To fix ideas, let $\theta_X(\tau)$ be the coefficient in an autoregression characterizing the time series $\{X_t\}_{t=1}^{\tau=\lfloor nr \rfloor}$, which may be recursively estimated by least squares regression as $\hat{\theta}_X(\tau)$. We explicitly allow for the coefficient $\theta_X(\tau)$ to be time dependent so that it captures any structural changes in the coefficient arising from exuberance and collapse. The goal is to explain potential migrationary effects of these changes on the behavior of a second time series Y_t.

Suppose our dating mechanism identifies a bubble in X_t at $\tau_{eX} = \lfloor nr_{eX} \rfloor$ and the autoregressive estimate $\hat{\theta}_X(\tau)$ peaks at $\tau_{pX} = \lfloor nr_{pX} \rfloor$. In a similar way, define $\theta_Y(\tau)$, $\hat{\theta}_Y(\tau)$, r_{eY}, and r_{pY} for the time series $\{Y_t\}_{t=1}^{\tau=\lfloor nr \rfloor}$. It is assumed that $r_{pY} > r_{pX}$ and that this inequality is confirmed by the dating algorithm. In practice, we will be working with date estimators obtained by recursive regression.

Let $m = \tau_{pY} - \tau_{pX} = \lfloor nr_{pY} \rfloor - \lfloor nr_{pX} \rfloor$ be the number of observations in the interval $(\tau_{pX}, \tau_{pY}]$. We consider formulating over the interval $(\tau_{pX}, \tau_{pY}]$ an empirical model in which the null generating mechanism of Y_t involves an autoregressive coefficient θ_Y that transitions from a unit root to a mildly explosive root at $\tau_{eY} = \lfloor nr_{eY} \rfloor$ for some $r_{eY} \in (r_{pX}, r_{pY})$, namely

$$\theta_Y = \theta_Y(\tau) = \begin{cases} 1, & \tau < \tau_{eY} = \lfloor nr_{eY} \rfloor, \\ 1 + \dfrac{c_Y}{n^\alpha}, & \tau \geq \tau_{eY} = \lfloor nr_{eY} \rfloor. \end{cases}$$

In this model, $\theta_Y(\tau)$ has a structural break that produces exuberance at $\tau = \tau_{eY}$ and the localizing coefficient c_Y of the mildly explosive root is constant. The alternative hypothesis of interest is that the autoregressive coefficient θ_Y transitions to a mildly explosive root whose recursive value, $\theta_Y(\tau)$, depends on the corresponding recursive autoregressive coefficient, $\theta_X(\tau)$ for the series X_t. In this event, the value of θ_Y is in part determined by changes in θ_X, so that as the bubble in the series X_t collapses and $\theta_X(\tau)$ falls, the bubble migrates to the series Y_t and manifests in an increasing coefficient $\theta_Y(\tau)$ that exceeds unity. In effect, the bubble collapse in X_t influences and possibly augments exuberance in Y_t.

We allow the autoregressive coefficient for X_t to be local to unity upon the collapse of the bubble in X_t, namely $\theta_X(\tau) = 1 + c_X(\frac{\tau - \tau_{pX}}{m})/n$ for $\tau > \tau_{pX}$, which introduces a nonzero and possibly negative localizing coefficient function so that $c_X(\cdot) < 0$ to deliver a mean reverting effect during the collapse in X_t over $\tau \in (\tau_{pX}, \tau_{pY})$. Note that for $\tau = \lfloor nr \rfloor$, we have

$$\begin{aligned} c_X\left(\frac{\tau - \tau_{pX}}{m}\right) &= c_X\left(\frac{\tau - \tau_{pX}}{n}\frac{n}{m}\right) = c_X\left(\frac{r - r_{pX}}{r_{pY} - r_{pX}}\{1 + O(n^{-1})\}\right) \\ &\sim c_X\left(\frac{r - r_{pX}}{r_{pY} - r_{pX}}\right). \end{aligned}$$

The dependence of θ_Y on the changing behavior of $\theta_X(\tau)$ can be captured through the localizing coefficients as

$$\theta_Y(\tau) = 1 + \frac{c_Y + dc_X\left(\dfrac{\tau - \tau_{pX}}{m}\right)}{n^\alpha}, \quad \tau = \lfloor nr \rfloor, r \geq r_{eY}, \tag{18}$$

so that exuberance in Y_t, which is measured by the second term of (18), is influenced by the evolving pattern of the autoregressive behavior in X_t, represented by the localizing coefficient function $c_X(\cdot)$ of X_t. Thus, as the bubble in X_t collapses and X_t returns to near martingale behavior, the localizing coefficient c_X impacts the autoregressive parameter θ_Y that determines the behavior of Y_t. The simplest version of (18) involves a linear relation for θ_X in which c_X is constant and negative ($c_X < 0$). Then $\theta_Y(\tau) = 1 + \frac{c_Y}{n^\alpha} + d\frac{c_X}{n^\alpha}(\frac{\tau - \tau_{pX}}{m})^2$ and this specification is useful in what follows.

The null hypothesis is no migration ($d = 0$) and the alternative hypothesis is bubble migration ($d \neq 0$). When interest focuses, as here, on the possibility of a collapse in X_t inducing or increasing subsequent exuberance in Y_t, the alternative may be signed ($d < 0$) so that $dc_X > 0$. Based on (18), we formulate the interactive model

$$\begin{aligned} &\theta_Y(\tau) - 1 = \beta_{0n} + \beta_{1n}(\theta_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m} + \text{error}, \\ &\tau = \lfloor nr_{pX} \rfloor + 1, \dots, \lfloor nr_{pY} \rfloor, \end{aligned} \tag{19}$$

for data over the time interval $(\lfloor nr_{pY} \rfloor, \lfloor nr_{pY} \rfloor)$ of length $m = \lfloor nr_{pX} \rfloor - \lfloor nr_{pY} \rfloor$ which covers the period of collapse of X_t and the emergence of exuberance in Y_t. The null and alternative hypotheses in (19) are

$$\mathcal{H}_0: \beta_{1n} = 0, \qquad \mathcal{H}_1: \beta_{1n} < 0. \tag{20}$$

The coefficients (β_{0n}, β_{1n}) in (19) may be fitted by linear regression (using recursive estimates $(\hat{\theta}_Y(\tau), \hat{\theta}_X(\tau))$ of the autoregressive coefficients), giving $(\hat{\beta}_{0n}, \hat{\beta}_{1n})$. The fitted coefficient $\hat{\beta}_{1n}$ is then tested for significance. The limit behavior of $\hat{\beta}_{1n}$ under the null can be used to construct a suitable test.

The empirical regression form of (19) involves recursive estimates of the variables $\theta_Y(\tau) - 1$ and $\theta_X(\tau) - 1$. The fitted slope coefficient is then

$$\hat{\beta}_{1n} = \frac{\sum_{\tau = \lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_Y(\tau)\tilde{Z}_X(\tau)}{\sum_{\tau = \lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_X(\tau)^2},$$

where

$$Z_Y(\tau) = \hat{\theta}_Y(\tau) - 1, \qquad Z_X(\tau) = (\hat{\theta}_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m},$$

and

$$\tilde{Z}_a(\tau) = Z_a(\tau) - \frac{1}{m}\sum_{s = \lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} Z_a(s) \quad \text{for } a = X, Y.$$

The limit theory for $\hat{\beta}_{1n}$ can be obtained using methods similar to those in Phillips, Shi, and Yu (2011). Under $\mathcal{H}_0$ as $n \to \infty$, we find that $\hat{\beta}_{1n} = O_p(1)$. An explicit expression for the limiting form of $\hat{\beta}_{1n}$ has been obtained and the limit depends on c_X and the form of the generating mechanism for X_t during the collapse period. Under $\mathcal{H}_1$ as $n \to \infty$, we find that $\hat{\beta}_{1n} = O_p(n^{1-\alpha})$ and is divergent. Details of these results are provided in the Appendix.

TABLE 1. Values of $L(m)$.

	$a=1/3$	$a=1$	$a=3$
$m=20$	0.43	1.30	3.90
$m=40$	0.53	1.60	4.81

Given the limit behavior under the null and alternative, we construct an asymptotically conservative and consistent test of $\mathcal{H}_0:\beta_{1n}=0$ against $\mathcal{H}_1:\beta_{1n}<0$ based on the standardized statistic

$$Z_\beta=\hat{\beta}_{1n}/L(m), \quad \text{where } \frac{1}{L(m)}+\frac{L(m)}{n^\varepsilon}\to 0 \text{ as } n\to\infty \text{ for any } \varepsilon>0$$

for some slowly varying function $L(m)$, such as $a\log(m)$ with $a>0$ and where $m=\lfloor nr_{pY}\rfloor-\lfloor nr_{pX}\rfloor=O(n)$. The test uses critical values from a standard $N(0,1)$ distribution and rejects $\mathcal{H}_0$ (no bubble migration from X to Y) in favor of $\mathcal{H}_1$ (bubble migration from X to Y) if $|Z_\beta|>cv_\alpha$ where cv_α is the $100\alpha\%$ critical value of the standard normal. This test has asymptotically zero size (because $\hat{\beta}_{1n}/L(m)\to_p 0$ under the null) and unit power (because $Z_\beta=O_p(n^{1-\alpha}/L(m))$ diverges under the alternative). The test relies on standard normal distribution critical values cv_α and all the usual settings of α (i.e., 10%, 5%, and 1%) will control size to zero asymptotically as $n\to\infty$. Values of Z_β may be reported for a range of values of $L(m)$ such as $\{a\log m:\frac{1}{3}\le a\le 3\}$, reflecting the impact of different normalization factors. Table 1 reports values of $L(m)$ when $m=20,40$ and $a=1/3$, 1, and 3.

The test has the advantage that it does not require knowledge of the precise model for X_t following the collapse of the bubble in X_t. In particular, knowledge of the localizing coefficient c_X (or its precise functional time dependence in the more general case of $c_X(\cdot)$) is not required to mount this test. The test is robust to any model for X_t that has an autoregressive root that is local to unity, even with a localizing coefficient function $c_X(\cdot)$ that is nonconstant, and further allows for a wide range of potential (conditional and unconditional) volatility in the error process. Moreover, this test does not require standard error computations for implementation as it is a coefficient based test.

One weakness of the test is that it is consistent against alternatives in which Y_t displays changing levels of exuberance that might be proxied by the regressor $(\hat{\theta}_X(\tau)-1)\times\frac{\tau-\tau_{pX}}{m}$ in the empirical regression (21). This is a form of trend determined spurious regression phenomena that arises because both dependent variable and regressor are computed recursively and have limits that trend in the recursive argument (r) of the regression sample size $\tau=\lfloor nr\rfloor$. In this event, of course, rejection of the null signals that exuberance in Y_t is evolving and that the pattern of the evolution is such that it can be partly explained by the course of the collapse in the time series X_t following τ_{pX}. Rejection of $\mathcal{H}_0$ therefore establishes an association but does not confirm a causal link.

3. DATA

Two data sets are studied in the empirical work reported here. The primary data constitute three financial time series: the monthly U.S. house price index divided by the rental measure from January 1990 to January 2009; monthly crude oil prices (in U.S. dollars) normalized by the oil supply that is approximated by the U.S. inventory from January 1999 to January 2009; and the daily spread between the Baa and Aaa bond rates from January 3, 2006 to July 2, 2009.

A secondary data set is studied to check whether the empirical bubble characteristics found in the primary series apply to other commodities. The secondary data include some commodity prices such as monthly heating oil, coffee, cotton, cocoa, sugar, and feeder cattle prices, all measured in U.S. dollars, from January 1999 to January 2009.

The choice of the sampling periods is guided by CFG (2008a) because we aim to match the empirical analysis with predictions they made. The CFG story begins with the Internet bubble in the Nasdaq in the 1990s (see p. 7 in CFG (2008a)) and ends with the collapse of all financial bubbles when the economy goes seriously into recession. For the Baa bond rates, it is well known that a relevant event that signaled the effects of the credit crunch is the failure of Lehman Brothers on September 15, 2008. The sampling period is chosen so that we have enough observations before September 15 for the bubble test to have good power. Similar arguments apply to the choice of the sampling period for the exchange rates. Longer sampling intervals, all covering the subprime crisis period, have been used and the empirical findings reported here are robust to the choice of the sample period. This is because earlier observations are discarded by the proposed procedure of selecting the initial observation.

The house price index is the S&P Case–Shiller Composite-10 index obtained from Shiller's website. We measure fundamental values by standardizing the use of rental data. The quarterly rental data are imputed using the method of Davis, Lehnert, and Martin (2008) and are linearly interpolated to a monthly frequency.[5] The ratio between the two series is the first time series we use. The crude oil price series is based on WTI–Cushing, Oklahoma spot prices obtained from the Energy Information Administration website. We measure fundamental values using a measure of the oil supply based on the inventory of crude oil in the United States.[6] The ratio between the two series is the second time series we use. The Baa (Aaa) bond rates are averages of Baa (Aaa) industrial bond rates and are obtained from the Federal Reserve Board. The Baa bond rates measure the credit risk level and are particularly relevant because, as the crisis unfolded, the drop in the prices and market liquidity of all mortgage-backed securities led a sharp increases in the price of risk and in spreads. Unsurprisingly, mistrust among financial counterparties surged and bond rates jumped. While the same argument may apply to the Aaa bond rate, the effect is believed to be much less serious for this bond grade. As a result, the spread between the Baa and Aaa bond rates is a measure of the credit risk level.

For the secondary data set, all the commodity prices are downloaded from EconStats at http://www.econstats.com/index.htm and deflated using the Consumer Price Index

[5]The rental data can be downloaded from http://www.lincolninst.edu/subcenters/land-values/rent-price-ratio.asp.

[6]The inventory data can be downloaded from the Department of Energy, Monthly Energy Review, U.S. crude oil ending stocks non-SPR (thousands of barrels).

Table 2. Summary statistics.

Data	Sample Size	Freq	Min	Date (min)	Max	Date (max)	DF_1^t
House	229	M	0.5405	Sep 1996	1.1418	Feb 2006	−0.3024
Oil	121	M	12.086	Feb 1999	146.188	June 2008	−1.5152
Baa/Aaa	878	D	0.75	May/2/06	3.50	Dec/3/08	−0.6694

(CPI) obtained from the Department of Labor. Figure 1 plots the three series in the primary data set. Table 2 reports some summary descriptive statistics for these three time series, including sample size, sample frequency, sample minimum, date of the minimum, sample maximum, date of the maximum, as well as the DF t statistic (DF_1^t) based on the entire sample.

The rental-adjusted house price index troughed in September 1996 and peaked in February 2006. The supply-adjusted crude oil price has its minimum and wanders around in the early part of the sample, reaching its maximum in mid-2008. The spread between Baa and Aaa moves within a narrow range (between 0.75 and 1.0) in the first half of the sample and reaches its highest point (3.5) on December 3, 2008, shortly after the failure of Lehman Brothers on September 15. At the 5% level, in all cases the unit root null cannot be rejected in favor of an explosive alternative for the full sample (the 5% asymptotic critical value is −0.08 for the unit root test statistic DF_1^t). That is, the full sample analysis indicates that no bubble can be found in any of the three time series.

4. Empirical findings

Three phases have been identified in connection with the subprime crisis. According to CFG (2008a), each phase involves a specific hypothesis that concerns related bubble activity. In the first phase (A), before the subprime crisis publicly erupted, bubbles had emerged and burst in the stock market, the housing market, and the mortgage market. These bubbles all played a role in global imbalances. The following hypothesis is central.

Hypothesis A. *A price bubble arises in the housing market. The property price bubble originated before the subprime crisis surfaced in August 2007 and collapsed as the subprime crisis broke.*

During the second phase (B), the subprime crisis erupted and funds flowed selectively to assets in other markets with lower perceived risk or greater opportunity. In consequence, bubbles emerged in certain commodity markets and credit risk perceptions rapidly elevated, leading to the following hypothesis:

Hypothesis B. *Following the public eruption of the subprime crisis, new bubbles emerged in* (i) *selected commodity price markets and* (ii) *bond markets.*

In the third phase (C), perceptions increased that the financial crisis and associated credit crunch might seriously impact real economic activity both in the United States

Table 3. Critical values of $\max DF_r^t$ obtained in simulations.

Sample Size	Test Statistic	10%	5%	1%
500	$\max DF_r^t$	1.1800	1.4603	2.0043
100	$\max DF_r^t$	1.1914	1.5073	2.1899

and worldwide. Recognition of the global recessionary effects of the financial crisis precipitated a collapse in the commodity price and the bond market bubbles.

Hypothesis C. *Bubbles in commodity prices and the bond market collapsed as the global economic implications of the crisis became apparent.*

These hypotheses provide a timeline of market bubble phenomena that can be subjected to empirical evaluation. To do so, we need to identify a bubble in each of the relevant time series within the sample period, fit the dating of these bubbles to the given timeline, and, if possible, test the migration of the bubbles from the housing market to commodity prices and the bond market.

We now report and discuss our empirical findings in relation to these hypotheses. First, we check for statistical evidence of the presence of bubble activity in each time series using the recursively calculated sup statistic $\max DF_r^t$.[7] Table 3 reports critical values for the statistic obtained by simulation for the two sample sizes, 100 and 500. The critical values for $\max DF_r^t$ are essentially identical to those reported in PWY. Note that these critical values are substantially larger than those of the DF_r^t statistic.

The first row in Table 4 reports the $\max DF_r^t$ statistic for the three time series using endogenized initializations. All cases show overwhelming evidence for the presence of bubbles. The p values are substantially below 1% for each of these time series, suggesting that the bubble characteristics are strong in all three cases, in sharp contrast to the result from the full sample analysis.

Time series plots of the recursively calculated DF_r^t statistic are shown in Figures 3–5. Superposed on these plots are the critical value paths, $-0.08 + \ln(n)/5$ (where the constant C in $cv_{\beta_n}^{df} = -0.08 + \ln(\lfloor nr \rfloor)/C$ is chosen as $C = 5$ to give a conservative test), the estimated dates $\hat{\tau}_e(\hat{\tau}_0)$ and $\hat{\tau}_f(\hat{\tau}_0)$, and the onset date for the subprime crisis. Recall that Figure 1 plots $\hat{\tau}_e(\hat{\tau}_0)$ and $\hat{\tau}_f(\hat{\tau}_0)$, together with the time series data. In all these cases, we clearly identify an explosive subperiod in the data.

Table 4. Testing the presence of bubbles and date stamping.[a]

	Home Price	Oil	Baa/Aaa
$\max DF_r^t$	14.867	3.0849	8.7515
$\hat{\tau}_e(\hat{\tau}_0)$	February/02	March/08	September/22/08
$\hat{\tau}_f(\hat{\tau}_0)$	December/07	July/08	April/20/09

[a]The reported estimates of τ_e and τ_f are based on DF_t^t.

[7]Similar results were obtained with the $\max DF_r^\delta$ statistic and, for brevity, they are not reported.

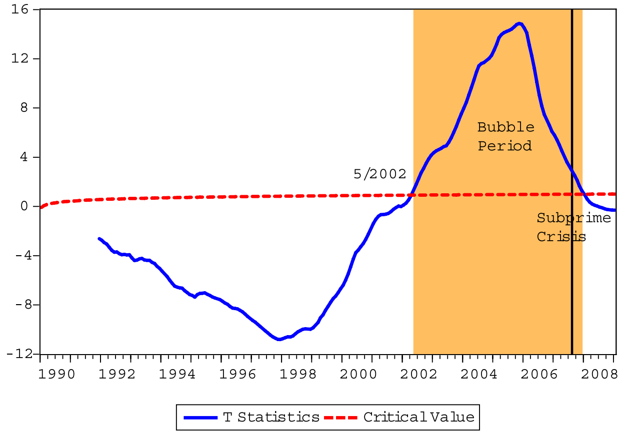


FIGURE 3. Recursive values of the t statistic for the rental-adjusted home price Composite-10 index from January 1990 to January 2009 obtained from forward recursive regressions.

Some general conclusions can be drawn from the estimates and Figures 1 and 3–5. First, the estimated origination and collapse dates seem to cover a subperiod of significant price run-up in each of the time series. Second, the estimated origination dates are not the same as the apparent beginning of these run-up periods. This may be because a unit root process as well as processes with very mildly explosive roots that are closer to unity than an $O(n^{-1})$ neighborhood can also generate mild run-ups but the latter are indistinguishable from a unit root process. The present tests have substantial discriminatory power for mildly explosive roots beyond those of $O(n^{-1})$ neighborhoods and are consistent against such alternatives.

The following conclusions can be drawn for the individual time series and Hypotheses A–C.

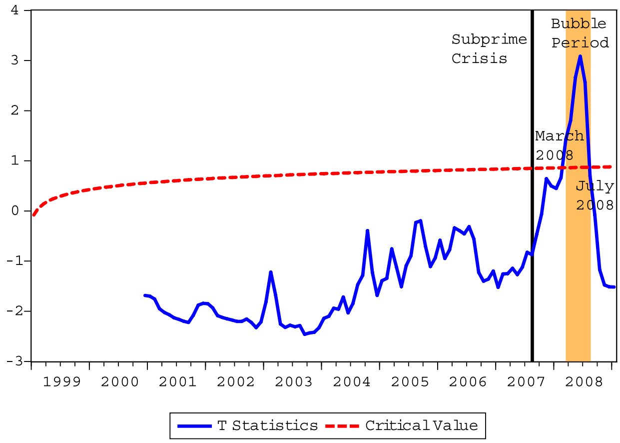


FIGURE 4. Recursive values of the t statistic for the supply-adjusted crude oil price from January 1999 to January 2009 obtained from forward recursive regressions.

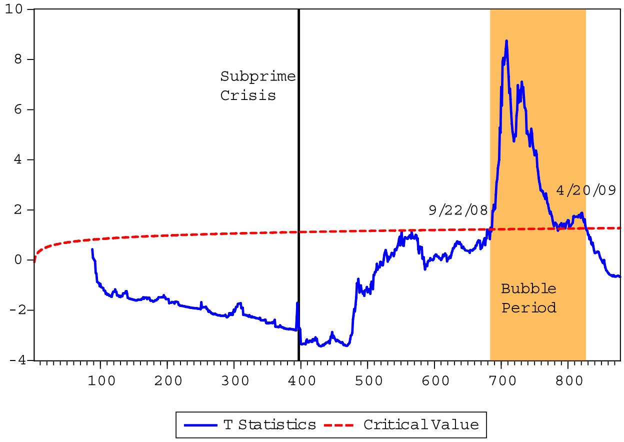
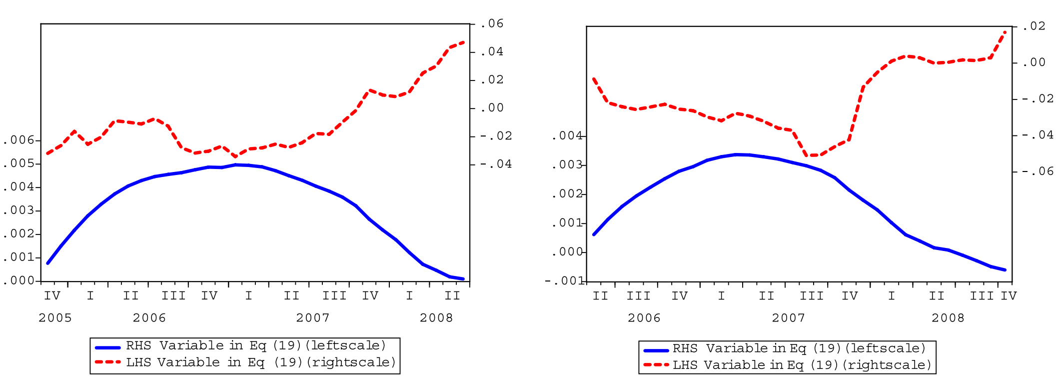


FIGURE 5. Recursive calculation of the t statistic for the spread between Baa and Aaa bond rates from January 3, 2006 to July 2, 2009, obtained from forward recursive regressions.

- For the rental-adjusted house price series, a significant bubble is found by the DF_r^t statistic during the early part of 2000. Our estimate of the bubble origination date in May 2002 strongly supports the argument by Baker (2002), who claimed that there was a housing bubble at that time. In addition, according to DF_r^t, the bubble collapsed in December 2007, soon after the subprime crisis erupted, which is consistent with Hypothesis A.

- For the supply-adjusted crude oil price, DF_r^t does not identify a bubble before the subprime crisis broke. However, a significant bubble is found by DF_r^t from March to July 2008. In the left panel of Figure 6, we plot $(\hat{\theta}_X(\tau) - 1)^{\frac{\tau - \tau_{pX}}{m}}$, where $\hat{\theta}_X(\tau)$ is obtained from the property market, and $\hat{\theta}_Y(\tau) - 1$, where $\hat{\theta}_Y(\tau)$ is obtained from the crude oil market. To test bubble migration from house prices to oil prices, we



(a) Migration from Housing to Oil

(b) Migration from Housing to Bonds

FIGURE 6. Bubble migration between markets. Panel (a) plots $(\hat{\theta}_X(\tau) - 1)^{\frac{\tau - \tau_{pX}}{m}}$ for the housing market and $(\hat{\theta}_Y(\tau) - 1)$ from the oil market between November 2005 and June 2008. Panel (b) plots $(\hat{\theta}_X(\tau) - 1)^{\frac{\tau - \tau_{pX}}{m}}$ for the house market and $(\hat{\theta}_Y(\tau) - 1)$ from the bond market between May 2006 and October 2008.

TABLE 5. Values of Z_β to test migration from the housing market to the oil market.

	$a=1/3$	$a=1$	$a=3$
$m=32$	−20.85	−6.95	−2.33

ran the empirical regression (19) with r_p being selected by the t statistic. The point estimate of β_1 is -10.46. Table 5 shows the values of Z_β for $L(m)=a\log(m)$ with $a=1/3, 1$, and 3. These values are all significant at the 1% level. The results suggest very strong evidence of bubble migration from the housing market to the oil price market, consistent with Hypothesis B(i).

- For the time series spread between the Baa and Aaa bond rates, the DF_r^t statistic suggests random wandering behavior in the series for much of the period but reveals a significant bubble from September 22, 2008 to April 20, 2009. This period corresponds to the rapid acceleration of financial distress in the weeks following the Lehman Brothers bankruptcy on September 15, 2008. To test bubble migration from house prices to bond prices, we ran the empirical regression (19) with r_p being selected by the t statistic. In the right panel of Figure 6, we plot $(\hat{\theta}_X(\tau)-1)\frac{\tau}{n}$, where $\hat{\theta}_X(\tau)$ relates to the house market, and $\hat{\theta}_Y(\tau)-1$, where $\hat{\theta}_Y(\tau)$ relates to the bond market. The point estimate of β_1 is -12.00. Table 6 shows the values of Z_β for $L(m)=a\log(m)$ with $a=1/3, 1$, and 3. These values are all significant at the 1% level. The results suggest very strong evidence of bubble migration from the housing market to the bond market, consistent with Hypothesis B(ii).

- The bubble in the oil price market collapsed in July 2008 and the bubble in the bond market collapsed on April 20, 2009. Both findings are consistent with Hypothesis C.

In sum, the tests reveal bubble characteristics in the data that are consistent with Hypotheses A–C. The empirical estimates of the crisis timeline broadly support the predictions made in the CFG (2008a) model. Figure 7 shows the complete timeline of the bubble phenomena in a profile using the recursive tests. The timeline shows how bubbles migrated from the property market following the subprime crisis to certain goods in the commodity markets and then to the bond market.

To assess whether these bubble characteristics were generic or specific features of commodity and financial markets during the subprime crisis and its aftermath, we applied the methods more broadly to many series in a secondary data set. To preserve

TABLE 6. Values of Z_β to test migration from the housing market to the bond market.

	$a=1/3$	$a=1$	$a=3$
$m=30$	−24.37	−8.12	−2.71

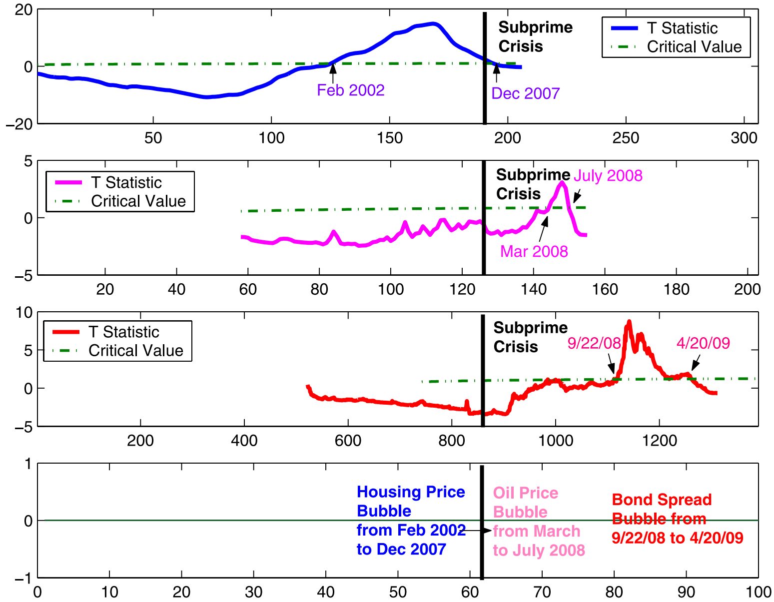


FIGURE 7. Timeline of financial bubbles in the real estate, commodity, and bond markets. The panels show recursive calculations of the *DF t* statistic and critical values highlighting the successive bubble episodes.

space, we present only summary empirical results of these findings in Table 7 without plotting the recursive test statistics.

Although it is clear from the empirical results obtained earlier that funds moved across markets during the crisis period for flight-to-quality and flight-to-liquidity reasons, the results in Table 7 suggest that investors were selective in transferring assets. For example, in the commodity market, we identify a bubble in heating oil prices, with similar origination and collapsing dates as those for crude oil prices. However, we find no evidence of bubbles in coffee, cotton, sugar, and feeder cattle prices.

5. CONCLUSIONS

This paper provides an empirical study of the bubble characteristics in several key financial variables over an historical time period that includes the subprime crisis and its

TABLE 7. Test results for the presence of bubbles and date stamps.

	$\max DF_r^t$	$\hat{\tau}_e$	$\hat{\tau}_f$
Heating oil	2.2416	March/08	August/08
Coffee	−0.7002	NA	NA
Cotton	−0.0866	NA	NA
Cocoa	0.9872	NA	NA
Sugar	−0.2220	NA	NA
Feeder cattle	0.4327	NA	NA

sequel, including global effects. The econometric methods employed are based on recursive regression, right-sided unit root tests, and a newly developed dating technology and associated limit theory from Phillips and Yu (2009) and Phillips, Shi, and Yu (2011). The methods are complemented by a mechanism for testing potential migration across markets. The dating techniques enable us to track the timeline of the crisis in terms of the individual series by empirically determining the origination and collapse of each of the bubbles. The dates are matched against the onset date for the subprime crisis as well as a specific sequential hypothesis concerning bubble migrations that are predicted in the theoretical model proposed by CFG (2008a). Our estimates suggest that bubbles emerged in the housing market before the subprime crisis and collapsed with the subprime crisis. The bubble then migrated from the housing market to selected commodity markets and the bond market after the crisis erupted into the public arena. All these bubbles collapsed as the financial crisis impacted real economic activity. The estimated sequence of the bubble migration phenomenon is broadly consistent with the predictions of CFG (2008a).

The methods used here can be used to provide early warning diagnostics for market exuberance as they provide consistent tests for mildly explosive behavior. Such diagnostics may assist policy makers in framing early monetary policy responses or other regulatory actions or interventions to combat speculative bubbles in financial markets.

Appendix: Limit theory for the bubble migration test

Consistent with the notion of bubble migration and to fix ideas, it is subsequently assumed that the period of bubble collapse in X_t is relatively short prior to the emergence of exuberance in Y_t. We therefore set $\tau_{pX} = \tau_{eY} + o(n)$. Then $\tau_{pX} = \lfloor nr_{pX} \rfloor = \lfloor nr_{eY} \rfloor \{1 + o(1)\}$ and we define $m = \lfloor nr_{pY} \rfloor - \lfloor nr_{pX} \rfloor$.

The empirical regression version of (19) has the form

$$\hat{\theta}_Y(\tau) - 1 = \hat{\beta}_{0n} + \hat{\beta}_{1n}(\hat{\theta}_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m} + \text{error} \tag{21}$$

over the interval $\tau \in (\tau_{pX}, \tau_{pY}]$. Our first interest is in the null behavior of the fitted slope coefficient

$$\hat{\beta}_{1n} = \frac{\displaystyle\sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_Y(\tau)\tilde{Z}_X(\tau)}{\displaystyle\sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_X(\tau)^2},$$

where

$$Z_Y(\tau) = \hat{\theta}_Y(\tau) - 1, \qquad Z_X(\tau) = (\hat{\theta}_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m}$$

and

$$\tilde{Z}_a(\tau) = Z_a(\tau) - m^{-1} \sum_{s=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor} Z_a(s) \quad \text{for } a = X, Y.$$

Under the null hypothesis $\theta_Y(\tau) = 1 + \frac{c_Y}{n^\alpha}$, the estimation error $\hat{\theta}_Y(\tau) - \theta_Y(\tau)$ has the asymptotic behavior (as shown in Phillips, Shi, and Yu (2011))

$$\frac{n^\alpha \theta_n^{\tau-\tau_{eY}}}{2c_Y}(\hat{\theta}_Y(\tau) - \theta_Y(\tau)) \Rightarrow \frac{\frac{1}{r}\int_0^{r_{eY}} B_Y(s)\,ds}{B_Y(r_e)} =: \frac{1}{r}\mathbb{C} \quad \text{say.} \tag{22}$$

Here $\mathbb{C} = \int_0^{r_{eY}} B_Y(s)\,ds / B_Y(r_e)$ is a Cauchy-like random variable, being a ratio of correlated normal variables, and $B_Y(s)$ is the Brownian motion (or more general Gaussian process) corresponding to the limit under weak convergence of the standardized series $n^{-1/2}Y_{\lfloor ns\rfloor}$ prior to exuberance in Y_t (i.e., for $s < r_{eY}$). Then

$$\begin{aligned} nZ_Y(\lfloor nr\rfloor) &= n(\hat{\theta}_Y(\tau) - 1) = n(\theta_Y(\tau) - 1) + n(\hat{\theta}_Y(\tau) - \theta_Y(\tau)) \\ &= n^{1-\alpha}c_Y + \frac{n^{1-\alpha}2c_Y}{r\theta_Y^{\tau-\tau_{eY}}}\mathbb{C}\{1 + o_p(1)\} \quad \text{for } r > r_{eY}. \end{aligned} \tag{23}$$

It follows that

$$\begin{aligned} n\tilde{Z}_Y(\tau) &= nZ_Y(\tau) - n\frac{\sum_{k=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor} Z_Y(k)}{\lfloor nr_{pY}\rfloor - \lfloor nr_{pX}\rfloor} \\ &= n^{1-\alpha}2c_Y\mathbb{C}\left\{\frac{1}{r\theta_Y^{\tau-\tau_{eY}}} - \frac{\sum_{k=\tau_{eY}}^{\tau_{pY}} \frac{1}{\theta_Y^{k-\tau_{eY}}(k/n)}}{\lfloor nr_{pY}\rfloor - \lfloor nr_{pX}\rfloor}\right\}\{1 + o_p(1)\} \\ &= n^{1-\alpha}2c_Y\mathbb{C}\left\{\frac{1}{r\theta_Y^{\tau-\tau_{eY}}} - \frac{1}{(r_{pY} - r_{eY})}\int_{r_{eY}}^{r_{pY}} \frac{e^{-c_Y(s-r_{eY})n^{1-\alpha}}}{s}\right\} \\ &\quad \times \{1 + o_p(1)\} \\ &\sim n^{1-\alpha}2c_Y\mathbb{C}\left\{\frac{1}{r\theta_Y^{\tau-\tau_{eY}}} - \frac{1}{n^{1-\alpha}(r_{pY} - r_{eY})r_{eY}c_Y}\right\}, \end{aligned} \tag{24}$$

since as $n \to \infty$,

$$\frac{1}{n}\sum_{k=\tau_{eY}}^{\tau_{pY}} \frac{1}{\theta_Y^{k-\tau_{eY}}(k/n)} = \int_{r_{eY}}^{r_{pY}} \frac{e^{-c_Y(s-r_{eY})n^{1-\alpha}}}{s}\,ds\{1 + o(1)\}$$

$$
\begin{aligned}
&= \int_0^{r_{pY}-r_{eY}} \frac{e^{-c_Y p n^{1-\alpha}}}{p + r_{eY}}\, dp\{1+o(1)\} \\
&= \frac{1}{n^{1-\alpha}} \int_0^{(r_{pY}-r_{eY})n^{1-\alpha}} \frac{e^{-c_Y q}}{r_{eY} + q/n^{1-\alpha}}\, dq\{1+o(1)\} \\
&= \frac{1}{n^{1-\alpha} r_{eY} c_Y}\{1+o(1)\}.
\end{aligned}
\tag{25}
$$

A more general version of this type of asymptotic approximation is developed in (28) below.

To obtain the corresponding limit theory for $\hat{\theta}_X(\tau)$, we assume that from the bubble peak $\tau = \lfloor nr_{pX} \rfloor$, the autoregressive coefficient $\theta_X(\tau)$ falls rapidly during the collapse period to the immediate vicinity of unity. We use data from $[\tau_{pX}, \tau_{pY}]$ to compute $\hat{\theta}_X(\tau)$ recursively for $\tau = \lfloor nr \rfloor$ by least squares regression, giving data for the regressor in (21), namely

$$
\begin{aligned}
nZ_X(\lfloor nr \rfloor) &= n(\hat{\theta}_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m} \\
&= n(\hat{\theta}_X(\tau) - \theta_X(\tau))\frac{\tau - \tau_{pX}}{m} + n(\theta_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m} \\
&\sim \frac{\dfrac{r - r_{pX}}{r_{pY} - r_{pX}} \displaystyle\int_{r_{pX}}^{r} \tilde{W}_X(s)\,dW_X(s)}{\displaystyle\int_{r_{pX}}^{r} \tilde{W}_X^2(s)} + \frac{r - r_{pX}}{r_{pY} - r_{pX}} c_X \\
&=: \frac{r - r_{pX}}{r_{pY} - r_{pX}} h_X(r) + \frac{r - r_{pX}}{r_{pY} - r_{pX}} c_X =: g_X(r),
\end{aligned}
\tag{26}
$$

where $\tilde{W}_X(s) = W_X(s) - \frac{1}{r}\int_0^r W_X(p)\,dp$ for $s \in [0, r]$ is a Gaussian process (a demeaned Brownian motion if $\theta_X(\tau) = 1$ and a demeaned diffusion if $\theta_X = 1 + c_X/n$ for some constant localizing coefficient c_X). Since the regressor in (21) is $Z_X(\tau) = (\hat{\theta}_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m}$, we define the demeaned and scaled variable for $\tau = \lfloor nr \rfloor$ over the sample period $(\tau_{pX}, \tau_{pY}]$ as

$$
\begin{aligned}
n\tilde{Z}_X(\tau) &= nZ_X(\tau) - n\frac{\displaystyle\sum_{k=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} Z_X(k)}{\lfloor nr_{pY} \rfloor - \lfloor nr_{pX} \rfloor} \\
&\sim g_X(r) - \frac{1}{r_{pY} - r_{pX}} \int_{r_{pX}}^{r_{pY}} g_X(s)\,ds =: \tilde{g}_X(r).
\end{aligned}
$$

An important aspect of the migration test is that it does not require explicit characterization of the limit process $\tilde{W}_X(s)$ or the demeaned process $\tilde{g}_X(r)$, thereby accommodating any value of θ_X that is local to unity and various forms of unconditional and conditional heterogeneity in the X_t error process. The null hypothesis is therefore highly composite in terms of potential generating mechanisms leading to the limit process $\tilde{W}_X$.

Under the null, we have

$$
\begin{aligned}
\hat{\beta}_{1n} &= \frac{\displaystyle\sum_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor} \tilde{Z}_Y(\tau)\tilde{Z}_X(\tau)}{\displaystyle\sum_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor} \tilde{Z}_X(\tau)^2} = \frac{\displaystyle\frac{1}{n}\sum_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor} \{n\tilde{Z}_Y(\tau)\}\{n\tilde{Z}_X(\tau)\}}{\displaystyle\frac{1}{n}\sum_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor} \{n\tilde{Z}_X(\tau)\}^2} \\
&\sim \frac{\displaystyle n^{1-\alpha}2c_Y\mathbb{C}\frac{1}{n}\sum_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor}\{n\tilde{Z}_X(\tau)\}\left\{\frac{1}{\theta_Y^{\tau-\tau_e}\tau/n} - \frac{1}{n^{1-\alpha}(r_{pY}-r_{eY})r_{eY}c_Y}\right\}}{\displaystyle\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)^2\,ds} \\
&= \frac{\displaystyle n^{1-\alpha}2c_Y\mathbb{C}\int_{r_{eY}}^{r_{pY}} \tilde{g}_X(s)s^{-1}e^{-c_Y n^{1-\alpha}(s-r_{eY})}\,ds}{\displaystyle\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)^2\,ds} \\
&\sim \frac{n^{1-\alpha}2c_Y\mathbb{C}}{\displaystyle\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)^2\,ds}\frac{\tilde{g}_X(r_{eY})}{n^{1-\alpha}c_Y r_{eY}} \\
&\sim 2\frac{\mathbb{C}\tilde{g}_X(r_{eY})}{\displaystyle r_{eY}\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)^2\,ds}.
\end{aligned}
\tag{27}
$$

Here we use the fact that

$$
1/\theta_Y^{\tau-\tau_e} = 1\Big/\left(1+\frac{c_Y}{n^\alpha}\right)^{\tau-\tau_e} \sim e^{-c_Y n^{1-\alpha}(r-r_{eY})}
$$

and employ the asymptotic approximation

$$
\begin{aligned}
\int_a^b f(s)e^{-cn^\gamma(s-a)}\,ds &= \int_0^{b-a} f(p+a)e^{-cn^\gamma p}\,dp \\
&= \int_0^{(b-a)n^\gamma} f\left(a+\frac{q}{n^\gamma}\right)e^{-cq}\,dq/n^\gamma \\
&\sim \frac{f(a)}{n^\gamma}\int_0^\infty e^{-cq}\,dq = \frac{f(a)}{n^\gamma c},
\end{aligned}
\tag{28}
$$

to the integral in the numerator of (27), of which (25) above is a special case and for which there are other uses. It follows that

$$
\hat{\beta}_{1n} \Rightarrow 2\frac{\mathbb{C}\tilde{g}_X(r_{eY})}{r_{eY}\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)^2\,ds}, \tag{29}
$$

which gives the limit distribution of $\hat{\beta}_{1n}$ under the null hypothesis. The form of this limit distribution depends on the components $(\mathbb{C}, \tilde{g}_X)$, which in turn depend on the precise form of the generating mechanism for X_t, including the localizing coefficient c_X, and the generating mechanism for Y_t. Thus, as it stands $\hat{\beta}_{1n}$ is not well suited for testing because of the difficulty in constructing critical values for the nonpivotal limit theory. However, it is possible to construct a conservative test, as explained in the text using the statistic $Z_\beta = \hat{\beta}_{1n}/L(m)$, where $L(m) \to \infty$ is slowly varying at infinity and satisfies $L(m) = o(n^\varepsilon)$ for all $\varepsilon > 0$ as $n \to \infty$.

To develop a limit theory under the alternative, we consider the specific form

$$\theta_Y(\tau) = \begin{cases} 1, & \tau < \tau_{eY}, \\ 1 + \dfrac{c_Y}{n} + d\dfrac{c_X}{n^\alpha}\left(\dfrac{\tau - \tau_{pX}}{m}\right)^2, & \tau \geq \tau_{eY}, \end{cases} \tag{30}$$

where $\theta_X(\tau) = 1 + \frac{c_X}{n}\frac{\tau-\tau_{pX}}{m}$, possibly with $c_X < 0$, so that for $\tau \geq \tau_{eY}$ we have

$$\theta_Y(\tau) - 1 = \frac{c_Y}{n^\alpha} + d\frac{c_X}{n^\alpha}\left(\frac{\tau - \tau_{pX}}{m}\right)^2 = \beta_{0n} + \beta_{1n}(\theta_X(\tau) - 1)\frac{\tau - \tau_{pX}}{m}, \tag{31}$$

with $\beta_{1n} = dn^{1-\alpha} \neq 0$ and $\beta_{0n} = \frac{c_Y}{n^\alpha}$. For Y_t to be mildly explosive from $\tau \geq \tau_{eY}$, we require that

$$\min_{\tau \in [\tau_e, \tau_p]}\left(c_Y + dc_X\left(\frac{\tau - \tau_{pX}}{m}\right)^2\right) > 0$$

and, in general, we expect $dc_X > 0$ or $d < 0$ if $c_X < 0$, so that under the alternative hypothesis, the collapse in X_t has a migratory effect that raises exuberance in Y_t. The model (31) allows for explicit evolution in the autoregressive coefficient $\theta_Y(\tau)$ that depends on the presence of the localizing autoregressive coefficient c_X and the trend factor $\frac{\tau-\tau_{pX}}{m}$.

The specific alternative (31) involves a linear dependence of $\theta_Y(\tau)$ on c_X but more general dependencies might be formulated as indicated above in (18). In (18), we functionalize the dependence as $c_X(\frac{\tau-\tau_{pX}}{m})$ to allow for changes in the autoregressive structure of X_t during the collapse period in which X_t returns to a unit root process, with an implied impact on the generating mechanism of $\theta_Y(\tau)$. In such a case, the localizing coefficient $c_X = c_X(\frac{\tau-\tau_{pX}}{m})$ captures changes in the autoregressive process for X_t as the bubble in X_t collapses, and this evolution is mapped into the generating mechanism for Y_t by means of a localizing coefficient of the form

$$\theta_Y(\tau) = 1 + \frac{c_Y}{n^\alpha} + d\frac{c_X\left(\dfrac{\tau - \tau_{pX}}{m}\right)}{n^\alpha}\frac{\tau - \tau_{pX}}{m}.$$

Development of the limit theory under this more general functional alternative is, of course, more complicated than under (31) and is not pursued here.

The model (31) does not hold exactly for the empirical regression which uses estimated versions of $\theta_Y(\tau) - 1$ and $\theta_X(\tau) - 1$. However, it can be verified that since the formulation given in (31) still produces a mildly explosive process in Y_t, we have the relationship, (just as in the basic case (22) where $\beta_{1n}=0$ and $\theta_Y(\tau)=1+\frac{c_Y}{n^\alpha}$)

$$\begin{aligned}\hat{\theta}_Y(\tau)-1&=\theta_Y(\tau)-1+(\hat{\theta}_Y(\tau)-\theta_Y(\tau))\\&=\theta_Y(\tau)-1+O_p\left(\frac{1}{n^\alpha\theta_{n,\min}^{\tau-\tau_{eY}}}\right),\end{aligned}$$

with

$$\theta_{n,\min}=1+\frac{\min\limits_{\tau\in[\tau_e,\tau_p]}\left\{c_Y+dc_X\left(\frac{\tau-\tau_{pX}}{m}\right)^2\right\}}{n^\alpha},$$

which involves a negligible estimation error. The estimation error in $\hat{\theta}_X(\tau)$ is, as before,

$$\hat{\theta}_X(\tau)-1=(\theta_X(\tau)-1)+(\hat{\theta}_X(\tau)-\theta_X(\tau))\sim(\theta_X(\tau)-1)+\frac{1}{n}h_X\left(\frac{\tau}{n}\right),$$

since from (26) we have $n(\hat{\theta}_X(\tau)-\theta_X(\tau))\Rightarrow h_X(r)$ for $\tau=\lfloor nr\rfloor$. So the fitted model can be written in the form

$$\begin{aligned}\hat{\theta}_Y(\tau)-1&=\frac{c_Y}{n^\alpha}+d\frac{n(\theta_X(\tau)-1)}{n^\alpha}\frac{\tau-\tau_{pX}}{m}+O_p\left(\frac{1}{n^\alpha\theta_{n,\min}^{\tau-\tau_{eY}}}\right)\\&=\frac{c_Y}{n^\alpha}+d\frac{n(\hat{\theta}_X(\tau)-1)}{n^\alpha}\frac{\tau-\tau_{pX}}{m}+d\frac{(\theta_X(\tau)-\hat{\theta}_X(\tau))n}{n^\alpha}\frac{\tau-\tau_{pX}}{m}\\&\quad+O_p\left(\frac{1}{n^\alpha\theta_{n,\min}^{\tau-\tau_{eY}}}\right)\\&\sim\frac{c_Y}{n^\alpha}+d\frac{n(\hat{\theta}_X(\tau)-1)}{n^\alpha}\frac{\tau-\tau_{pX}}{m}-\frac{d}{n^\alpha}h_X\left(\frac{\tau}{n}\right)\frac{\tau-\tau_{pX}}{m}\\&=\beta_{0n}+\beta_{1n}\left\{(\hat{\theta}_X(\tau)-1)\frac{\tau-\tau_{pX}}{m}-\frac{1}{n}h_X\left(\frac{\tau}{n}\right)\frac{\tau-\tau_{pX}}{m}\right\}.\end{aligned}$$

If $d=0$, then $\beta_{1n}=0$ and null behavior is the same as that described above. If $d\neq 0$, then the limit behavior under the alternative is

$$\hat{\beta}_{1n}=\frac{\sum\limits_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor}\tilde{Z}_Y(\tau)\tilde{Z}_X(\tau)}{\sum\limits_{\tau=\lfloor nr_{pX}\rfloor}^{\lfloor nr_{pY}\rfloor}\tilde{Z}_X(\tau)^2}$$

$$= \beta_{1n} \left\{ \frac{\sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_X(\tau)^2}{\sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_X(\tau)^2} - \frac{\frac{1}{n} \sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_X(\tau) h_X\left(\frac{\tau}{n}\right) \frac{\tau - \tau_{pX}}{m}}{\sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} \tilde{Z}_X(\tau)^2} \right\}$$

$$= \beta_{1n} \left\{ 1 - \frac{\frac{1}{n} \sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} (n\tilde{Z}_X(\tau)) h_X\left(\frac{\tau}{n}\right) \frac{\tau - \tau_{pX}}{m}}{\frac{1}{n} \sum_{\tau=\lfloor nr_{pX} \rfloor}^{\lfloor nr_{pY} \rfloor} (n\tilde{Z}_X(\tau))^2} \right\}$$

$$\sim dn^{1-\alpha} \left\{ \frac{\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)\left(\tilde{g}_x(s) - h_X(s) \frac{s - r_{pX}}{r_{pY} - r_{pX}}\right) ds}{\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)^2 \, ds} \right\}.$$

From (26), we have $h_X(s)\frac{s-r_{pX}}{r_{pY}-r_{pX}} + c_X \frac{s-r_{pX}}{r_{pY}-r_{pX}} = g_X(s)$. Then

$$\int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s)\left(\tilde{g}_X(s) - h_X(s) \frac{s - r_{pX}}{r_{pY} - r_{pX}}\right) ds = c_X \int_{r_{pX}}^{r_{pY}} \tilde{g}_X(s) \frac{s - r_{pX}}{r_{pY} - r_{pX}} \, ds \neq 0,$$

provided that $\int_{r_{pX}}^{r_{eY}} \tilde{g}_X(s)\frac{s-r_{pX}}{r_{pY}-r_{pX}} \, ds \neq 0$ and $c_X \neq 0$. It follows that the statistic $\hat{\beta}_{1n} = O_p(n^{1-\alpha})$ and is divergent under the alternative hypothesis $\mathcal{H}_1 : \beta_{1n} \neq 0$. We deduce that the test statistic $Z_\beta = \hat{\beta}_{1n}/L(m)$ leads to an asymptotically conservative test under the null when a fixed critical value from the standard normal distribution is used because $Z_\beta \to_p 0$ and the test is consistent under the alternative $\mathcal{H}_1 : \beta_{1n} \neq 0$, provided that $c_X \neq 0$.

Figures 8 and 9 show densities of $Z_\beta = \hat{\beta}_{1n}/\ln(m)$ under $\mathcal{H}_0$ and $\mathcal{H}_1$ for various sample sizes for a model with localizing coefficients $c_X = -2$ (for the period of collapse for the X_t series), $c_Y = 1$, $\alpha = 0.85$, and $d = -3$ for the mildly explosive au-

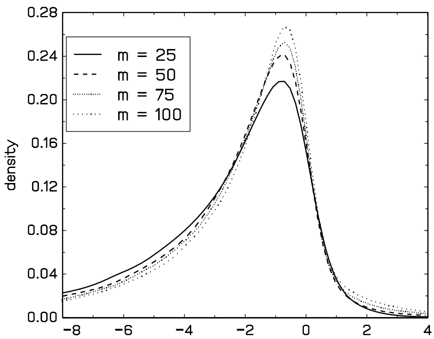


FIGURE 8. Densities of Z_β under the null.

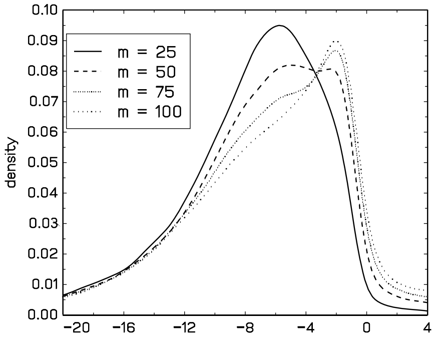


FIGURE 9. Densities of Z_β under the alternative.

toregressive parameter settings for Y_t (i.e., $\theta_Y = 1 + c_Y/n^\alpha$ under the null and $\theta_Y = 1 + \frac{c_Y}{n} + d\frac{c_X}{n^\alpha}\frac{\tau-\tau_{pX}}{m}$ under the alternative in (30)) and date settings $r_{eX} = 0.25$, $r_{pX} = 0.50$, $r_{eY} = 0.625$, and $r_{pY} = 0.875$. The densities are computed from 50,000 replications with $m = 25, 50, 75, 100$.

References

Baker, D. (2002), "The run-up in home prices: Is it real or is it another bubble?" Working paper, Center for Economic and Policy Research. [480]

Brunnermeier, M. K. (2009), "Deciphering the 2007-08 liquidity and credit crunch." *Journal of Economic Perspectives*, 23, 77–100. [464]

Caballero, R. J., E. Farhi, and P.-O. Gourinchas (2008a), "Financial crash, commodity prices and global imbalances." *Brookings Papers on Economic Activity*, Fall, 1–55. [455, 457, 459, 465, 476, 477, 481, 483]

Caballero, R. J., E. Farhi, and P.-O. Gourinchas (2008b), "An equilibrium model of "global imbalances" and low interest rates." *American Economic Review*, 98, 358–393. [465]

Callis, R. and L. Cavanaugh (2007), "Census Bureau Reports on residential vancancies and homeownership." U.S. Census Bureau. [465]

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay (1997), *The Econometrics of Financial Markets*. Princeton University Press, Princeton, New Jersey. [457]

Campbell, J. Y. and R. J. Shiller (1988), "Stock prices, earnings, and expected dividends." *Journal of Finance*, 43, 661–676. [460]

Davis, M. A., A. Lehnert, and R. F. Martin (2008), "The rent-price ratio for the aggregate stock of owner-occupied housing." *Review of Income and Wealth*, 54, 279–284. [476]

Economist Newspaper Limited (2005), "After the fall." Economist, June 18. [466]

Economist Newspaper Limited (2008), "The end of the affair." Economist, November 22. [465]

Evans, G. W. (1991), "Pitfalls in testing for explosive bubbles in asset prices." *American Economic Review*, 81, 922–930. [457, 468]

Greenlaw, D., J. Hatzius, A. Kashyap, and H. S. Shin (2008), "Leveraged losses: Lessons from the mortgage market meltdown." U.S. Monetary Policy Forum Report 2. [464]

Hull, J. (2008), "The credit crunch of 2007: What went wrong? Why? What lessons can be learned?" Working paper, Rotman School of Management, University of Toronto. [464]

Lee, J.-H. and P. C. B. Phillips (2011), "Asset pricing with financial bubble risk." Working paper, Yale University. [460]

Phillips, P. C. B. (1987), "Time series regression with a unit root." *Econometrica*, 55, 277–301. [467]

Phillips, P. C. B. (1996), "Econometric model determination." *Econometrica*, 64, 763–812. [471]

Phillips, P. C. B. (2008), "Unit root model selection." *Journal of the Japan Statistical Society*, 38, 65–74. [472]

Phillips, P. C. B. and T. Magdalinos (2007a), "Limit theory for moderate deviations from a unit root." *Journal of Econometrics*, 136, 115–130. [457, 466, 467, 469]

Phillips, P. C. B. and T. Magdalinos (2007b), "Limit theory for moderate deviations from unity under weak dependence." In *The Refinement of Econometric Estimation and Test Procedures: Finite Sample and Asymptotic Analysis* (G. D. A. Phillips and E. Tzavalis, eds.) 123–162. Cambridge University Press, Cambridge. [457, 466, 469]

Phillips, P. C. B. and T. Magdalinos (2009), "Unit root and cointegrating limit theory when initialization is in the infinite past." *Econometric Theory*, 25, 1682–1715. [471]

Phillips, P. C. B., S. Shi, and J. Yu (2011), "Testing for multiple bubbles." Working paper, Yale University. [458, 466, 471, 474, 483, 484]

Phillips, P. C. B., Y. Wu, and J. Yu (2011), "Explosive behavior in the 1990s Nasdaq: When did exuberance escalate asset values?" *International Economic Review*, 52, 201–226. [455, 457, 466, 468]

Phillips, P. C. B. and J. Yu (2009), "Limit theory for dating the origination and collapse of mildly explosive periods in time series data." Unpublished manuscrip, Sim Kee Boon Institute for Financial Economics, Singapore Management University. [458, 466, 468, 469, 483]

Schwarz, G. (1978), "Estimating the dimension of a model." *Annals of Statistics*, 6, 461–464. [471]

Submitted May, 2010. Final version accepted July, 2011.